Multiple State Models

Lecture: Weeks 6-7

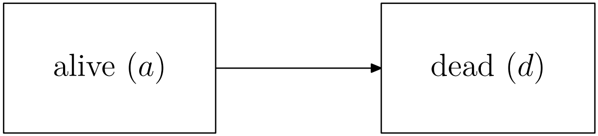
Chapter summary

- Multiple state models (also called transition models)
  - what are they?
  - actuarial applications - some examples
- State space
- Transition probabilities
  - continuous and discrete time space
- Markov chains
  - time homogeneous versus non-homogeneous Markov chains
- Cash flows and actuarial present value calculations in multiple state models
- Chapter 8 (Dickson, et al.)

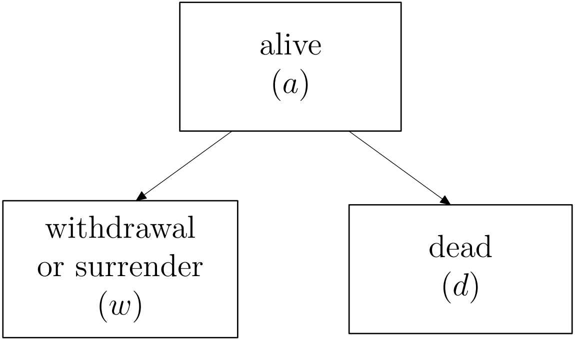
Introduction

- Multiple state models are probability models that describe the random movements of:
    - a subject (often a person, but could be a machinery, organism, etc.)
    - among various states
- Discrete time or continuous time and discrete state space
- Examples include:
    - basic survival model
    - multiple decrement models
    - health-sickness model
    - disability model
    - pension models
    - multiple life models
    - long term care (or continuing care retirement communities, CCRCs) models

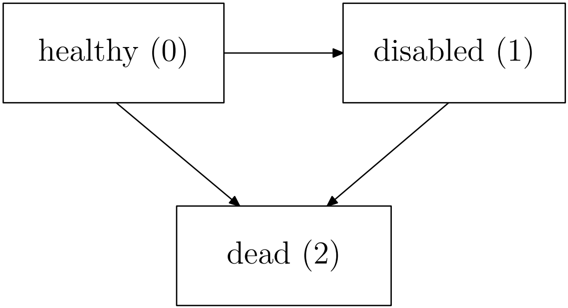
The basic survival model



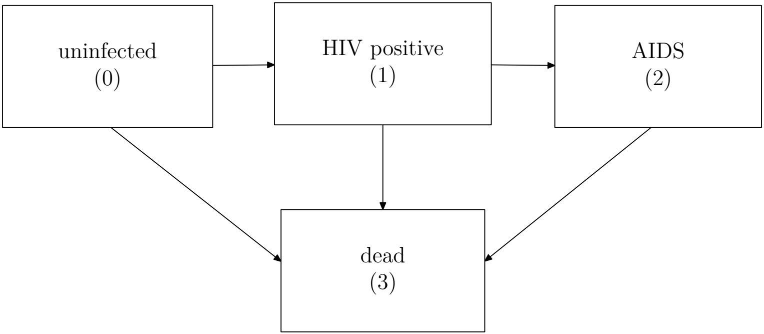
The withdrawal-death model



The permanent disability model



The HIV-AIDS progression model



Notation

- Assume a finite state space (total of $n+1$ states): $\{0, 1, \dots, n\}$
- In most actuarial applications, we need a reference age.
  - Denote by x the age at which the multiple state process begins.
  - x is the age at time $t = 0$.
- Denote by $Y_x(t)$ the state of the process at time t.
  - This can take on possible values in the state space.
  - The process can be denoted by $\{Y_x(t),\ t \geq 0\}$.

Continuous time Markov chain models

Transition probabilities and forces of transition

- Transition probabilities:

$$ {}_t p_x^{ij} = \Pr\left[Y_x(t) = j | Y_x(0) = i\right] $$

  - This is the probability that a life age x at time 0 is in state i and will be in state j after t periods.

- Force of transition (also called transition intensity):

$$ \mu_x^{ij} = \lim_{h \to 0^+} \frac{1}{h} {}_h p_x^{ij}, \quad \text{for } i \neq j $$

  - This is defined only in the case where we have a continuous time process.
  - Analogous to the force of mortality in the basic survival model.
  - It is understood that $\mu_x^{ij} = 0$ if it is not possible to transition from state i to state j at any time.

Some assumptions

- Assumption 1: The Markov property holds.

$$\Pr\big[Y_x(s+t)=j|Y_x(s)=i, Y_x(u)=k, 0\leq u<s\big]$$
$$=\Pr\big[Y_x(s+t)=j|Y_x(s)=i\big]$$

- Assumption 2: For any positive interval of time length (generally very small) h,

$$\Pr[\text{2 or more transitions within a time period of length } h]=o(h)$$

- Assumption 3: For all states i and j and all ages $x\geq 0$, ${}_tp_x^{ij}$ is a differential function of t.

Some useful approximation

We can express the transition probabilities in terms of the forces of transition as

$$ {}_h p_x^{ij} = h\,\mu_x^{ij} + o(h), $$

so that for very small values of h, we have the approximation

$$ {}_h p_x^{ij} \approx h\,\mu_x^{ij}. $$

The occupancy probability

When a person currently age x and is currently in state i, the probability that the person continuously remains in the same state for a length of t periods is called an occupancy probability.

For any state i in a multiple state model, the probability that (x) now in state i will remain in state i for t years can be computed using:

$$
{}_t p_x^{\overline{ii}} = \exp\left[-\int_0^t \sum_{j=0, j \neq i}^{n} \mu_{x+s}^{ij} ds \right].
$$

Sketch of proof will be done in class - also on pages 239 - 240.

Kolmogorov's forward equations

For a Markov process, transition probabilities can be expressed as

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij} + h \sum_{k=0, k\neq j}^{n} \left({}_tp_x^{ik} \mu_{x+t}^{kj} - {}_tp_x^{ij} \mu_{x+t}^{jk} \right) + o(h). $$

This leads us to the Kolmogorov's Forward Equations (KFE):

$$ \frac{d}{dt} {}_tp_x^{ij} = \sum_{k=0, k\neq j}^{n} \left({}_tp_x^{ik} \mu_{x+t}^{kj} - {}_tp_x^{ij} \mu_{x+t}^{jk} \right). $$

This set of differential equations is used to solve for transition probabilities.

Numerical evaluation of transition probabilities

To solve for the set of KFE's for the transition probabilities, we can equate $o(h) \to 0$, especially if h is small, or equivalently use the approximation

$$\frac{d}{dt} {}_t p_x^{ij} \approx \frac{1}{h} \left({}_{t+h} p_x^{ij} - {}_t p_x^{ij} \right)$$

This is a similar approach used to approximate the solution to the Thiele's differential equation for reserves.

Method is called the Euler's method. The primary differences are:

- solution is performed recursively going forward with the boundary conditions:

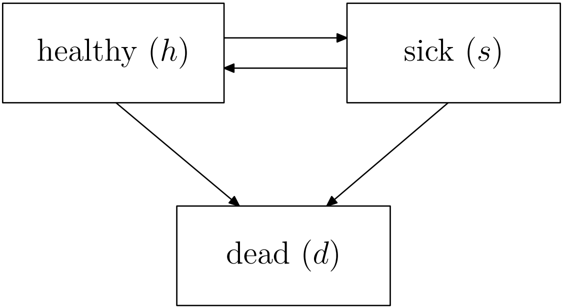
$${}_0 p_x^{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

- the process usually requires solving a number of equations.

Illustrative example from book

- Consider Example 8.4 on pages 254-255

The health-sickness model



Example 8.5 from the book

Consider the health-sickness insurance model illustrated in Example 8.5 with

$$\begin{aligned}
\mu_x^{01} &= a_1 + b_1 \exp(c_1 x) \\
\mu_x^{10} &= 0.10\mu_x^{01} \\
\mu_x^{02} &= a_2 + b_2 \exp(c_2 x) \\
\mu_x^{12} &= \mu_x^{02}
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 4 \times 10^{-4}, \quad b_1 = 3.4674 \times 10^{-6}, \quad c_1 = 0.138155 \\
a_2 &= 5 \times 10^{-4}, \quad b_2 = 7.5868 \times 10^{-5}, \quad c_2 = 0.087498
\end{aligned}$$

Verify the calculations of ${}_{10}p_{60}^{00}$ and ${}_{10}p_{60}^{01}$, and follow the same procedure to calculate ${}_{10}p_{60}^{02}$.

Numerical process of solutions

One can verify that to solve for the desired probabilities, one solves the set of Kolmogorov's forward equations

$$\begin{aligned}
\frac{d}{dt}\,{}_tp_{60}^{00} &= {}_tp_{60}^{01}\,\mu_{60+t}^{10} - {}_tp_{60}^{00}\left(\mu_{60+t}^{01} + \mu_{60+t}^{02}\right) \\
\frac{d}{dt}\,{}_tp_{60}^{01} &= {}_tp_{60}^{00}\,\mu_{60+t}^{01} - {}_tp_{60}^{01}\left(\mu_{60+t}^{10} + \mu_{60+t}^{12}\right) \\
\frac{d}{dt}\,{}_tp_{60}^{02} &= {}_tp_{60}^{00}\,\mu_{60+t}^{01} + {}_tp_{60}^{01}\,\mu_{60+t}^{12}
\end{aligned}$$

Then use the numerical approximations:

$$\begin{aligned}
{}_{t+h}p_{60}^{00} &\approx {}_tp_{60}^{00} + h\left[{}_tp_{60}^{01}\,\mu_{60+t}^{10} - {}_tp_{60}^{00}\left(\mu_{60+t}^{01} + \mu_{60+t}^{02}\right)\right] \\
{}_{t+h}p_{60}^{01} &\approx {}_tp_{60}^{01} + h\left[{}_tp_{60}^{00}\,\mu_{60+t}^{01} - {}_tp_{60}^{01}\left(\mu_{60+t}^{10} + \mu_{60+t}^{12}\right)\right] \\
{}_{t+h}p_{60}^{02} &\approx {}_tp_{60}^{02} + h\left[{}_tp_{60}^{00}\,\mu_{60+t}^{02} - {}_tp_{60}^{01}\,\mu_{60+t}^{12}\right]
\end{aligned}$$

with initial boundary conditions: ${}_0p_{60}^{00} = 1$, ${}_0p_{60}^{01} = {}_0p_{60}^{02} = 0$

Detailed results with step size $h = 1/12$

t	μ^{01}_{60+t}	μ^{02}_{60+t}	μ^{10}_{60+t}	μ^{12}_{60+t}	${}_tp^{00}_{60}$	${}_tp^{01}_{60}$	${}_tp^{02}_{60}$
0	0.01420	0.01495	0.00142	0.01495	1.00000	0.00000	0.00000
$1/12$	0.01436	0.01506	0.00144	0.01506	0.99757	0.00118	0.00125
$2/12$	0.01453	0.01517	0.00145	0.01517	0.99512	0.00238	0.00250
$3/12$	0.01469	0.01527	0.00147	0.01527	0.99266	0.00358	0.00376
$4/12$	0.01485	0.01538	0.00149	0.01538	0.99018	0.00479	0.00503
$5/12$	0.01502	0.01549	0.00150	0.01549	0.98769	0.00601	0.00630
$6/12$	0.01519	0.01560	0.00152	0.01560	0.98518	0.00723	0.00759
$7/12$	0.01536	0.01571	0.00154	0.01571	0.98265	0.00847	0.00888
$8/12$	0.01554	0.01582	0.00155	0.01582	0.98011	0.00972	0.01017
$9/12$	0.01571	0.01593	0.00157	0.01593	0.97755	0.01097	0.01148
$10/12$	0.01589	0.01605	0.00159	0.01605	0.97497	0.01224	0.01279
$11/12$	0.01607	0.01616	0.00161	0.01616	0.97238	0.01351	0.01411
1	0.01625	0.01628	0.00162	0.01628	0.96977	0.01479	0.01544
2	0.01860	0.01772	0.00186	0.01772	0.93713	0.03089	0.03198
3	0.02129	0.01929	0.00213	0.01929	0.90200	0.04833	0.04967
4	0.02439	0.02101	0.00244	0.02101	0.86432	0.06712	0.06856
5	0.02794	0.02289	0.00279	0.02289	0.82407	0.08722	0.08872
6	0.03202	0.02493	0.00320	0.02493	0.78127	0.10855	0.11018
7	0.03671	0.02717	0.00367	0.02717	0.73601	0.13100	0.13299
8	0.04209	0.02961	0.00421	0.02961	0.68846	0.15435	0.15719
9	0.04826	0.03227	0.00483	0.03227	0.63886	0.17835	0.18279
10	0.05535	0.03517	0.00554	0.03517	0.58756	0.20263	0.20981

Additional problem

When you have the moment, try to calculate (using some software or a spreadsheet) to estimate the transition probabilities given that at age 60, the person is sick: ${}_{10}p_{60}^{10}$ and ${}_{10}p_{60}^{11}$, and ${}_{10}p_{60}^{12}$

Illustrative example 1

Consider the health-sickness insurance model with:

$$\begin{aligned} \mu_{50+t}^{hs} &= 0.040, \\ \mu_{50+t}^{sh} &= 0.005, \\ \mu_{50+t}^{hd} &= 0.010, \text{ and} \\ \mu_{50+t}^{sd} &= 0.020, \end{aligned}$$

for all $t \geq 0$. Do the following:

1. Calculate ${}_{10}p_{50}^{\overline{hh}}$ and ${}_{10}p_{50}^{\overline{ss}}$.
2. Write out the Kolmogorov's forward equations for solving the t-year transition probabilities for a person age 50 who is currently healthy. (consider all possible transitions; do not solve)
3. Write out the Kolmogorov's forward equations for solving the t-year transition probabilities for a person age 50 who is currently sick. (consider all possible transitions; do not solve)

Illustrative example 2

Suppose that an insurer uses the health-sickness model to price a policy that provides both sickness and death benefits to healthy lives aged 40. You are given:

- The term of the policy is 25 years.
- If the individual dies during the term of the policy, there is a death benefit of $20,000 payable at the moment of death. An additional $10,000 is payable if the individual is sick at the time of death.
- If the individual becomes sick during the term of the policy, there is a sickness benefit at the rate of $3,000 per year. No waiting period before benefits are payable.
- The premium rate is $600 payable annually by healthy policyholders.

Express the following in integral form using standard notation of transition probabilities and forces of transitions:

1. the actuarial present value at issue of future premiums;
2. the actuarial present value at issue of future death benefits; and
3. the actuarial present value at issue of future sickness benefits.

Policy values and Thiele's differential equations

Consider the health-sickness insurance model where we have a disability income policy with a term for n years issued to a healthy life (x):

- Premiums are payable continuously throughout the policy term at the rate of P per year, while healthy.
- Benefit in the form of an annuity is payable continuously at the rate of B per year, while sick.
- A lump sum benefit of S is payable immediately upon death within the term of the policy.

Give an expression for the:

1. policy value at time t for a healthy policyholder;
2. policy value at time t for a sick policyholder; and
3. Thiele's differential equations for solving these policy values.

Generalization of Thiele's differential equations

- Section 8.7.2, pages 266-267
- General situation of an insurance contract issued within a more general multiple state model

SOA question #12, Spring 2012

Employees in Company ABC can be in: State 0: Non-executive employee; State 1: Executive employee; or State 2: Terminated from employment.

John joins Company ABC as a non-executive employee at age 30.

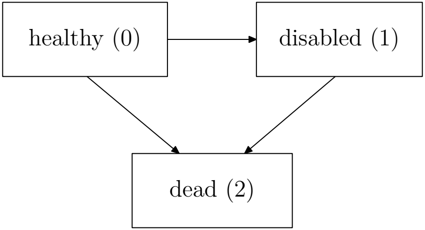
You are given:

- $\mu^{01} = 0.01$ for all years of service
- $\mu^{02} = 0.006$ for all years of service
- $\mu^{12} = 0.002$ for all years of service
- Executive employees never return to the non-executive employee state.
- Employees terminated from employment never get rehired.
- The probability that John lives to age 65 is 0.9, regardless of state.

Calculate the probability that John will be an executive employee of Company ABC at age 65.

SOA question #10, Fall 2013

For a multiple state model, you are given:



The following forces of transition:

$$\mu^{01} = 0.02 \qquad \mu^{02} = 0.03 \qquad \mu^{12} = 0.05$$

Calculate the conditional probability that a Healthy life on January 1, 2004 is still Healthy on January 1, 2014, given that this person is not Dead on January 1, 2014.

Discrete time Markov chain models

Transition probabilities - Markov Chains

- Assume a finite state space: $\{0, 1, 2, \ldots, n\}$ and let $Y_x(k)$ be the state at time k.
- Basic Markov chain assumption:

$$\mathsf{Pr}\big[Y_x(k+1) = j | Y_x(k) = i, Y_x(k-1), \ldots, Y_x(0)\big]$$
$$= \mathsf{Pr}\big[Y_x(k+1) = j | Y_x(k) = i\big]$$

- Notation of transition probabilities:

$$\mathsf{Pr}\big[Y_x(k+1) = j | Y_x(k) = i\big] = Q_k^{(i,j)} = Q_k^{ij}.$$

- Transition probability matrix:

$$\mathbf{Q}_k = \begin{pmatrix} Q_k^{00} & Q_k^{01} & \cdots & Q_k^{0,n} \\ Q_k^{10} & Q_k^{11} & \cdots & Q_k^{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_k^{n,0} & Q_k^{n,1} & \cdots & Q_k^{n,n} \end{pmatrix}$$

Homogeneous and non-homogeneous Markov chains

- If the transition probability matrix $\mathbf{Q}_k$ depends on the time k, it is said to be a non-homogeneous Markov Chain.
- Othewise, it is called a homogeneous Markov Chain, and we shall simply denote the transition probability matrix by $\mathbf{Q}$.
- Define

$$ {}_r\mathbf{Q}_k = \begin{pmatrix} {}_rQ_k^{00} & {}_rQ_k^{01} & \cdots & {}_rQ_k^{0,n} \\ {}_rQ_k^{10} & {}_rQ_k^{11} & \cdots & {}_rQ_k^{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ {}_rQ_k^{n,0} & {}_rQ_k^{n,1} & \cdots & {}_rQ_k^{n,n} \end{pmatrix} $$

where

$$ {}_rQ_k^{ij} = \Pr\left[Y_x(k+r) = j | Y_x(k) = i\right] $$

is the probability of going from state i to state j in r steps. It is sometimes written as ${}_rQ_k^{(i,j)}$.

Chapman-Kolmogorov equations

- Discrete analogue of the Kolmogorov's forward equations.
- Theorem:

$$ {}_r\mathbf{Q}_k = \mathbf{Q}_k \times \mathbf{Q}_{k+1} \times \cdots \times \mathbf{Q}_{k+r-1} $$

- Chapman-Kolmogorov equations:

$$ {}_{m+p}Q_k^{ij} = \sum\nolimits_s \ {}_mQ_k^{is} \times \ {}_pQ_{k+m}^{sj} $$

- In the case of homogeneous Markov Chains, we drop the subscript k and simply write

$$ {}_r\mathbf{Q} = \mathbf{Q} \times \cdots \times \mathbf{Q} = \mathbf{Q}^r. $$

Example 1

- Consider a critical illness model with 3 states: healthy (H), critically ill (I) and dead (D).
- Suppose you have the homogeneous Markov Chain with transition matrix

$$
\begin{array}{cc}
 & \begin{array}{ccc} \mathsf{H} & \mathsf{C} & \mathsf{D} \end{array} \\
\begin{array}{c} \mathsf{H} \\ \mathsf{C} \\ \mathsf{D} \end{array} &
\begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.00 & 0.76 & 0.24 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}
\end{array}.
$$

- What are the probabilities of being in each of the state at times $t = 1, 2, 3$?

Example 2

- Suppose that an auto insurer classifies its policyholders according to Preferred (State #0) or Standard (State #1) status, starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year.
- You are given the following t-th year non-homogeneous transition matrix:

$$\mathbf{Q}_t = \begin{pmatrix} 0.65 & 0.35 \\ 0.50 & 0.50 \end{pmatrix} + \frac{1}{t+1} \begin{pmatrix} 0.15 & -0.15 \\ -0.20 & 0.20 \end{pmatrix}$$

- Given that an insured is Preferred at the start of the second year:
  1. Find the probability that the insured is also Preferred at the start of the third year.
  2. Find the probability that the insured transitions from being Preferred at the start of the third year to being Standard at the start of the fourth year.

Cash flows and actuarial present values

- We are interested in the actuarial present value of cash flows

$$ {}_{t+k+1}C^{ij} $$

  which are the cash flows at time $t+k+1$ for movement from state i (at time $t+k$) to state j (at time $t+k+1$).

- Discount typically by v^{k+1}.

- Theorem: Suppose that the subject is in state s at time t. The actuarial present value (APV) of cash flows from state i to state j is given by

$$ \mathsf{APV}_{s@t} = \sum_{k=0}^{\infty} \left({}_kQ_t^{si} \cdot Q_{t+k}^{ij} \right) \ {}_{t+k+1}C^{ij} \times v^{k+1}. $$

Illustrative example no. 1

An insurer issues a special 3-year insurance contract to a high risk individual with the following homogeneous Markov Chain model:

- States: $0 =$ active, $1 =$ disabled, $2 =$ withdrawn, and $3 =$ dead.
- Transition probability matrix:

$$\begin{array}{c c} & \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.0 & 0.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

- Changes in state occur only at the end of the year.
- The death benefit is $\$1,000$, payable at the end of the year of death.
- The insured is disabled at the end of year 1.
- Assuming interest rate of 5% p.a., Calculate the actuarial present value of the prospective death benefits at the beginning of year 2.

Illustrative example no. 2

Consider a special three-year term insurance:

- Insureds may be in one of three states at the beginning of each year: active, disabled or dead. All insureds are initially active.
- The annual transition probabilities are as follows:

	Active	Disabled	Dead
Active	0.8	0.1	0.1
Disabled	0.1	0.7	0.2
Dead	0.0	0.0	1.0

- A $\$100,000$ benefit is payable at the end of the year of death whether the insured was active or disabled.
- Premiums are paid at the beginning of each year when active. Insureds do not pay annual premiums when they are disabled.
- Interest rate $i = 10\%$.
- Calculate the level annual net premium for this insurance.

Illustrative example no. 3

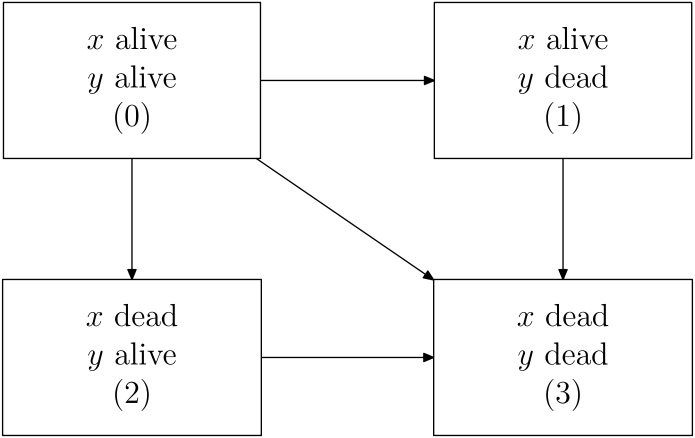
- A machine can be in one of four possible states, labeled *a*, *b*, *c*, and *d*. It migrates annually according to a Markov Chain with transition probabilities:

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0.25 & 0.75 & 0.00 & 0.00 \\ 0.50 & 0.00 & 0.50 & 0.00 \\ 0.80 & 0.00 & 0.00 & 0.20 \\ 1.00 & 0.00 & 0.00 & 0.00 \end{pmatrix} \end{array}$$

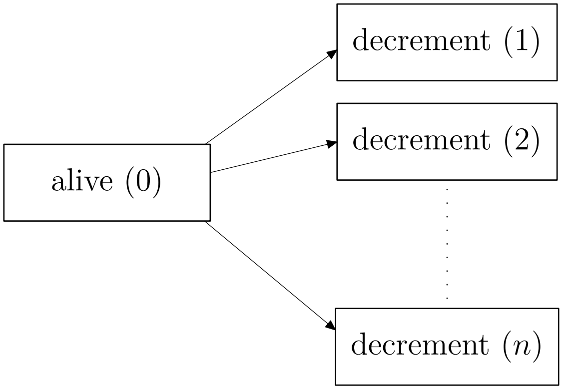
- At time $t = 0$, the machine is in State *a*. A salvage company will pay 100 at the end of 3 years if the machine is in State *a*.
- Assuming $v = 0.90$, calculate the actuarial present value at time $t = 0$ of this payment.

Other transition models with actuarial applications

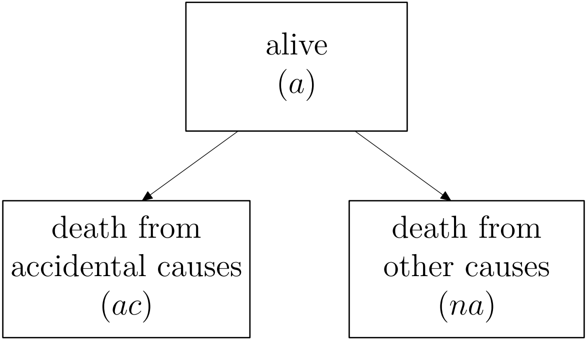
Joint life model



Multiple decrement model



Accidental death model



A simple retirement model