Exercise 9.12

(a) First we note that

$$\begin{aligned} {}_{(t-1)/m}p_{xy} - {}_{t/m}p_{xy} &= {}_{(t-1)/m}p_{xy}\left(1 - {}_{1/m}p_{xy}\right) \\ &= {}_{(t-1)/m}p_{xy} \times {}_{1/m}q_{xy} \\ &= {}_{\frac{t-1}{m}|\frac{1}{m}}q_{xy}, \end{aligned}$$

gives the probability that the joint life (xy) dies in the interval between $(t-1)/m$ and t/m. Thus, the following APV

$$\sum_{t=1}^{m} v^{t/m}\left[{}_{(t-1)/m}p_{xy} - {}_{t/m}p_{xy}\right] = \sum_{t=1}^{m} v^{t/m} \times {}_{\frac{t-1}{m}|\frac{1}{m}}q_{xy}$$

gives the expected present value of an insurance that pays \$1 at the end of the m-th interval in the period of death of the joint life (xy) within one year. The failure of the joint life (xy) is when the first death of (x) and (y) occurs. We can write this one-year APV as

$$A^{(m)}_{\overset{1}{xy}:\overline{1}|} = \sum_{t=1}^{m} v^{t/m} \times {}_{\frac{t-1}{m}|\frac{1}{m}}q_{xy}$$

(b) We can write the APV $A^{(m)}_{xy}$ as a double sum

$$\begin{aligned} A^{(m)}_{xy} &= \sum_{k=0}^{\infty}\left[\sum_{t=1}^{m} v^{k+(t/m)} \times {}_{k+\frac{t-1}{m}|\frac{1}{m}}q_{xy}\right] \\ &= \sum_{k=0}^{\infty} v^k\, {}_kp_{xy}\left[\sum_{t=1}^{m} v^{t/m} \times {}_{\frac{t-1}{m}|\frac{1}{m}}q_{x+k:y+k}\right] \\ &= \sum_{k=0}^{\infty} v^k\, {}_kp_{xy} \times A^{(m)}_{\overset{1}{x+k:y+k}:\overline{1}|} \end{aligned}$$

which indicates as a sum of deferred one-year term insurances as in (a).

(c) It is best we start with the RHS of the equation and work our way to the LHS.

$$\begin{aligned} \frac{1}{m}\left(1 - p_{xy}\right) + \frac{m-2t+1}{m^2}q_xq_y &= \frac{1}{m}\left[1 - (1-q_x)(1-q_y)\right] + \frac{1}{m}q_xq_y + \frac{1-2t}{m^2}q_xq_y \\ &= \frac{1}{m}\left(q_x + q_y\right) + \frac{t^2-2t+1}{m^2}q_xq_y - \frac{t^2}{m^2}q_xq_y \\ &= \frac{1}{m}\left(q_x + q_y\right) + \left(\frac{t-1}{m}\right)^2 q_xq_y - \left(\frac{t}{m}\right)^2 q_xq_y \\ &= \frac{1}{m}\left(q_x + q_y\right) + {}_{(t-1)/m}q_x\,{}_{(t-1)/m}q_y - {}_{t/m}q_x\,{}_{t/m}q_y \end{aligned}$$

Consider the last two terms in the equation above:

$$\begin{aligned}
{}_{(t-1)/m}q_x \, {}_{(t-1)/m}q_y - {}_{t/m}q_x \, {}_{t/m}q_y &= \left(1 - {}_{(t-1)/m}p_x\right)\left(1 - {}_{(t-1)/m}p_y\right) - \left(1 - {}_{t/m}p_x\right)\left(1 - {}_{t/m}p_y\right) \\
&= 1 - {}_{(t-1)/m}p_x - {}_{(t-1)/m}p_y + {}_{(t-1)/m}p_{xy} \\
&\quad -1 + {}_{t/m}p_x + {}_{t/m}p_y - {}_{t/m}p_{xy} \\
&= \left({}_{(t-1)/m}p_{xy} - {}_{t/m}p_{xy}\right) \\
&\quad + \left({}_{(t-1)/m}p_x - {}_{t/m}p_x\right) + \left({}_{(t-1)/m}p_y - {}_{t/m}p_y\right)
\end{aligned}$$

It is not difficult to show that under the UDD assumption, the sum of the terms

$$\left({}_{(t-1)/m}p_x - {}_{t/m}p_x\right) + \left({}_{(t-1)/m}p_y - {}_{t/m}p_y\right) = -\frac{1}{m}\left(q_x + q_y\right),$$

so that we have

$$\frac{1}{m}\left(1 - p_{xy}\right) + \frac{m - 2t + 1}{m^2}\, q_x q_y = {}_{(t-1)/m}p_{xy} - {}_{t/m}p_{xy}$$

which is exactly what we wanted to prove. In addition, we have

$$\begin{aligned}
\sum_{t=1}^{m} v^{t/m}\left[{}_{(t-1)/m}p_{xy} - {}_{t/m}p_{xy}\right] &= \left(1 - p_{xy}\right)\sum_{t=1}^{m}\frac{1}{m}v^{t/m} + q_x q_y \sum_{t=1}^{m} v^{t/m}\,\frac{m-2t+1}{m^2} \\
&= \left(1 - p_{xy}\right)\frac{1}{m}\frac{v^{1/m}(1-v)}{1-v^{1/m}} + q_x q_y \sum_{t=1}^{m} v^{t/m}\,\frac{m-2t+1}{m^2} \\
&= \left(1 - p_{xy}\right)\frac{1-v}{m[(1+i)^{1/m}-1]} + q_x q_y \sum_{t=1}^{m} v^{t/m}\,\frac{m-2t+1}{m^2} \\
&= \left(1 - p_{xy}\right)\frac{iv}{i^{(m)}} + q_x q_y \sum_{t=1}^{m} v^{t/m}\,\frac{m-2t+1}{m^2}.
\end{aligned}$$

(d) Thus, under the assumptions in part (c), we approximate

$$A^{(m)}_{\overset{1}{xy}:\overline{1|}} \approx \frac{i}{i^{(m)}} v\left(1 - p_{xy}\right) = \frac{i}{i^{(m)}} v q_{xy}$$

so that from part (b), we have

$$\begin{aligned}
A^{(m)}_{xy} &\approx \sum_{k=0}^{\infty} v^k \, {}_kp_{xy} \times \frac{i}{i^{(m)}} v q_{x+k:y+k} \\
&= \frac{i}{i^{(m)}} \sum_{k=0}^{\infty} v^{k+1} \, {}_kp_{xy} \times q_{x+k:y+k} \\
&= \frac{i}{i^{(m)}} \sum_{k=0}^{\infty} v^{k+1} \, {}_{k|}q_{xy} \\
&= \frac{i}{i^{(m)}} A_{xy}
\end{aligned}$$