Exercise 5.15

Recall from basic interest theory the following relationships:

$$i = e^{\delta} - 1, \quad i^{(m)} = m\left(e^{\delta/m} - 1\right),$$

$$d = 1 - e^{-\delta}, \quad \text{and} \quad d^{(m)} = m\left(1 - e^{-\delta/m}\right).$$

(a) Write $\alpha(m)$ as a function of δ:

$$\alpha(m) = \frac{i}{i^{(m)}} \cdot \frac{d}{d^{(m)}} = \frac{e^{\delta} - 1}{m(e^{\delta/m} - 1)} \cdot \frac{1 - e^{-\delta}}{m(1 - e^{-\delta/m})} = \frac{1}{m^2} e^{\delta[(1/m)-1]} \left(\frac{e^{\delta} - 1}{e^{\delta/m} - 1}\right)^2.$$

Let this be $g_1(\delta)$ and use Taylor's series expansion to express g_1 in terms of powers of δ. It is a very tedious exercise to even show that $g_1(0) = 1$, $g_1'(0) = 0$ and $g_1''(0) = (m^2 - 1)/(6m^2)$ so that we can write

$$\alpha(m) = g_1(0) + g_1'(0)\delta + \frac{1}{2} g_1''(0)\delta^2 + \cdots = 1 + \frac{m^2 - 1}{12m^2}\delta^2 + \cdots$$

Thus, we see that removing powers of 2 and higher, we get the approximation $\alpha(m) \approx 1$. One can also verify, using Mathematica for example, that

$$\alpha(m) = 1 + \frac{m^2 - 1}{12m^2}\delta^2 + \frac{2m^4 - 5m^2 + 3}{720m^4}\delta^4 + \cdots$$

(b) Similarly, write $\beta(m)$ as a function of δ:

$$\beta(m) = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} = \frac{(e^{\delta} - 1) - [m(e^{\delta/m} - 1)]}{m(e^{\delta/m} - 1) \cdot m(1 - e^{-\delta/m})} = \frac{1}{m^2} e^{\delta/m} \cdot \frac{(e^{\delta} - 1) - [m(e^{\delta/m} - 1)]}{(e^{\delta/m} - 1)^2}.$$

Let this be $g_2(\delta)$ and again use Taylor's series expansion. It is equally very tedious to show that $g_2(0) = (m - 1)/2m$, $g_2'(0) = (m^2 - 1)/(6m^2)$ so that we can write

$$\beta(m) = g_2(0) + g_2'(0)\delta + \cdots = \frac{m - 1}{2m} + \frac{m^2 - 1}{6m^2}\delta + \cdots$$

Thus, we see that removing powers of 1 and higher, we get the approximation $\beta(m) \approx (m - 1)/2m$. For additional terms in the series expansion, one can verify, again with Mathematica for example, that we have

$$\beta(m) = \frac{m - 1}{2m} + \frac{m^2 - 1}{6m^2}\delta + \frac{24m^2 - 1}{24m^2}\delta^2 + \frac{3m^4 - 5m^2 + 2}{360m^4}\delta^3 + \cdots$$

The results in this problem indeed lead us to the common approximation

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m - 1}{2m}.$$