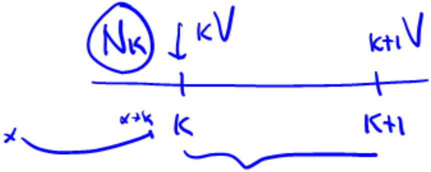
Reserves

Policy Values - additional topics

Lecture: Week 5

Chapter summary - additional topics

- Other topics needed to be covered from this chapter:
    - analysis of profit or loss and analysis by source (mortality, interest, expenses)
    - asset shares
    - Thiele's differential equation for reserve calculation
    - policy alterations
    - modified reserve systems
- Chapters 7 (Dickson, et al.): Sec 7.3.5, 7.5, 7.6, 7.9



$$ {}_{k+1}V = \frac{({}_kV + P - e_k)(1+i) - B\, q_{x+k}}{1 - q_{x+k}} $$

$$ {}_{k+1}V = ({}_kV + P - e_k)(1+i) - (B - {}_{k+1}V)\, q_{x+k} $$

$$ N_k\ {}_{k+1}V = {}_{k+1}V^{E} \longleftrightarrow {}_{k+1}V^{A} $$

$$ \text{Gain/Profit} = {}_{k+1}V^{A} - {}_{k+1}V^{E} $$

Profit defined

Consider the period between years k and $k+1$ and our block of policies at the beginning of this period has a total of N_k (active) policies.

Now denote by ${}_kV$ and ${}_{k+1}V$ the gross premium reserves at the beginning and end of the period, on a per policy basis. Thus, on an expected basis, the ending (total) gross premium reserve for our block of policies is

$$ {}_{k+1}V^{\mathsf{E}} = N_k \cdot {}_{k+1}V $$

Applying the recursion equation, we can express this total reserve as

$$ {}_{k+1}V^{\mathsf{E}} = \big(N_k\, {}_kV + N_k G - N_k e_k\big)(1+i) - \big(B - {}_{k+1}V\big) N_k \cdot q_{x+k} $$

where clearly $N_k \cdot q_{x+k}$ is the expected number of deaths for the period.

If we denote the ending (total) actual gross premium reserve held for this block of policies by ${}_{k+1}V^{\mathsf{A}}$, then the insurer's profit for the period is the difference:

$$ \text{Profit}_k = {}_{k+1}V^{\mathsf{A}} - {}_{k+1}V^{\mathsf{E}} $$

Sources of profit

Clearly, the profit (or loss) earned during the period can be derived from essentially three sources:

- Interest
  - there is a gain if we earn an interest higher than expected (and vice versa)
- Mortality
  - there is a gain if we have fewer deaths than expected (and vice versa)
- Expenses
  - there is a gain if we have lesser expenses than expected (and vice versa)

Gain from interest

$i' = \text{actual}$

Suppose that during the period, the insurer earned an interest rate of i' instead of i.

For simplicity (for now), suppose that the rest of the actual experience (i.e. mortality and expenses) are as expected.

The (total) actual reserve at the end of the period is

$$_{k+1}V^{\mathsf{A}} = \big(N_{k\,k}V + N_k G - N_k e_k\big)(1+i') - \big(B - {}_{k+1}V\big)N_k \cdot q_{x+k}$$

$$\rightarrow {}_{k+1}V^{E} = \big(N_{k\,k}V + N_k G - N_k e_k\big)(1+i) - \big(B - {}_{k+1}V\big)N_k \cdot q_{x+k}$$

The difference between the actual and expected reserves then can be written as

$$\text{gain from interest} = \big(N_{k\,k}V + N_k G - N_k e_k\big)(i' - i)$$

Gain from expenses

$e_k' = \text{actual}$

Suppose that during the period, the insurer's actual expenses is e_k' on a per policy basis.

Again for simplicity (for now), suppose that the rest of the actual experience (i.e. mortality and interest) are as expected.

The (total) actual reserve at the end of the period is

$$_{k+1}V^{\mathsf{A}} = \left(\cancel{N_k\,{}_kV} + \cancel{N_k G} - N_k e_k'\right)(1+i) - \left(B - \cancel{{}_{k+1}V}\right)N_k \cdot q_{x+k}$$

$$-)\quad {}_{k+1}V^{E} = \left(\cancel{N_k\,{}_kV} + \cancel{N_k G} - N_k e_k\right)(1+i) - \left(B - \cancel{{}_{k+1}V}\right)\cancel{N_k \cdot q_{x+k}}$$

The difference between the actual and expected reserves then can be written as

$$\text{gain from expenses} = \left(N_k e_k - N_k e_k'\right)(1+i) = N_k\left(e_k - e_k'\right)(1+i)$$

Gain from mortality

$N_k \, q'_{x+k} = D'_k$

Suppose that during the period, the actual (total) number of deaths is D'_k. Note that the expected number of deaths from the N_k lives is $N_k \cdot q_{x+k}$.

Again for simplicity (for now), suppose that the rest of the actual experience (i.e. interest and expenses) are as expected.

The (total) actual reserve at the end of the period is

$$_{k+1}V^{\mathsf{A}} = \big(N_k \, _kV + N_k G - N_k e_k\big)(1+i) - \big(B - \, _{k+1}V\big) D'_k$$

$\rightarrow \, _{k+1}V^E = (N_k \, _kV + \ldots - (B - \, _{k+1}V) N_k q_{x+k}$

The difference between the actual and expected reserves then can be written as

$$\text{gain from mortality} = \big(B - \, _{k+1}V\big)\big(N_k \cdot q_{x+k} - D'_k\big)$$

$= (B - \, _{k+1}V) N_k (q_{x+k} - q'_{x+k})$

Putting all the sources together

Suppose that during the period, the insurer earned an interest rate of i', the insurer's actual expenses is e'_k and the actual (total) number of deaths is D'_k.

The (total) actual reserve at the end of the period is

$$ {}_{k+1}V^{\mathsf{A}} = \big(N_k\, {}_kV + N_k G - N_k e'_k\big)(1+i') - \big(B - {}_{k+1}V\big)D'_k $$

$$\Rightarrow \; {}_{k+1}V^{E} = (N_k\, {}_kV + N_k G - N_k e_k)(1+i) - (B - {}_{k+1}V)\, N_k\, q_{x+k}$$

In this case, one should be able to have the relation:

$$\begin{aligned} \text{Profit}_k &= \text{gain from interest} \\ &\quad + \text{gain from expenses} \\ &\quad + \text{gain from mortality} \end{aligned}$$

interest → expenses → mortality

top → assumed
↓
bottom → actual once you accounted for it

Sources of profit with the following ordering: interest → expenses → mortality.

- gain from interest $= \big(N_k\, {}_kV + N_k G - N_k e_k\big)(i' - i)$
- gain from expenses $= \big(N_k e_k - N_k e_k'\big)(1+i') = N_k\big(e_k - e_k'\big)(1+i')$
- gain from mortality $= \big(B - {}_{k+1}V\big)\big(N_k \cdot q_{x+k} - D_k'\big)$

expenses → interest → mortality

Sources of profit with the following ordering: expenses → interest → mortality.

- gain from expenses $= \big(N_k e_k - N_k e'_k\big)(1+i) = N_k\big(e_k - e'_k\big)(1+i)$
- gain from interest $= \big(N_k\, {}_kV + N_k G - N_k e'_k\big)(i' - i)$
- gain from mortality $= \big(B - {}_{k+1}V\big)\big(N_k \cdot q_{x+k} - D'_k\big)$

E_{k+1} = claim/death expense

Remark on gain from mortality

Notice from the previous slides that the order where mortality is does not matter because its calculation neither involves interest nor expenses.

However, there are instances where there may be death-related expenses, usually denoted by E_{k+1}.

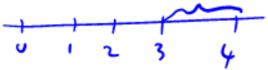
In this case, it matters where you order the mortality.

However, so long as you go with the following principle: when calculating gain of a source from the top, always use the expected experience of any sources unaccounted for, and once gain for a source is accounted for, use the actual experience.

Illustrative example from book

- Consider Example 7.8

Illustrative example 1



For a fully discrete 20-year term life insurance of $10,000 on (40), you are given:

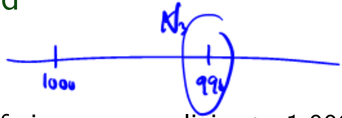
- The following actual and expected experience in year 4:

Experience	actual	expected	
Gross annual premium	\$ 90	\$ 90	
Expenses as a percent of premium	2.5%	3.0%	gain
$1000 \times q_{43}$	2	3	gain
Annual effective rate of interest	0.04	0.05	loss

- Profits are calculated based on the following (per policy) gross premium reserves:

$$ {}_3V = 100 \qquad {}_4V = 125 $$

Illustrative example 1 - continued



A company issued (such) 20-year term life insurance policies to 1,000 lives age 40 with independent future lifetimes.

At the end of the 3rd year, 990 (of these) insurances remain in force.

1. Calculate the total gain in year 4.
2. Allocate this total gain from the following sources (in the given order): interest, expenses, and mortality.
3. Allocate this total gain from the following sources (in the given order): expenses, interest, and mortality.

$$\underline{{}_{k+1}V^{A}} - \underline{{}_{k+1}V^{E}}$$

$${}_4V^A = \underbrace{990\big(100+90-.025(90)\big)(1.04)-(10{,}000-125)\,990\cdot\tfrac{2}{1000}}_{173{,}754.90}$$

$$-)\quad {}_4V^E = \underbrace{990\big(100+90-.03(90)\big)(1.05)-(10{,}000-125)\,990\cdot\tfrac{3}{1000}}_{165{,}369.60}$$

$$\text{Gain} = 173\,754.90 - 165\,369.60 = 8385.30$$

(b) interest → expenses → mortality

gain from interest: $990\big(100+90-.03(90)\big)(.04-.05) = -1854.27$

gain from expenses: $990\big(.03(90)-.025(90)\big)(1.04) = 463.32$

gain from death: $990(.003-.002)(10{,}000-125) = 9776.25$

$$(+) = 8385.30$$

(c) expenses → interest → mortality

gain from expenses: $990(.03(90) - .025(90))(1.05) = 467.775$

" " interest: $990(100 + 90 - .025(90))(.04 - .05) = -1858.725$

" " death: 9776.25

$\oplus = \underline{\underline{8385.30}}$

Example 7.8 - book

Friday -

20 year endowment to (60)

\$100,000 fully discrete

premium $G = 5200$

assume: $i = 5\%$

Standard Select Table

expenses: 10% 1st yr

5% subsequent yrs

\$200 on payment of death benefit

---

actual: $i' = 6.5\%$

mortality = 1 death

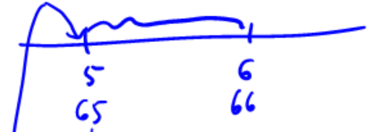
expenses = 6% of premium

expenses on death = \$250

${}_5V = 29068$

${}_6V = 35324$

gross reserve per policy



$q_{65} = .0059$

100 active

7.8

$$cV^{A} = 100\left(29068 + 5200 - .06(5200)\right)(1.065) - \left(100{,}000 + 250 - 35324\right)\cdot 1$$

$$\underbrace{}_{3{,}551{,}388}$$

$$cV^{E} = 100\left(29068 + 5200 - .05(5200)\right)(1.05) - \left(100{,}000 + 200 - 35324\right) * \begin{matrix}100 * \\ .10059\end{matrix}$$

$$\underbrace{}_{3{,}532{,}563}$$

$$\text{Gain} = 3{,}551{,}388 - 3{,}532{,}563 = +\ \underline{\underline{18{,}824.84}}$$

mortality → interest → expenses

$\overset{29068}{{}_5V}$, $\overset{35324}{{}_6V}$

gain from mortality: $\underbrace{(100{,}000 + \underline{200} - {}_6V) * (100 * .0059 - 1)}_{= -26{,}599.12}$

gain from interest: $\underbrace{100 * ({}_5V + G - .05G) * (.065 - .05)}_{51{,}012}$

gain from expenses: $100 * (.05G - .06G) * 1.065$

$\underbrace{(200 - 250) * 1}_{\sim 5588}$

$= 18{,}824.84$

expenses → interest → mortality

expenses: $\underbrace{100*(.05G-.06G)*1.05 + (200-250)*100*.0059}$

~~[illegible]~~ -5489.5

interest: $\underbrace{100*({}_5V+G-.06G)*(.065-05)}$

$50,934$

mortality: $\underbrace{(100{,}000+\underline{250}-{}_6V)*(100*.0059-1)}$

$\underline{-26,619.65}$ ✓

$\oplus = 18,824.84$ ✓

try different combin$\underline{\text{ation}}$s

SOA question #17, Spring 2012

${}_{10}V$ ${}_{11}V$

10 11

Your company issues fully discrete whole life policies to a group of lives age 40. For each policy, you are given:

- The death benefit is \$ 50,000.
- Assumed mortality and interest are the `Illustrative Life Table` at 6%.
- Annual gross premium equals 125% of benefit premium.
- Assumed expenses are 5% of gross premium, payable at the beginning of each year, and \$ 300 to process each death claim, payable at the end of the year of death.
- Profits are based on gross premium reserves.

During year 11, actual experience is as follows:

- There are 1,000 lives in force at the beginning of the year.
- There are five deaths.
- Interest earned equals 6%.
- Expenses equal 6% of gross premium and \$ 100 to process each death claim.

SOA question #17, Spring 2012 - continued

For year 11, you calculate the gain due to mortality and then the gain due to expenses.

1. Calculate the total gain in year 11.
2. Calculate the gain due to mortality during year 11.
3. Calculate the gain due to expenses during year 11. [only this question was asked in the exam]

${}_{10}V \qquad {}_{11}V$

$$P = 50000 \frac{\overset{.16132}{A_{40}}}{\underset{14.8166}{\ddot{a}_{40}}} = 544.3894$$

$$G = 1.25P = 640.4868 \checkmark$$

$${}_{10}V = APVFB_{10} - APVFP_{10} + APVFE_{10} = 3950.727$$

$$\downarrow$$

$$50300 \underset{.24905}{A_{50}} - G \underset{13.0803}{\ddot{a}_{50}} + .05 G \ddot{a}_{50}$$

$${}_{11}V = APVFB_{11} - APVFP_{11} + APVFE_{11} = \underline{\underline{4602.461}}$$

$$= 50300 \underset{.25961}{A_{51}} - G \ddot{a}_{51} + .05 G \underset{13.0803}{\ddot{a}_{51}}$$

$${}_{11}V^A = \underbrace{1000({}_{10}V + G - .06G)(1.06) - (50000+100-{}_{11}V)*5}$$

$$= 4{,}638{,}320$$

$${}_{11}V^E = 1000({}_{10}V + G - .05G)(1.06) - (50{,}000+300-{}_{11}V)*$$

$$1000 * q_{50}$$

$$= \underbrace{1000\Big[\underbrace{({}_{10}V + G - .05G)(1.06) - (50300-{}_{11}V)\,q_{50}}_{{}_{11}V}\Big]}_{4{,}602{,}461}$$

$$q_{50} = \frac{5.92}{1000}$$

(a) ${}_{11}V^A - {}_{11}V^E = \underline{\underline{35{,}859}}$

mortality → expenses

(b) Gain from mortality

$$\underbrace{\left(50{,}000 + 300 - {}_{11}V\right) * \left(1000 * \frac{5.92}{1000} - 5\right)}_{= \; 42{,}041.74}$$

$${}_{11}V \downarrow 4602.461$$

(c) Gain from expenses

$$\underbrace{1000\left(.05G - .06G\right) * (1.06) + (300 - 100) * 5}_{-6213.16}$$

check $= \oplus \; 35{,}859$

Asset shares

AS. AS.

An asset share, generally denoted by AS, is the "share of the insurer's assets attributable to each policy in force at any given time".

It is calculated using the (year by year) experience that actually emerge (over time).

For our purposes, any symbol with a prime ($'$) denotes actual experience. Thus the asset share for a single policy at the end of year $k+1$ is equal to

$$\mathsf{AS}_{k+1} = \frac{\big(\mathsf{AS}_k + G_k - e'_k\big)(1 + i'_k) - \big(B_{k+1} + E'_{k+1}\big)q'_{x+k}}{1 - q'_{x+k}}.$$

It is not difficult to see that if the actual experience is equal to expected experience in all years (which is highly unlikely), this is exactly equal to the gross premium reserve.

$AS_0 \quad AS_1 \quad AS_2$

$AS_0 \downarrow \phi$

$AS_1 \downarrow$

$$\frac{(AS_0 + G_0 - e_0')(1+i_0') - (B_1 + E_1')(q'_{x+\text{[illegible]}})}{1 - q'_{x+\text{[illegible]}}}$$

$$\frac{(AS_1 + G_1 - e_1')(1+i_1') - (B_2' + E_2')\, q'_{x+1}}{1 - q'_{x+1}}$$

$\vdots$

Illustrative example from book

deferred annuity

$G = 11{,}900$

0 10

DB = return of premium without interest

- Consider Example 7.9

	i'	e'	E'	q'
1	4.8%	15% G		
2	5.6%	6%	120	.0015
3	5.2			
4	4.9			
5	4.7			

$$AS_0 = \phi$$

$$AS_1 = \underbrace{\frac{(\phi + 11900(1-.15))(1.048) - (11900+120)(.0015)}{1-.0015}}_{\underline{\underline{10,598.39}}}$$

$$AS_2 = \underbrace{\frac{(10598.39 + 11900(1-.06))(1.056) - (2(11900)+120)(.0015)}{1-.0015}}_{23,002.94}$$

$$AS_3 = \underbrace{\frac{(23002.94 + 11900(1-.06))(1.052) - (3(11900)+120)(.0015)}{1-.0015}}_{35,966.98}$$

$$\vdots$$

$$AS_4 = 49,466.10$$

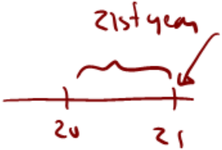
$$AS_5 = 63,508.58$$

Illustrative example 2

For a portfolio of fully discrete whole life insurances of \$100 each on (x), you are given:

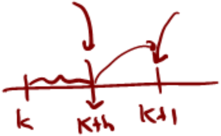
- The annual contract premium per policy is \$0.98.
- Expenses incurred at the beginning of year 21 is \$0.15.
- The annual effective interest rate earned in year 21 is 8%.
- Out of the remaining 950 policies at the beginning of year 21, there were a total of 6 deaths during the year.
- The asset share at the end of year 20 is \$15.

Calculate the asset share at the end of year 21.



$$AS_{21} = \frac{(AS_{20} + G_{20} - e'_{20})(1+i'_{20}) - (B_{20})(q'_{x+20})}{1 - q'_{x+20}}$$

$$= \underbrace{\frac{(15 + .98 - .15)(1.08) - 100\left(\frac{6}{950}\right)}{1 - 6/950}}_{16.56947}$$



$$16.56947$$

$${}_{k+1}V = \frac{({}_kV + G - e)(1+i) - (B+E)(q_{x+k})}{1 - q_{x+k}}$$

$${}_{k+h}V = \frac{({}_kV + G - e)(1+i)^h - (B+E)\,{}_hq_{x+k}\,v^{1-h}}{1 - {}_hq_{x+k}}$$

$$\frac{d\,{}_tV}{dt} = \underbrace{\lim_{h\to 0} \frac{{}_{t+h}V - {}_tV}{h}}$$

$$\boxed{\frac{d\,{}_tV}{dt} = \left(\delta_t\,{}_tV + G_t - e_t\right) - \left(B_t + E_t - {}_tV\right)^* \mu_{[x]+t}}$$

how do I solve this?

numerically Euler method

boundary conditions

- term: ${}_nV = \phi$
- endowment: ${}_nV = S > 0$

$$\cancel{\lim_{h\to 0}}\ \frac{{}_{t+h}V - {}_tV}{h}$$

$h \to 0$

h = step size

Thiele's differential equation for reserves

The Thiele's differential equation is the continuous analogue of the recursive relation between policy years for reserves:

$$\frac{d}{dt}\,{}_tV = \delta_t\,{}_tV + G_t - e_t - \big(B_t + E_t - {}_tV\big)\mu_{[x]+t}$$

Note:

- Proof can be found in Section 7.5.1 of DHW book.
- Interpretation (which is quite similar to the discrete analogue) can be found on page 210.
- Notice some differences in symbols used: we have consistently used G for gross premiums and B for benefit amount.

Numerical solution to Thiele's

The numerical approximation to the solution to Thiele's differential equation is based on what is often referred to as the Euler's method.

According to this method, we approximate the derivative using the following (for some very small h):

$$\frac{d}{dt}\,{}_tV \approx \frac{{}_{t+h}V - {}_tV}{h}$$

Note:

- h is sometimes referred to as a step size.
- The procedure is to solve backward equations using boundary conditions such as:
  - term insurance: ${}_nV = 0$
  - endowment insurance: ${}_nV = S$ where S denotes the benefit upon survival at maturity

- continued

We then have the following approximate formula:

$$ {}_{t+h}V - {}_{t}V \approx h\big[\delta_t\, {}_tV + G_t - e_t - \big(B_t + E_t - {}_tV\big)\mu_{[x]+t}\big] $$

Starting with the boundary condition for ${}_nV$:

$$ {}_nV - {}_{n-h}V = h\big[\delta_{n-h}\, {}_{n-h}V + G_{n-h} - e_{n-h} - \big(B_{n-h} + E_{n-h} - {}_{n-h}V\big)\mu_{[x]+n-h}\big] $$

We solve for ${}_{n-h}V$ from this approximate formula. Using this value, we then proceed to calculate for reserve at $n-2h$, and so forth.

Illustrative example from book

- Consider Example 7.12

Illustrative example 3

For a 15-year term insurance policy issued to age 50, you are given:

- Death benefit of $10,000 is payable at the moment of death.
- Expense rate incurred continuously during each year is 10% of the annual premium rate; annual premium rate payable continuously throughout the year is equal to $61.47.
- The force of mortality for age 50 is given by

$$\mu_{50+t} = A + B \times c^{50+t}, \quad \text{for } t \geq 0,$$

  where $A = 0.0003$, $B = 2.7 \times 10^{-6}$ and $c = 1.14$.

- $\delta = 4.5\%$

With step size of $h = 0.05$, estimate the reserve at the end of year 14 using the Euler's method.

$$ {}_nV - {}_{n-h}V = h\left[\left(\delta\, {}_{n-h}V + G' - .10G'\right) - \left(10{,}000 - {}_{n-h}V\right)\mu_{50+n}\right] $$

$h = .05$

$$ {}_{n-h}V = \frac{{}_nV - h\left[G' - .10G' - 10{,}000\,\mu_{50+n}\right]}{1 + \delta h + h\,\mu_{50+n}} $$

$$ {}_{n-.05}V = \frac{{}_nV' - .05\left(.90(61.47) - 10{,}000\,\mu_{50+n}'\right)}{1 + .05(.045) + .05\,\mu_{50+n}'} $$

$$ {}_{15}V = 0 $$

μ_{50+95}

$$ {}_{14.95}V = \frac{-.05\left(.90(61.47) - 10{,}000(.013795)\right)}{1 + .05(.045) + .05(.013795)} $$

Illustrative example 3 - continued

Starting with ${}_{15}V = 0$, use the equation (derived from the Euler's method): [1]

$$ {}_tV = \frac{{}_{t+h}V - h\left(0.9G - 10000\mu_{50+t}\right)}{1 + h\delta + h\mu_{50+t}} $$

With steps of $h = 0.05$, one can verify the following calculations:

t	μ_{50+t}	${}_tV$	t	μ_{50+t}	${}_tV$
13.95	0.012061	74.5368	14.50	0.012939	38.3120
14.00	0.012138	71.4853	14.55	0.013022	34.7194
14.05	0.012216	68.3868	14.60	0.013106	31.0751
14.10	0.012294	65.2407	14.65	0.013190	27.3784
14.15	0.012373	62.0467	14.70	0.013275	23.6291
14.20	0.012452	58.8044	14.75	0.013360	19.8266
14.25	0.012532	55.5134	14.80	0.013446	15.9704
14.30	0.012612	52.1732	14.85	0.013533	12.0602
14.35	0.012693	48.7834	14.90	0.013620	8.0953
14.40	0.012775	45.3435	14.95	0.013707	4.0754
14.45	0.012857	41.8532	15.00	0.013795	0.0000

verify these numbers

[1] corrected: Feb 22, 2012

7.12

20-year endowment   100,000 = DB = endowment

no expenses $e = E = 0$

$x=30$

$\delta = .04 \longrightarrow$ Standard Select

$P = 2500$ $\mu_{x+t} = A + BC^{x+t}$

A, B, C's

$_{10}V = ?$ $h = .05$ $A = .00022$ $B = 2.7 \times 10^{-6}$

$C = 1.124$

$_{20}V = 100{,}000$ $B = 100{,}000$

$$\frac{d}{dt}\,{}_tV = (\delta\,{}_tV + P) - (B - {}_tV)\mu_{x+t}$$

$$\frac{{}_{t+h}V - {}_tV}{h} = (.04\,{}_tV + 2500) - (100{,}000 - {}_tV)\mu_{30+t}$$

$$_tV = \frac{{}_{t+h}V - 2500(.05) + 100{,}000(.05)\mu_{30+t}}{1 + .04(.05) + .05\mu_{30+t}}$$

$t+h=20 \quad h=.05$

$${}_{19.95}V = \underbrace{\frac{\cancel{{}_{20}V}^{\,100,000} - .05(2500) + .05\,(100,000)\,\mu_{49.95}}{1+.04(.05)+.05\,\underbrace{\mu_{49.95}}_{\underbrace{A+BC^{49.95}}_{.001147}}}}$$

$$= 99,675.6673$$

$${}_{19.90}V = \underbrace{\frac{{}_{19.95}V - .05(2500) + .05\,(100,000)\,\mu_{49.90}}{1+.04(.05)+.05\,\underbrace{\mu_{49.90}}_{\underbrace{A+BC^{49.90}}_{.001142}}}}$$

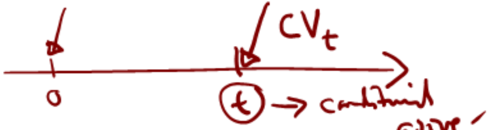
$$= 99,352.0003$$

$\vdots$

${}_{10}V = 46,635.1245$

Surrender values

long term validity



- Many policies, especially those considered long term, contain non-forfeiture clauses which provide for cash surrender values.
- Minimum cash surrender values are generally imposed by regulation - called non-forfeiture laws.
- Terminal reserves are used to determine the appropriate policy values, hence cash surrender values are computed quite similar to reserves.
- However, the cash surrender value is generally smaller than the terminal reserve: $\mathrm{CV}_t \leq {}_tV$.
- The difference

$$ {}_tV - \mathrm{CV}_t $$

→ AS_t something

is indeed called the surrender charge.

Computing surrender values

- They may be computed quite similar to reserve calculations.
- For example on a prospective basis, we may have

$$\mathrm{CV}_t = \mathrm{APV(Future\ Benefits)} - \mathrm{APV(Future\ Adjusted\ Premiums)}$$

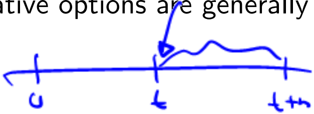
- Some possible differences may include:
  - The calculation basis (mortality/interest assumptions) may be different from that used in reserve calculation - for conservatism.
  - Premiums may also be adjusted to recoup expenses, especially the large first-year initial expense.

$CV_t = \text{cash}$

Surrender options

lower benefit

- In lieu of receiving cash, some alternative options are generally available:
  - (reduced) paid-up insurance
  - extended term insurance



- Because these are generally initiated by the policyholder, these are sometimes called policy alterations.
- In a (reduced) paid-up insurance surrender option, the idea is to provide for a reduced amount of insurance which becomes paid-up. No further premium is required from surrender date onwards.
- In an extended term insurance, the idea is to provide for a term insurance protection, the length of which is determined depending on the cash surrender value. The amount of insurance is therefore maintained, but the duration of coverage is reduced.
- These will be illustrated in lectures.

Calculating altered contracts

Assume at time t the policyholder decides to alter the contract. At that time, he has the option to receive the cash surrender value, CV_t, or use it to buy an altered contract.

To determine the new benefits under an altered contract, one may use the following equation of value:

$$\mathsf{CV}_t + \mathsf{APV}(\mathsf{FP}_t^*) = \mathsf{APV}(\mathsf{FB}_t^*),$$

where FP_t^* and FB_t^* respectively denote future premiums and future benefits under the altered (new) contract.

Illustrative example 4

Consider a fully discrete whole life policy with $B = 100,000$ issued to (45). You are given the following table of applicable cash values for various duration for this policy:

k	x	A_x	${}_2E_x$	${}_kV$	CV_k	k	x	A_x	${}_2E_x$	${}_kV$	CV_k
0	45	0.34740	0.84040	0.00	0.00	13	58	0.52294	0.78730	26898.33	22863.58
1	46	0.35995	0.83785	1923.76	1154.26	14	59	0.53699	0.78082	29051.90	24694.11
2	47	0.37272	0.83511	3880.50	2522.33	15	60	0.55102	0.77389	31201.57	26521.33
3	48	0.38569	0.83216	5868.22	4107.75	16	61	0.56500	0.76647	33343.88	28342.30
4	49	0.39885	0.82898	7884.76	5913.57	17	62	0.57891	0.75854	35475.36	30154.06
5	50	0.41219	0.82556	9927.80	7942.24	18	63	0.59273	0.75006	37592.55	31953.67
6	51	0.42567	0.82189	11994.85	10195.62	19	64	0.60643	0.74101	39692.03	33738.23
7	52	0.43930	0.81794	14083.27	11970.78	20	65	0.61999	0.73137	41770.41	35504.85
8	53	0.45305	0.81369	16190.30	13761.75	21	66	0.63340	0.72109	43824.40	37250.74
9	54	0.46691	0.80913	18313.00	15566.05	22	67	0.64662	0.71015	45850.79	38973.17
10	55	0.48084	0.80423	20448.35	17381.10	23	68	0.65964	0.69852	47846.47	40669.50
11	56	0.49484	0.79898	22593.20	19204.22	24	69	0.67245	0.68618	49808.49	42337.22
12	57	0.50888	0.79334	24744.30	21032.65	25	70	0.68501	0.67309	51734.02	43973.92

Illustrative example 4 - continued

$B = 100,000$ -

$\downarrow$

R = reduced paid up amount

Do the following:

1. The policyholder decides to lapse the policy after 10 years and wishes to purchase a paid-up insurance. Calculate the amount of (reduced) paid-up insurance.

2. The policyholder decides to lapse the policy after 20 years and wishes to purchase a paid-up insurance. Calculate the amount of (reduced) paid-up insurance.

3. The policyholder decides to lapse the policy after 10 years and wishes to purchase an extended term insurance. Estimate from the table the range of extended term while maintaining the same amount of insurance.

$x = 45$

R R R . . . .

① R

$$CV_{10} = R\,A_{55} \Rightarrow R = \frac{CV_{10}}{A_{55}}$$

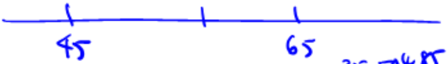
17381.1

.48084

< 100K

$\underline{\underline{36{,}147.37}}$

② R = new benefit

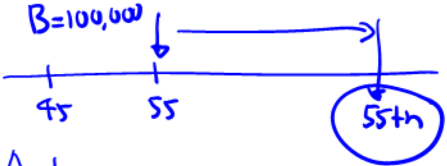


$$CV_{20} = R\,A_{65} \Rightarrow R = \frac{CV_{20}}{A_{65}} = \underline{\underline{57{,}226.81}}$$

35,504.85

.61999

3



$$CV_{10} = 100{,}000\, \underbrace{A^{\,1}_{55:\overline{n}|}}_{(A_{55} - {}_nE_{55}\, A_{55+n})}$$

CV_{10}: 17381.10

A_{55}: .48084

$${}_nE_5 A_{55+n} = .307029$$

n	${}_nE_{55}\, A_{55+n}$
2	${}_2E_{55}\, A_{55+2} = .409254$
4	.3426146
6	.2814745

$\Rightarrow$ around 5.*

Deferred acquisition cost

Recall that in most circumstances, there is usually an (extra) large expense item at point of sale.

This expense item can either be:

- Immediately recognized as expenses in the year of sale: this will reduce income and therefore surplus.
- Amortize (or spread) this expense item over the life of the contract: this gives rise to the concept of deferred acquisition cost (DAC).

A | L ↓⊖

Deferred acquisition cost - continued

negative liability -

A - L = S

The difference between the gross premium reserve and the benefit premium reserve

with — without

$$ {}_tV^e = {}_tV^g - {}_tV^n $$

is called the (negative) expense reserve and is referred to as the deferred acquisition cost.

Mathematically, if we define the difference between the gross premium and the benefit premium as the expense loading

$$ P^e = P^g - P^n, $$

then it is not difficult to show that

$$ \text{DAC} = {}_tV^e = \text{APV(future expenses)} - \text{APV(future expense loadings)} $$

Let $e_t = G_t - P_t =$ $\qquad G_t = P_t + e_t$ ✓

$$_tV^g = \cancel{APV(FB)_t} + APV(FE)_t - APV(FG)_t$$

$$-)\ _tV^n = \cancel{APV(FB)_t} - APV(FP)_t \qquad e = \text{loading}$$

$$_tV^e = DAC = APV(FE)_t - [\cancel{APV(FP)_t} + APV(Fe)_t] + \cancel{APV(FP)_t}$$

$$_tV^e = APV(FE)_t - APV(Fe)_t$$

Illustrative example from ~~study note~~ book

to [50]

fully discrete whole life
benefit = 100,000
premium b.o.y.
Standard Select Table
@ 4%

- Consider Example 7.17

$\ddot{a}_{[50]} = 19.35185$

$A_{[50]} = 1 - d\,\ddot{a}_{[50]}$

$d = \frac{.04}{1.04}$

$= .2556981$

Expenses
1st year 50% of gross premium plus 250

Renewal years 3% of gross prem plus 25

Calculate the expense loading.

$$P = 100{,}000\,\frac{A_{[50]}}{\ddot{a}_{[50]}} = 1{,}321.311$$

$$APV(FG) = APV(FB) + APV(FE)$$

$$\downarrow$$

$[50]+10$

$$G\,\ddot{a}_{[50]} = 100{,}000\,A_{[50]} + (.47G + 225) + (.03G + 25)\,\ddot{a}_{[50]}$$

$$G = \frac{100{,}000\,A_{[50]} + 225 + 25\,\ddot{a}_{[50]}}{.97\,\ddot{a}_{[50]} - .47} = 1{,}435.888$$

$$G - P = 1435.888 - 1321.311 = 114.5770$$

$${}_{t}V^{e} = \underbrace{APV(FE)_t}_{} - APV(Fe)_t$$

$$(25 + .03G)\,\ddot{a}_{60} - 114.5770\,\ddot{a}_{60}$$

$t=10$

Exercise: fully discrete $B = 1000$ (45) Friday

Mortality ILT @ 6%

Expenses: $\begin{cases} 10 & \text{1st year} \\ 2 & \text{renewal years} \end{cases}$

Expense loading $= 2 + \dfrac{8}{\ddot{a}_{45}}$

Find DAC (or expense reserve) at end of 15 years

$$DAC_{15} = {}_{15}V^e = \cancel{2}\ddot{a}_{60} - \left(\cancel{2} + \frac{8}{\ddot{a}_{45}}\right)\ddot{a}_{60}$$

$$= -8\,\frac{\ddot{a}_{60}}{\ddot{a}_{45}} = -6.318209$$

Modified reserve systems

$${}_tV^g - {}_tV^n = {}_tV^e = DAC$$

$${}_tV^s = {}_tV^n + DAC$$

In most jurisdictions, authorities specify the extent to which the level of expenses that can be amortized and reserved. * solvency

There are clear explanations to this:

- If expenses are spread over the contract life, there is the possibility that the DAC may not be recovered especially when policy lapse.
- DAC may also lead to negative reserves in the first year.

This gives rise to what are sometimes referred to as modified reserve systems.

The most common of these methods is called the Full Preliminary Term or FPT.

The Full Preliminary Term (FPT) method

P P . . . .
α β . . .

The general idea with modified reserve method is to replace the level benefit premiums, P, with a first year premium of α and increased renewal premiums of β.

For instance, in the case of a fully discrete whole life insurance of $1 on (x), we solve the equation of value:

$(\ddot{a}_x - 1)$ $\beta = 1$

$$P\ddot{a}_x = \alpha + \beta a_x = \beta\ddot{a}_x - (\beta - \alpha)$$

Therefore, we see that

$$\boxed{\beta = P + \frac{\beta - \alpha}{\ddot{a}_x}.}$$

EA.

Then use α and β to calculate reserves. Because we know that to avoid negative reserves in the first year, the value of α must be at least the first year cost of insurance: vq_x.

1st year cost of insurance
COI

Under FPT, set $\alpha = vq_x$.

$$_tV^{mod} = APV(FB)_t - APV(FP)_t$$

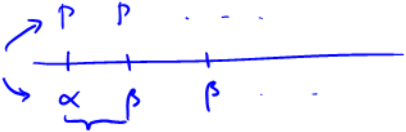
uses α, β's

mod = FPT

$$_tV^{FPT}$$

$$_0V = \emptyset$$

$$_1V =$$



P_{x+1}

$$= \underbrace{A_{x+1+(t-1)} - P_{x+1}\,\ddot{a}_{x+1+(t-1)}}_{_{t-1}V \text{ for } \underline{\underline{x+1}}}$$

$\alpha = v q_x$

$$\beta = \underset{\downarrow \frac{A_x}{\ddot{a}_x}}{P} + \frac{\beta - \alpha}{\ddot{a}_x} \qquad \alpha = v q_x$$

$$\beta \ddot{a}_x - \beta = A_x - v q_x = v \cancel{p_x} A_{x+1}$$

$$\cancel{\beta(\ddot{a}_x - 1)} = \frac{A_x - v q_x}{\ddot{a}_x - 1} = \frac{A_{x+1}}{\ddot{a}_{x+1}} = P_{x+1}$$

$$\ddot{a}_x - 1 \rightarrow v \cancel{p_x} \ddot{a}_{x+1}$$

Illustrative example 5

Calculate ${}_{10}V^{\mathsf{FPT}}$ for each of the following cases:

1. A fully discrete whole life insurance of \$100,000 issued to (40).
2. A fully discrete 25-pay whole life insurance of \$100,000 issued to (40).

Assume mortality follows the `Illustrative Life Table` with $i = 6\%$.

① $x = 40$ $\quad B = 100{,}000$

$${}_{10}V^{FPT} = {}_{9}V@41$$

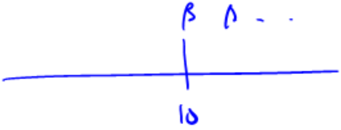
α, β

$$\alpha = 100{,}000\, \underbrace{v}_{\frac{1}{1.06}} \cdot \underbrace{q_{40}}_{\frac{2.78}{1000}} = 262.2642$$

$$\beta = 100{,}000\, P_{41}$$

$$P_{41} = \frac{A_{41}}{\ddot{a}_{41}} \qquad A_{41} = .16869 \qquad \ddot{a}_{41} = 14.6864$$

1148.614

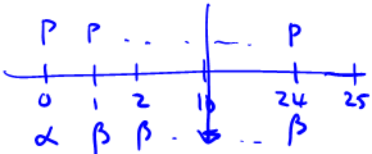


β's

$${}_{10}V^{FPT} = APV(FB)_{10} - APV(FP)_{10}$$

$$= 100{,}000\, \underbrace{A_{50}}_{.24905} - 1148.614\, \underbrace{\ddot{a}_{50}}_{13.2668} = \underline{\underline{9666.764}}$$

② 25 pay, to (40) FPT



$$\alpha = 100{,}000\, v\, q_{40} = 262.2642$$

$$\beta = P + \frac{\beta - \alpha}{\ddot{a}_{40:\overline{25}|}} \qquad \beta = 100{,}000 \frac{\overset{.16869}{A_{41}}}{\underbrace{\ddot{a}_{41:\overline{24}|}}}$$

$$P = 100{,}000 \frac{A_{40}}{\ddot{a}_{40:\overline{25}|}}$$

$$\ddot{a}_{41:\overline{24}|} = \underbrace{\ddot{a}_{40:\overline{25}|}} - 1$$

$$\ddot{a}_{40:\overline{25}|} = \underbrace{\ddot{a}_{40} - {}_{25}E_{40}\, \ddot{a}_{65}}_{12.70353}$$

$$\beta = 1327.899$$

$${}_{10}V^{FPT} = 100{,}000\, A_{50} - \beta\, \ddot{a}_{50:\overline{15}|}$$

let you do the calc!

SOA question #8, Spring 2012

Whole life $\ {}_tV = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$

$\alpha = v\, q_{80} * 1000$

$\beta = P_{81} = \frac{A_{81}}{\ddot{a}_{81}} * 1000$

For a fully discrete whole life insurance of $ 1,000 on (80):

- $i = 0.06$
- $\ddot{a}_{80} = 5.89$
- $\ddot{a}_{90} = 3.65$
- $q_{80} = 0.077$

${}_tV = 1000\left(1 - \frac{\ddot{a}_{90}}{\ddot{a}_{80}}\right)$

${}_{10}V^{FPT} = \underbrace{1000 A_{90} - \beta \ddot{a}_{90}}_{{}_9V \text{ @ } 81}$

Calculate the Full Preliminary Term (FPT) reserve for this policy at the end of year 10.

$\ddot{a}_{80} = 1 + v p_{80} \ddot{a}_{81}$

$5.89 = 1 + \frac{1}{1.06}(1 - .077)\ddot{a}_{81}$

$\ddot{a}_{81} = 5.615818$

$= \underbrace{1000 A_{81} - P\ddot{a}_{81}}$

$\leftarrow \ = 1000\left(1 - \frac{\ddot{a}_{90}}{\ddot{a}_{81}}\right) = 350.02$

3.65

Spring 2014

FPT

30-year endowment of 1000 on (40)

ILT @ 6%

$_{10}V^{FPT} = ?$

$\Downarrow$

$_9V @ 51 = 1000\, A_{51:\overline{19}|} - P\, \ddot{a}_{51:\overline{19}|}$

$P = 1000 \dfrac{A_{40:\overline{30}|}}{\ddot{a}_{40:\overline{30}|}}$

$\alpha = 1000\, v\, q_{40} = 2.622642$

$\alpha + \beta\, {}_1E_{40}\, \ddot{a}_{41:\overline{29}|} = 1000\, A_{40:\overline{30}|} = P\, \ddot{a}_{40:\overline{30}|}$

$\Rightarrow \beta = 17.14355$

$${}_{10}V^{FPT} = 1000\,\underbrace{A_{50:\overline{20|}}}_{.3608395} - \beta\,\underbrace{\ddot{a}_{50:\overline{20|}}}_{11.29184}$$

40 50 70

167.2573

---

old → P P .... P

new → α β ... β

$$FPT \Rightarrow \text{set } \alpha = B * v q_x$$