Exercise 9.13

(a) Recall that

$$\frac{d}{dt}v^t = -\delta v^t$$

and that

$$\frac{d}{dt}{}_tp_x\,{}_tp_y = \frac{d}{dt}{}_tp_{xy} = -{}_tp_{xy}\,\mu_{x+t:y+t} = -{}_tp_x\,{}_tp_y\,\mu_{x+t:y+t}$$

Apply product rule of derivatives and we get the desired result in part (a).

(b) Apply Equation (5.40) to the function

$$g(t) = v^t\,{}_tp_x\,{}_tp_y = v^t\,{}_tp_{xy}.$$

Let $h = 1$ and ignore second and higher order derivatives to get:

$$\begin{aligned} \sum_{k=0}^{\infty} g(k) - \frac{1}{2} + \frac{1}{12}g'(0) &= \sum_{k=0}^{\infty} v^k\,{}_kp_{xy} - \frac{1}{2} + \frac{1}{12}\big(-\delta - \mu_{xy}\big) \\ &= \ddot{a}_{xy} - \frac{1}{2} - \frac{1}{12}\big(\delta + \mu_{xy}\big) \end{aligned}$$

Let $h = 1/m$ and ignore second and higher order derivatives to get:

$$\begin{aligned} \frac{1}{m}\sum_{k=0}^{\infty} g(k/m) - \frac{1}{2m} + \frac{1}{12m^2}g'(0) &= \sum_{k=0}^{\infty} \frac{1}{m}v^k\,{}_{k/m}p_{xy} - \frac{1}{2m} - \frac{1}{12m^2}\big(\delta + \mu_{xy}\big) \\ &= \ddot{a}_{xy}^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}\big(\delta + \mu_{xy}\big) \end{aligned}$$

Since both approximate the same formula, we equate the two:

$$\ddot{a}_{xy} - \frac{1}{2} - \frac{1}{12}\big(\delta + \mu_{xy}\big) \approx \ddot{a}_{xy}^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}\big(\delta + \mu_{xy}\big)$$

so that solving for $\ddot{a}_{xy}^{(m)}$, we have the Woolhouse's approximate formula expressed as

$$\ddot{a}_{xy}^{(m)} \approx \ddot{a}_{xy} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\big(\delta + \mu_{xy}\big)$$