ON GENERALIZED SUM RULES FOR JACOBI MATRICES

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Abstract. This work is in a stream (see e.g. [4], [8], [10], [11], [7]) initiated by a paper of Killip and Simon [9], an earlier paper [5] also should be mentioned here. Using methods of Functional Analysis and the classical Szegö Theorem we prove sum rule identities in a very general form. Then, we apply the result to obtain new asymptotics for orthonormal polynomials.

1. Introduction

Probably the paper [12] of M.G. Krein started the so called trace formula

$$\mathrm{tr}\{\phi(J)-\phi(J_0)\} = ...,$$

where ϕ is a function of a certain class, defined on the union of spectra of self-adjoint operators J and J_0, and in the right hand side one has an explicit functional of ϕ. In [12] and in the huge subsequent trace formula literature $J-J_0$ was of trace class, and the functional of ϕ had a form of a certain entropy. Of course one cannot hope to have this kind of equality with a wide class of ϕ unless the perturbation $J-J_0$ is of trace class. There are many reasons for that (the most important among them is the result of Krein himself that his trace formula tested on many ϕ's implies that the perturbation is of trace class). But we would like to indicate another reason: the classical trace formulae have the functional of ϕ in the right hand side, and this functional is very much enracinated in the absolutely continuous spectrum of J that turns out to be *the same* as the absolutely continuous spectrum of J_0. But the absolutely continuous spectrum of the self-adjoint operator can be completely destroyed by perturbations of the class worse than the trace class, for example, by the perturbations of the Hilbert-Schmidt class as it claimed by the von Neumann Theorem. However, a very important effect holds: for some ϕ and for special types of perturbations $J-J_0$ of not the trace class, the absolutely continuous spectrum is preserved and a certain trace formula exists. This effect was observed by Deift and Killip [5] in their study of the absolutely continuous spectrum of discrete Schrödinger operator. For the continuous Schrödinger operator we refer to paper of Christ and Kiselev, where the preservation of absolutely continuous spectrum was proved [3]. In all these examples as well as in the results of the present paper the "special" nature of the perturbation is grasped by the fact that both J and J_0 are Jacobi matrices (and so the perturbation $J-J_0$ "has the same form" as unperturbed operator J_0). In such a situation the Hilbert-Schmidt perturbation

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Date: July 1, 2004.

This work supported by [1]the NSF grant DMS–0200713 and [2]the Austrian Founds FWF, project number: P16390–N04.

Keywords: orthogonal polynomials, asymptotics, Szegö condition.

2000 AMS Subject classification: primary 47B36, secondary 42C05.

was fully treated in a beautiful paper of Killip and Simon [9]. And a trace formula of entropy nature was obtained.

In this paper we show that a quite general entropy integrals always give rise to certain (and it seems to us meaningful) trace formulae. We also show that the meaning of these formulae lies in the asymptotics of orthogonal polynomials with respect to measures, whose entropy (the form of which is chosen by us) is finite.

The reader should be warned that in the introduction, just for the sake of brevity, we focus our attention to the contribution to the trace formula from the absolutely continuous spectrum only, and forget about the point spectrum. However, the paper itself treats these both spectra simultaneously, so the main result has an important "point spectrum" part, and our trace formula always take it into consideration. But in the introduction we skip it to be more concise.

For Jacobi matrices in a mentioned breakthrough paper [9] of Killip and Simon the assumption that $J - J_0$ is in the trace class was replaced by the requirement that $J - J_0$ were in the Hilbert–Schmidt class, and the entropy got the following form

$$\int_{-2}^{2} \sqrt{4-x^2}\,\log \sigma'_{a.c.}(x)\,dx, \tag{1}$$

where $\sigma'_{a.c.}$ denotes the density of the absolutely continuos part of the spectral measure of J. The work of Killip and Simon gave rise to an ever–growing literature proving that certain sets of measures correspond to certain sets of Jacobi matrices whose coefficients obey explicit estimates (sum rules). The structure of these sums over matrix entries in general is enigmatic and their explicit form becomes very fast a quagmire of combinatorial nature. However, we show here that, in a sense, finding the exact form of such sums is not necessary for establishing some interesting properties of Jacobi matrices and associated orthogonal polynomials. Also replacing the constructive approach to finding sum rules (quite puzzling so far and not known in general) by a non–constructive existence theorem may be interesting in its own right as it reveals the relationship between Killip–Simon type results, solvability of a classical homology equation, and trace formula for non–trace class perturbations $J - J_0$.

So let us start with a natural question: what happens if the entropy (1) is replaced by this entropy:

$$\int_{-2}^{2} r(x)\,\log \sigma'_{a.c.}(x)\,dx, \tag{2}$$

with a rather general weight $r(x)$ nonnegative on $[-2, 2]$? Does this entropy give rise to a trace formula? How important and usable is this trace formula (if exists)?

The essence of our main result is that under the assumption of finiteness of "entropy" (2) a certain meaningful trace formula always exists. This formula is also interesting in the sense that, for example, it gives a nontrivial asymptotics for polynomials orthogonal with respect to measure σ. Roughly speaking, this trace formula can be written as the equality of the entropy to a "naive" trace of the difference $\phi(J) - \phi(J_0)$ for perturbed and unperturbed operator (ϕ depends explicitly on r). By "naive" trace we understand the limit of partial sums of diagonal entries of the matrix, that is $\sum_0^N \langle (\phi(J) - \phi(J_0))e_k, e_k\rangle$ (of course, here $J - J_0$ is much worse than the trace class, so the existence of such limit is bound to be nontrivial even though our ϕ turns out to be a pretty nice function always).

Also the existence of such limit (and its equality with the entropy (2)) has several features, which, in our opinion, are interesting in their own right:

- this existence of the limit of partial sums turns out to be a quite general fact,
- the underlying reason for this wide generality of existence of the limit of the partial sums is in a solvability of a certain homology equation, which we write down explicitly (by the way the scope of generality is not clear to us at all, we use still the polynomial nature of r),
- let (Ω, μ) be a probability space, let τ be a measure preserving transformation, and let h be a function such that $\int h\, d\mu = 0$. The homology equation statements (at least the ones that one can meet, for example, in statistical mechanics, see [2]) are usually claiming that if the sums $\sum_{k=0}^{N} h \circ \tau^k$ are bounded μ a.e. then (under natural assumptions on (Ω, μ, τ)) h is homologous to zero, which means $h = \gamma \circ \tau - \gamma$. In our case we prove and use a "one–sided" homology theorem, which says that if sums are bounded from one side (say, they are positive), then h is homologous to a *positive function.* This positive function hides the sum rules! But we use only its existence rather than how it looks like to prove our trace formula and orthogonal polynomials asymptotics.

Let us finish by saying that the classical trace formula found lots of applications ranging from the completely integrable systems theory to a theory o hyponormal operators to name just a few. The "regularized trace formulas" also proved to be quite useful. Here we present one more creature of this nature and show one application (see also [7] for very interesting further applications).

1.1. **Finite dimensional perturbation of the Chebyshev matrix.** Let $\{e_n\}_{n\geq 0}$ be the standard basis in $l^2(\mathbb{Z}_+)$. Let J be a Jacobi matrix defining a bounded self–adjoint operator on $l^2(\mathbb{Z}_+)$:

$$Je_n = p_n e_{n-1} + q_n e_n + p_{n+1} e_{n+1}, \quad n \geq 1,$$

and

$$Je_0 = q_0 e_0 + p_1 e_1.$$

Under the condition $p_n > 0$, the vector e_0 is cyclic for J. The function

$$r(z) = \langle (J - z)^{-1} e_0, e_0 \rangle$$

is called the resolvent function. It has the representation

$$r(z) = \int \frac{d\sigma(x)}{x - z}.$$

The measure σ, $d\sigma \geq 0$, is called the spectral measure of J.

Using a three term recurrence relation for orthonormal polynomials $\{P_n(z)\}_{n\geq 0}$ with respect to σ one can restore the coefficient sequences of J

$$zP_n(z) = p_n P_{n-1}(z) + q_n P_n(z) + p_{n+1} P_{n+1}(z), \quad n \geq 1,$$

and

$$zP_0(z) = q_0 P_0(z) + p_1 P_1(z).$$

With a given J we associate a sequence $J(n)$ defined by

$$p(n)_k = \begin{cases} p_k, & k < n \\ 1, & k \geq n \end{cases},$$

$$q(n)_k = \begin{cases} q_k, & k < n \\ 0, & k \ge n \end{cases}.$$

$J(n)$ is a finite dimensional perturbation of the "free" (Chebyshev) matrix $J_0 = S_+ + S_+^*$, $S_+ e_n = e_{n+1}$.

Note that

$$r_0(z) = \langle (J_0 - z)^{-1} e_0, e_0 \rangle = -\zeta,$$

where $1/\zeta + \zeta = z,\ \zeta \in \mathbb{D}$, that is $\zeta = \frac{z-\sqrt{z^2-4}}{2}$. Further, in terms of orthonormal polynomials

$$r(n)(z) = \langle (J(n) - z)^{-1} e_0, e_0 \rangle = -\frac{p_n Q_n(z) - \zeta Q_{n-1}(z)}{p_n P_n(z) - \zeta P_{n-1}(z)},$$

where Q_n are so called orthonormal polynomials of the second kind

$$Q_n(z) := \int \frac{P_n(x) - P_n(z)}{x - z} d\sigma.$$

(They satisfy the same three term recurrence relation as P_n's but with a different initial condition.) What is important for us

$$\sigma'(n)_{a.c.}(x) = \frac{1}{\pi} \operatorname{Im} r(n)(x + i0) = \frac{1}{\pi} \frac{-\operatorname{Im} \zeta(x + i0)}{|p_n P_n(x) - \zeta(x + i0) P_{n-1}(x)|^2}. \tag{3}$$

and

$$\sigma(J(n)) \cap \{\mathbb{R} \setminus [-2, 2]\} = \{z \in \mathbb{C} \setminus [-2, 2] : p_n P_n(z) - \zeta(z) P_{n-1}(z) = 0\}. \tag{4}$$

The perturbation determinant of $J(n)$ with respect to J_0 is well defined and we can introduce a function

$$\Delta_n(\zeta) = \frac{1}{\prod_{j=1}^{n-1} p_j} \det((J(n) - z)(J_0 - z)^{-1}).$$

Then

$$\log \Delta_n(z) = -t(n)_0 - \sum_{k \ge 1} \frac{t(n)_k}{k z^k} \tag{5}$$

where

$$t(n)_0 = \sum_{j=1}^{n-1} \log p_j, \quad t(n)_k = \operatorname{tr}(J(n)^k - J_0^k),\ k \ge 1.$$

On the other hand one can find the determinant by a direct calculation:

$$\Delta_n(z) = (p_n P_n(z) - \zeta P_{n-1}(z)) \zeta^n,$$

where as before $1/\zeta + \zeta = z,\ \zeta \in \mathbb{D}$.

Therefore, $\Delta_n(z)$ has explicit representation (5) in terms of coefficients of $J(n)$, on the other hand it has nice analytic properties: its zeros in $\overline{\mathbb{C}} \setminus [-2, 2]$ are simple and related to the eigenvalues of $J(n)$ in this region (see (4)); it has no poles; and by (3)

$$|\Delta_n(x + i0)|^2 = \frac{1}{2\pi} \frac{\sqrt{4 - x^2}}{\sigma'(n)_{a.c.}}. \tag{6}$$

That is, we can restore $\Delta_n(z)$ only in terms of these (partial) spectral data (see the next subsection).

1.2. **The Killip–Simon functional via spectral data.**

Definition 1.1. *Let J be a Jacobi matrix with a spectrum on $[-2,2]\cup X$, where the only possible accumulation points of $X=\{x_k\}$ are ± 2. Following to Killip and Simon, to a given nonnegative polynomial A we associate the functional that might diverge only to $+\infty$*

$$\Lambda_A(J):=\sum_X F(x_k)+\frac{1}{2\pi}\int_{-2}^{2}\log\left(\frac{\sqrt{4-x^2}}{2\pi\sigma'_{a.c.}}\right)A(x)\sqrt{4-x^2}\,dx, \tag{7}$$

where

$$\begin{aligned} F(x)&=\int_2^x A(x)\sqrt{x^2-4}\,dx \quad \text{for} \quad x>2,\\ F(x)&=-\int_{-2}^x A(x)\sqrt{x^2-4}\,dx \quad \text{for} \quad x<-2. \end{aligned} \tag{8}$$

Let us point out that the Killip–Simon functional $\Lambda_A(J)$ is defined in terms of the spectral data of J only. Let us demonstrate how to obtain for a finite dimensional perturbation $J(n)$ of J_0 a representation of $\Lambda_A(J(n))$ in terms of the recurrence coefficients.

First, let us note that the function $\log\Delta_n(z)$ is well defined in the upper half plane, in fact, in the domain $\overline{\mathbb{C}}\setminus\sigma(J(n))$. Moreover, the boundary values of the real part $\operatorname{Re}\log\Delta_n(x+i0)$, $x\in[-2,2]$, are given by (6). For $x\ge 2$ the imaginary part of $\log\Delta_n(z)$ (that is the argument of $\Delta_n(\zeta)$) is of the form

$$\frac{1}{\pi}\arg\Delta_n(x+i0)=\#\{y\in\sigma(J(n)):y\ge x\}$$

and similarly,

$$\frac{1}{\pi}\arg\Delta_n(x+i0)=-\#\{y\in\sigma(J(n)):y\le x\}$$

for $x\le -2$. Therefore, multiplying $\log\Delta_n(z)$ by $A(z)\sqrt{z^2-4}$, where $A(z)$ is the given nonnegative polynomial, we get a function with the following representation

$$A(z)\sqrt{z^2-4}\log\Delta_n(z)=B_n(z)+\int_{\sigma(J(n))}\frac{d\lambda_n}{x-z}, \tag{9}$$

where $B_n(z)$ is a (real) polynomial of degree one bigger than A and

$$\lambda_n'(x)=\begin{cases}\frac{1}{2\pi}A(x)\sqrt{4-x^2}\log\frac{1}{2\pi}\frac{\sqrt{4-x^2}}{\sigma'(n)_{a.c.}}, & x\in[-2,2]\\ A(x)\sqrt{x^2-4}\#\{y\in\sigma(J(n)):y\ge x\}, & x\ge 2\\ A(x)\sqrt{x^2-4}\#\{y\in\sigma(J(n)):y\le x\}, & x\le -2\end{cases}.$$

Thus the functional $\Lambda_A(J(n))=\int d\lambda_n$.

Let us mention that the polynomial $B_n(z)$ is determined uniquely by (9) since

$$\int_{\sigma(J(n))}\frac{d\lambda_n}{x-z}=-\frac{\int d\lambda_n}{z}-\ldots=\underline{O}\left(\frac{1}{z}\right),\quad z\to\infty. \tag{10}$$

Lemma 1.2. *Uniformly in n*

$$\int d\lambda_n\ge C>-\infty\,.$$

Proof. We need to estimate from below

$$\sum_{y_k\in\sigma(J(n))} F(y_k) + \frac{1}{2\pi}\int_{-2}^{2} \log\Big(\frac{\sqrt{4-x^2}}{2\pi\sigma'_{a.c}}\Big)\, A(x)\sqrt{4-x^2}\,dx\,.$$

The sum is positive. We need to estimate only the $-\log^-$ part of the integral. But

$$\log^-\Big(\frac{\sqrt{4-x^2}}{2\pi\sigma'_{a.c}}\Big) \le \frac{2\pi\sigma'_{a.c}}{\sqrt{4-x^2}}\,,$$

and this is integrable with respect to $A(x)\sqrt{4-x^2}$.

□

Let us define

$$\Phi(z) = \text{Const} + a_1 z + \dots + a_{m+2} z^{m+2}$$

by

$$\Phi'(z) = zA(z) - \frac{1}{\pi}\int_{-2}^{2} \frac{A(x)-A(z)}{x-z}\sqrt{4-x^2}\,dx.$$

Note that

$$A(z)\sqrt{z^2-4} = \frac{1}{\pi}\int_{-2}^{2}\frac{A(x)}{x-z}\sqrt{4-x^2}\,dx + \Phi'(z). \tag{11}$$

Multiplying both sides by $\log\Delta_n(z) = -t(n)_0 - \sum_k \frac{t(n)_k}{kz^k}$ and comparing the "$\frac{1}{z}$–term" in both sides, using (9), (10), we get

$$\begin{aligned}\int d\lambda_n &= -\,at(n)_0 + a_1 t(n)_1 + 2a_2\frac{t(n)_2}{2} + \dots + (m+2)a_{m+2}\frac{t(n)_{m+2}}{(m+2)}\\ &= -\,at(n)_0 + \operatorname{tr}\{\Phi(J(n)) - \Phi(J_0)\},\end{aligned} \tag{12}$$

where we put

$$a = \frac{1}{\pi}\int_{-2}^{2} A(x)\sqrt{4-x^2}\,dx.$$

Note, if $A(z)=1$, that is $a=2$, $\Phi(z) = \text{Const} + z^2/2$, then we are in the Killip–Simon case [9]:

$$\int d\lambda_n = \frac{t(n)_2}{2} - 2t(n)_0 = -\frac{1}{2} + \sum_{k=1}^{\infty}(p(n)_k^2 - 1 - \log p(n)_k^2) + \frac{1}{2}\sum_{k=0}^{\infty} q(n)_k^2.$$

For a more general example see Sect. 6.

1.3. The Killip–Simon functional via coefficient sequences.

For a bounded operator G in $l^2(\mathbb{Z}_+)$ we denote $G^{(k)} := (S_+^*)^k G S_+^k$. In other words, we "obliterate" the first k columns and rows of the infinite matrix G.

Lemma 1.3. *For all $k \ge 1$ and $n \ge l-1$*

$$(J^{(k)})^l e_n = (J^l)^{(k)} e_n.$$

Proof. Let us mention that the decomposition of the vector $J^l e_{k+n}$ begins with the basic's vector e_{k+n-l}. Therefore the orthoprojector P_{k-1} onto the subspace spanned by $\{e_0, ...e_{k-1}\}$ annihilates this vector, $P_{k-1}J^l e_{k+n} = 0$. Thus, by induction,

$$\begin{aligned}(J^{(k)})^{l+1} e_n =& J^{(k)}(J^{(k)})^l e_n = J^{(k)}(J^l)^{(k)} e_n = (S_+^*)^k J S_+^k (S_+^*)^k J^l S_+^k e_n\\ =& (S_+^*)^k J(I - P_{k-1})J^l e_{k+n} = (S_+^*)^k J^{l+1} e_{k+n} = (J^{l+1})^{(k)} e_n.\end{aligned}$$

□

For a bounded Jacobi matrix J (and a polynomial A) let us define a function of a finite number of variables

$$h_A = h_A(J) := -a \log p_{m+2} + \langle \{\Phi(J) - \Phi(J_0)\} e_{m+1}, e_{m+1}\rangle .$$

Note that due to the previous lemma

$$\begin{aligned} h_A \circ \tau^k &= -a \log p_{m+k+2} + \langle \{\Phi(J^{(k)}) - \Phi(J_0)\} e_{m+1}, e_{m+1}\rangle \\ &= -a \log p_{m+k+2} + \langle \{\Phi(J) - \Phi(J_0)\} e_{m+k+1}, e_{m+k+1}\rangle , \end{aligned}$$

where τ acts just as a shift of indexes: $\tau(p_m) = p_{m+1}, \tau(q_m) = q_{m+1}$. In this case each term of the series

$$\sum_{k\ge 0} h_A \circ \tau^k$$

was just defined. It is a polynomial expression on matrix elements of J, and it is of "of finite window" type. This means that k–th term depends only on $p_s, q_s, |s-k| \le l$ in a polynomial way, and l is independent of k.

Definition 1.4. *With a given Jacobi matrix J and a polynomial A of degree m we associate the formal series*

$$H_A(J) := \sum_{k=0}^{m} (-a \log p_{k+1} + \langle \{\Phi(J) - \Phi(J_0)\} e_k, e_k\rangle) + \sum_{k\ge 0} h_A \circ \tau^k . \tag{13}$$

Note that $H_A(J(n))$ is just a finite sum, in fact $h \circ \tau^k$ vanishes starting with a suitable k, moreover $H_A(J(n)) = \Lambda_A(J(n))$.

Nobody however said that the series $\sum_{k\ge 0} h_A \circ \tau^k$ in the above definition converges for a given J.

Notice that for $J(n)$ we have uniformly in n (see Lemma 1.2)

$$H_A(J(n)) = -a\, t(n)_0 + \operatorname{tr}\{\Phi(J(n)) - \Phi(J_0)\} = \int d\lambda_n \ge C > -\infty . \tag{14}$$

1.4. **Results.**

Theorem 1.5. *Let A be a nonnegative polynomial. The spectral measure σ of a Jacobi matrix J with a spectrum of the form $[-2, 2] \cup X$, where ± 2 are the only possible accumulation points of the discrete set X, satisfies $\Lambda_A(J) < \infty$ if and only if series* (13) *converges; moreover $H_A(J) = \Lambda_A(J)$.*

In a sense our result is a kind of "existence theorem". The existence of what exactly will become clear when we discuss the homology equation in Lemma 3.1. One can understand the equality $H_A(J) = \Lambda_A(J)$ as a trace formula. In the right hand side we have a certain entropy of the spectral measure (see (7)), in the left hand side we basically have the limit of partial sums of the type $\sum_0^N \langle (\Phi(J) - \Phi(J_0)) e_m, e_m\rangle$ (actually with a logarithmic term). This limit of partial sums can be viewed as a regularized trace of $\Phi(J) - \Phi(J_0)$. This trace formula turns out to be useful. For example, it gives the following application to asymptotics of polynomials orthogonal with respect to measure σ of finite entropy. We call here the measure σ supported on $[-2, 2] \cup X$ (X accumulates only to points ± 2 and lies outside of

$[-2,2]$) the measure of finite A-entropy if $d\sigma=\sigma'_{a.c}dx+d\sigma_s+\sum_{x_k\in X}c_k\delta_{x_k}$, $d\sigma_s$ is any positive singular measure on $[-2,2]$, and

$$\sum_{x_k\in X}F(x_k)+\frac{1}{2\pi}\int_{-2}^{2}\log\Big(\frac{\sqrt{4-x^2}}{2\pi\sigma'_{a.c}}\Big)\,A(x)\sqrt{4-x^2}\,dx<\infty\,.$$

Theorem 1.6. *Let $A(x)$ be a nonnegative polynomial of degree m with all zeros on $[-2,2]$. Let a measure σ supported on $[-2,2]\cup X$ satisfy the condition $\int d\lambda<\infty$, where*

$$\lambda'(x)=\lambda'(x;\sigma)=\begin{cases}\frac{1}{2\pi}A(x)\sqrt{4-x^2}\log\left(\frac{1}{2\pi}\frac{\sqrt{4-x^2}}{\sigma'_{a.c.}(x)}\right), & x\in[-2,2]\\ A(x)\sqrt{x^2-4}\#\{y\in X:y\ge x\}, & x\ge 2\\ A(x)\sqrt{x^2-4}\#\{y\in X:y\le x\}, & x\le -2\end{cases}. \tag{15}$$

Then the sequence of orthonormal polynomials $P_n(z)=P_n(z;\sigma)$, normalized by

$$\zeta^{n+1}\sqrt{z^2-4}P_n(z)\exp\left(-\frac{\tilde B_n(z)}{A(z)\sqrt{z^2-4}}\right)=1+\underline{O}\left(\frac{1}{z^{m+2}}\right),$$

the polynomial $\tilde B_n(z)$ (of degree $m+1$) is determined uniquely by the condition

$$\log\{\zeta^{n+1}\sqrt{z^2-4}P_n(z)\}-\frac{\tilde B_n(z)}{A(z)\sqrt{z^2-4}}=\underline{O}\left(\frac{1}{z^{m+2}}\right),$$

converges uniformly on compact subsets of the domain $\overline{\mathbb{C}}\setminus[-2,2]$ to the holomorphic function

$$D(z):=\exp\left(\frac{1}{A(z)\sqrt{z^2-4}}\int\frac{d\lambda}{x-z}\right). \tag{16}$$

Note that as well as in the Szegö case the limit function $D(z)$ can be expressed only in terms of $\sigma'_{a.c.}$ and X.

2. Semicontinuity of Szegö type functional

For a measure μ on the unit circle $\mathbb{T}$ we denote by $\mathrm{Sz}(\mu)$ the functional

$$\mathrm{Sz}(\mu)=\int_{\mathbb{T}}\log\frac{d\mu_{a.c.}}{dm}\,dm.$$

Recall the main property of this functional

$$\mathrm{Sz}(\mu)=\inf\{\log\int_{\mathbb{T}}|1-f|^2\,d\mu(t):f\text{ is a polynomial}, f(0)=0\}.$$

Lemma 2.1. *Let μ_k converge weakly to μ. Then*

$$\limsup\mathrm{Sz}(\mu_k)\le\mathrm{Sz}(\mu). \tag{17}$$

Proof. Since for every ϵ there exists a polynomial g, $g(0)=0$, such that

$$\log\int_{\mathbb{T}}|1-g|^2\,d\mu(t)\le\mathrm{Sz}(\mu)+\epsilon,$$

starting from a suitable k we have

$$\log\int_{\mathbb{T}}|1-g|^2\,d\mu_k(t)\le\mathrm{Sz}(\mu)+2\epsilon.$$

But for every k

$$\begin{aligned}\mathrm{Sz}(\mu_k) &= \inf\{\log\int_{\mathbb{T}} |1-f|^2\, d\mu_k(t) : f \text{ is a polynomial}, f(0)=0\}\\ &\le \log\int_{\mathbb{T}} |1-g|^2\, d\mu_k(t).\end{aligned}$$

Thus (17) is proved. □

Lemma 2.2. *Let ρ be a normalized nonnegative weight, i.e., $\rho \ge 0$, $\int_{\mathbb{T}} \rho\, dm = 1$, such that $\rho\log\rho \in L^1$. Assume that μ_k converges weakly to μ. Then*

$$\liminf \int_{\mathbb{T}} \log\frac{dm}{d(\mu_k)_{a.c.}}\,\rho dm \ge \int_{\mathbb{T}} \log\frac{dm}{d\mu_{a.c.}}\,\rho dm. \tag{18}$$

Proof. Define a map $\psi : \mathbb{T} \to \mathbb{T}$ by $\psi(e^{i\theta}) = \exp\{i\int_0^\theta \rho(e^{i\theta})\, d\theta\}$ and denote by ϕ the inverse map, $\psi \circ \phi = \mathrm{id} : \mathbb{T} \to \mathbb{T}$. Let us apply Lemma 2.1 to the sequence $\tilde\mu_n := \mu_n \circ \phi$ that converges weakly to $\tilde\mu := \mu \circ \phi$.

$$\liminf \int_{\mathbb{T}} \log\frac{dm}{d(\tilde\mu_k)_{a.c.}}\, dm \ge \int_{\mathbb{T}} \log\frac{dm}{d\tilde\mu_{a.c.}}\, dm.$$

Making the inverse change of variable in each integral we have

$$\liminf \int_{\mathbb{T}} \log\frac{\rho dm}{d(\mu_k)_{a.c.}}\,\rho dm \ge \int_{\mathbb{T}} \log\frac{\rho dm}{d\mu_{a.c.}}\,\rho dm.$$

Since $\rho\log\rho \in L^1$ we get (18).

□

Corollary 2.3.

$$\liminf_{n\to\infty} \Lambda_A(J(n)) \ge \Lambda_A(J).$$

Proof. Outside of $[-2,2]$ we apply the Fatou Lemma, e.g. [16], p. 17, and on $[-2,2]$ we apply Lemma 2.2 □

3. LEMMA ON POSITIVENESS AND ITS CONSEQUENCES

Let I be a compact subset of $\mathbb{R}^n$, $0 \in I$. Let a function $h \in C(I^l)$ be such that $h(0,...,0) = 0$. Then

$$H(\underline{x}) = \sum_{i=0}^{\infty} h(x_{i+1}, x_{i+2}, ..., x_{i+l})$$

is well defined on the space of sequences:

$$I_0^\infty = \{\underline{x} : \underline{x} = (x_0, x_1, ..., x_N, 0, 0...),\ x_k \in I\}.$$

In this section we prove a lemma on the solvability of a homology equation mentioned in the Introduction.

Lemma 3.1. *Assume that H is bounded from below, $H(\underline{x}) \ge C$ for all $\underline{x} \in I_0^\infty$. Then there exists a function g of the form*

$$g(x_1, ..., x_l) = h(x_1, ..., x_l) + \gamma(x_2, ..., x_l) - \gamma(x_1, ..., x_{l-1}), \quad \gamma \in C(I^{l-1}),$$

such that $g \ge 0$.

We start with the following sublemma:

Lemma 3.2. *The set G, consisting of functions of the form*

$$G = \{g(x_1, ..., x_l) + \gamma(x_1, ..., x_{l-1}) - \gamma(x_2, ..., x_l)\},$$

where $g \in C(I^l), g \geq 0, g(0) = 0$, $\gamma \in C(I^{l-1})$, is closed in $C(I^l)$.

Proof. We give a proof in the case of three variables (the general case can be considered in the same way).

Let

$$h(x, y, z) = \lim\{g_n(x, y, z) + \gamma_n(x, y) - \gamma_n(y, z)\}. \tag{19}$$

Notice that, assuming the normalization $\gamma_n(0) = 0$, we have a uniform bound for γ_n, $n \geq n_0$,

$$\gamma_n(x, y) \leq h(x, y, 0) + h(y, 0, 0) + 2,$$

and

$$\gamma_n(y, z) \geq -h(0, y, z) - h(0, 0, y) - 2.$$

Let us reformulate the claim of the lemma in the following way: for the given $h \in C(I^3)$ there exists $\gamma \in C(I^2)$ such that

$$\gamma(x, y) - \gamma(y, z) \leq h(x, y, z). \tag{20}$$

First, let us construct a function γ_1 which is defined on I^2 and satisfies (20), that is we do not claim that γ_1 is a continuos function. We define

$$\gamma_1(x, y) = \limsup_{n\to\infty} \gamma_n(x, y).$$

Since

$$\limsup(a + b) \geq \limsup a + \liminf b$$

we get

$$\begin{aligned}\limsup \gamma_n(x, y) - \limsup \gamma_n(y, z) &= \limsup \gamma_n(x, y) + \liminf(-\gamma_n(y, z)) \\ &\leq \limsup(\gamma_n(x, y) - \gamma_n(y, z)) \leq h(x, y, z).\end{aligned}$$

Next, we construct an upper semicontinuous function

$$\gamma_2(x_0, y_0) = \limsup_{(x,y)\to(x_0,y_0)} \gamma_1(x, y)$$

The same reason shows that (20) holds with $\gamma = \gamma_2$. Now we are in a position to prove that there exists a continuous function γ that satisfies (20).

Let Γ be the set of upper semicontinuous functions defined on I^2 with normalization $\gamma(0) = 0$ that satisfy (20). The previous construction shows that $\Gamma \neq \emptyset$. Now, the key point is to consider the function

$$\gamma_3 := \sup\{\gamma : \gamma \in \Gamma\}.$$

It belongs to Γ since $\sup\{\beta_1, \beta_2\} \in \Gamma$ if only $\beta_1 \in \Gamma, \beta_2 \in \Gamma$.

Our first claim, concerning γ_3, is: $\gamma_3(x, y)$ is continuous on x uniformly on y, that is $\forall\epsilon$ $\exists\delta$ such that $|x - x_0| < \delta$ implies

$$|\gamma_3(x, y) - \gamma_3(x_0, y)| < \epsilon, \quad \forall y.$$

Assume, on the contrary, that it is not. This means that there exists $\epsilon > 0$, a point $x_0 \in I$ and a sequence $\{(x_n, y_n)\}$, $\lim x_n = x_0$, such that either

$$\gamma_3(x_0, y_n) \leq \gamma_3(x_n, y_n) - \epsilon, \tag{21}$$

or

$$\gamma_3(x_n, y_n) \le \gamma_3(x_0, y_n) - \epsilon. \tag{22}$$

Consider the first possibility (21). The function $h(x, y, z)$ is continuous therefore we can choose such N that

$$|h(x_N, y, z) - h(x_0, y, z)| \le \epsilon/2$$

for all $(y, z) \in I^2$. Note also that (x_0, y_N) is not the point of normalization, $(x_0, y_N) \neq 0$, since

$$-h(0, x, y) - h(0, 0, x) \le \gamma(x, y) \le h(x, y, 0) + h(y, 0, 0),$$

and hence $\lim_{(x,y)\to 0} \gamma(x, y) = 0 = \gamma(0)$ for all $\gamma \in \Gamma$.

Put

$$\gamma_4(x, y) = \begin{cases} \gamma_3(x, y), & (x, y) \neq (x_0, y_N) \\ \gamma_3(x_0, y_N) + \epsilon/2, & (x, y) = (x_0, y_N). \end{cases}$$

Let us check that $\gamma_4 \in \Gamma$. It is upper semicontinuous, $\gamma_4(0) = 0$. Further, for $(y_N, z) \neq (x_0, y_N)$ we have

$$\begin{aligned} \gamma_4(x_0, y_N) - \gamma_4(y_N, z) =& \gamma_3(x_0, y_N) + \epsilon/2 - \gamma_3(y_N, z) \\ \le& \gamma_3(x_N, y_N) - \epsilon/2 - \gamma_3(y_N, z) \\ \le& h(x_N, y_N, z) - \epsilon/2 \le h(x_0, y_N, z). \end{aligned}$$

Moreover the inequality $\gamma_4(x, y) - \gamma_4(y, z) \le h(x, y, z)$ holds for $(y, z) = (x_0, y_N)$ and for all other values of x, y, z.

On the other hand γ_4 could not be in the class, since

$$\gamma_4(x_0, y_N) > \sup\{\gamma(x_0, y_N),\ \gamma \in \Gamma\}.$$

Therefore we arrive to a contradiction.

In the second case (22) we get a contradiction using the function

$$\tilde{\gamma}_4(x, y) = \begin{cases} \gamma_3(x, y), & (x, y) \neq (x_N, y_N) \\ \gamma_3(x_N, y_N) + \epsilon/2, & (x, y) = (x_N, y_N), \end{cases}$$

and following the same line of proof. Thus, the first claim is proved.

Our second claim is: $\gamma_3(x, y)$ is continuous on both variables. If not, then there exists $\epsilon > 0$, and a sequence $\{(x_n, y_n)\}$, $\lim(x_n, y_n) = (x_0, y_0)$, such that

$$\gamma_3(x_n, y_n) \le \gamma_3(x_0, y_0) - \epsilon$$

(recall $\gamma_3(x, y)$ is upper semicontinuous).

Let us choose N such that

$$\begin{gathered} (x_N, y_N) \neq 0, \\ h(x_0, y_0, z) - \epsilon/3 \le h(x_N, y_N, z), \quad \forall z, \\ \gamma_3(y_0, z) - \epsilon/3 \le \gamma_3(y_N, z), \quad \forall z. \end{gathered} \tag{23}$$

The last inequality is satisfied since $\gamma_3(y, z)$ is continuous on y uniformly on z (the first claim).

Define

$$\gamma_5(x, y) = \begin{cases} \gamma_3(x, y), & (x, y) \neq (x_N, y_N) \\ \gamma_3(x_N, y_N) + \epsilon/3, & (x, y) = (x_N, y_N) \end{cases}$$

and check that $\gamma_5 \in \Gamma$. It is upper semicontinuous, $\gamma_5(0) = 0$. Using (23), for $(y_N, z) \neq (x_N, y_N)$, we have

$$\begin{aligned}\gamma_5(x_N, y_N) - \gamma_5(y_N, z) =& \gamma_3(x_N, y_N) + \epsilon/3 - \gamma_3(y_N, z)\\ \leq& \gamma_3(x_0, y_0) - 2\epsilon/3 - \gamma_3(y_N, z)\\ =& \gamma_3(x_0, y_0) - \gamma_3(y_0, z) - \epsilon/3\\ +& \gamma_3(y_0, z) - \gamma_3(y_N, z) - \epsilon/3\\ \leq& h(x_0, y_0, z) - \epsilon/3 \leq h(x_N, y_N, z).\end{aligned}$$

Moreover the inequality $\gamma_5(x, y) - \gamma_5(y, z) \leq h(x, y, z)$ holds for all other values of x, y, z.

On the other hand γ_5 could not be in the class, since

$$\gamma_5(x_N, y_N) > \sup\{\gamma(x_N, y_N),\ \gamma \in \Gamma\}.$$

Thus $\gamma_3(x, y)$ is a continuous function. The lemma is proved.

□

Proof of Lemma 3.1. Suppose that h does not belong to the closed convex set G. Therefore there exists a measure $\mu \in C(I^l)^*$, $d\mu \geq 0$, such that

$$\int_{I^l} h(x)\, d\mu(x) < 0 \tag{24}$$

and

$$\int_{I^l} (\gamma(x_2, ..., x_l) - \gamma(x_1, ..., x_{l-1}))\, d\mu(x) = 0, \quad \forall \gamma \in C(I^{l-1}).$$

In other words, define

$$d\nu(y) := \int_{z\in I} d\mu(y, z), \quad y \in I^{l-1}, \tag{25}$$

then

$$d\nu(y) = \int_{z\in I} d\mu(z, y). \tag{26}$$

We want to get a contradiction between (24) and $H \geq C$ by extending the functional related to μ on functions on I_0^∞.

We can normalize μ by the condition $\int_{I^l} d\mu = 1$. Let us think of μ as of the probability

$$d\mu(y) = \mathbb{P}\{\underline{x} : x_i \in (y_i, y_i + dy_i),\ i = 1, ..., l\},$$

and we want

$$\mathbb{P}\{\underline{x} : x_{i+k} \in (y_i, y_i + dy_i),\ i = 1, ..., l\} = d\mu(y), \text{ for all } k, \tag{27}$$

that is the probability should be shift invariant. Actually we will define on I^{N+l} step by step for increasing N probabilistic measures

$$d\mu^{(N)}(x_1, \ldots, x_{N+l})$$

such that

$$\int_{x\in I^{k_1}, z\in I^{k_2}} d\mu^{(N)}(x, y, z) = d\mu(y), \quad y \in I^l, \tag{28}$$

where $k_1 + k_2 = N$, $k_1 \geq 0, k_2 \geq 0$.

Define a system of probabilistic measures $\mu(x|x_1,...,x_{l-1})$ on I such that

$$\int f(x_1,...,x_l)\, d\mu = \int d\nu(x_1,...,x_{l-1})\{\int_I f(x_1,...,x_{l-1},x)\, d\mu(x|x_1,...,x_{l-1})\}, \tag{29}$$

actually this is the Fubini Theorem.

Let us make one–step extension of μ by using the conditional probabilities

$$d\mu^{(1)}(x_1,\dots,x_l,x) := d\mu(x_1,\dots,x_l)d\mu(x|x_2,...,x_l).$$

Now we have to check that (28) holds true. As $\mu(x|x_1,...,x_{l-1})$ is probabilistic we get

$$\int_{x\in I} d\mu^{(1)}(x_1,\dots,x_l,x) = d\mu(x_1,\dots,x_l).$$

By (26) and (29) we have

$$\begin{aligned}\int_{x\in I} d\mu^{(1)}(x,x_1,\dots,x_l) = &\int_{x\in I} d\mu(x,x_1,\dots,x_l)d\mu(x_l|x_1,...,x_{l-1})\\ =&d\nu(x_1,\dots,x_{l-1})d\mu(x_l|x_1,...,x_{l-1}) = d\mu(x_1,\dots,x_l).\end{aligned}$$

Continuing inductively in this way

$$d\mu^{(N+1)}(x_1,\dots,x_{N+l-1},x) := d\mu^{(N)}(x_1,\dots,x_{N+l-1})d\mu(x|x_N,...,x_{N+l-1}),$$

we get (28).

Now we are in a position to finish Lemma's proof. For $\underline{x}$'s of the form $\underline{x} = (x,0,...),\ x\in I^N$, we can integrate H against $\mu^{(N-l)}$:

$$\int_{x\in I^N} H(\underline{x})d\mu^{(N-l)}(x) \ge C.$$

On the other hand using the definition of H and the key property (28) of $\mu^{(N-l)}$ we get

$$C \le \int_{x\in I^N} H(\underline{x})d\mu^{(N-l)} \le (l-1)||h|| + (N-l+1)\int_{I^l} h(y)d\mu(y). \tag{30}$$

Since N is arbitrary large, (24) contradicts to (30). ☐

Corollary 3.3. *For a nonnegative polynomial A there exist continuous functions g_A and γ_A such that*

$$h_A = g_A + \gamma_A\circ\tau - \gamma_A \tag{31}$$

and $g_A \ge 0$.

Proof. Note that by (14) we know that $H_A(J(n))$ are uniformly bounded from below. ☐

Corollary 3.4. *Let J be such that $p_n \to 1$ and $q_n \to 0$. Then*

$$H_A(J) := \sum_{k=0}^{m}(-a\log p_{k+1} + \langle\{\Phi(J)-\Phi(J_0)\}e_k,e_k\rangle) - \gamma_A + \sum_{k\ge0} g_A\circ\tau^k. \tag{32}$$

That is the series with positive terms $\sum_{k\ge0} g_A\circ\tau^k$ converges if and only if the series $\sum_{k\ge0} h_A\circ\tau^k$ converges. In particular, the series $\sum_{k\ge0} g_A\circ\tau^k$ converges if and only if $H_A(J)<\infty$.

Proof. We use representation (31) and continuity of γ_A.

☐

4. Proof of the Main Theorem

Assume that for a given J its spectral measure σ is such that $\Lambda_A(J) < \infty$, see definition (7). Note that due to Denisov–Rakhmanov Theorem [6]

$$p_n(\sigma) \to 1, \quad q_n(\sigma) \to 0 \tag{33}$$

and we can use (32) as a definition of $H_A(J)$.

With the measure σ let us associate a measure σ_ϵ that we get by using the following two regularizations. First, we add to its absolutely continuous part the component $\epsilon\, dx$, that is $(\sigma'_\epsilon)_{a.c.} = \sigma'_{a.c.} + \epsilon$. Second, we leave just a finite number of the spectral points outside of $[-2, 2]$, say, that one that belongs to $\mathbb{R} \setminus [-2-\epsilon, 2+\epsilon]$. It is important that

$$p_n(\sigma_\epsilon) \to p_n(\sigma), \quad q_n(\sigma_\epsilon) \to q_n(\sigma) \tag{34}$$

for a fixed n as $\epsilon \to 0$. The measure σ_ϵ satisfies the conditions of Szegö's Theorem, and therefore $\zeta^n P_n(z, \sigma_\epsilon)$ converges uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-2, 2]$ to a certain function that can be expressed directly in terms of $(\sigma'_\epsilon)_{a.c.}$ and the mass–points outside of $[-2, 2]$, see e.g. [13]. We use the consequence of this statement in the form

$$\zeta^n (p_n(\sigma_\epsilon) P_n(z, \sigma_\epsilon) - \zeta P_{n-1}(z, \sigma_\epsilon)) \to \Delta(z; \sigma_\epsilon)$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-2, 2]$. Here $\Delta(z; \sigma_\epsilon)$ is defined by

$$\Delta(z; \sigma_\epsilon) = \exp\left\{ \sqrt{z^2 - 4} \int \frac{1}{x - z} \frac{d\lambda(x; \sigma_\epsilon)}{x^2 - 4} \right\}.$$

In other words

$$\log \Delta(z; J(n; \sigma_\epsilon)) \to \log \Delta(z; \sigma_\epsilon), \ n \to \infty,$$

uniformly on $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_\epsilon)$.

Finally, since (all) coefficients in decomposition (5) of $\log \Delta(z; J(n; \sigma_\epsilon))$ at infinity converge to the corresponding coefficients of $\log \Delta(z; \sigma_\epsilon)$ we get

$$H_A(J_\epsilon(n)) \to \Lambda_A(J_\epsilon), \quad n \to \infty.$$

Evidently $\Lambda_A(J_\epsilon) \le \Lambda_A(J)$. Therefore for every δ there exists n_0 such that

$$H_A(J_\epsilon(n)) \le \Lambda_A(J) + \delta$$

for all $n \ge n_0$. Since in the case under consideration H_A is (basically) a series with positive terms, we get that every partial sum is bounded

$$H_A^N(J_\epsilon(n)) \le \Lambda_A(J) + \delta.$$

Note that the left–hand side does not depend on n if n is big enough. Thus

$$H_A^N(J_\epsilon) \le \Lambda_A(J).$$

Now, for a fixed N let us pass to the limit as $\epsilon \to 0$. Due to (34) and continuity of g_A, for all N

$$H_A^N(J) \le \Lambda_A(J).$$

But this means that

$$\limsup H_A(J(n)) = \limsup \Lambda_A(J(n)) \le \Lambda_A(J).$$

Using Corollary 2.3 we get

$$H_A(J) = \lim H_A(J(n)) = \lim \Lambda_A(J(n)) = \Lambda_A(J).$$

Finally, starting with the condition that series (13) converges we conclude that $\limsup H_A(J(n)) = \limsup \Lambda_A(J(n)) < \infty$. Therefore, due to Corollary 2.3, we have $\Lambda_A(J) < \infty$ and this completes the proof.

5. Asymptotic of orthonormal polynomials

Proof of Theorem 1.6. First let us mention that simultaneously with the convergence

$$\Lambda(J(n)) = \int d\lambda_n \to \Lambda(J) = \int d\lambda,$$

we proved

$$\lim_{n\to\infty} \int P(x) d\lambda_n(x) = \int P(x) d\lambda(x) \tag{35}$$

for every $P(x) = Q^2(x)$ and hence (35) holds for all polynomials. Since the variations of λ_n's are uniformly bounded and since there is a finite interval $[\alpha_1, \alpha_2]$ containing the support of each measure λ_n in the family, λ_n converges weakly to λ.

We will estimate the difference

$$\left| \int \frac{d\lambda_n}{x-z} - \int \frac{d\lambda}{x-z} \right|$$

on a system of contours of the form

$$\tau = \{z = x + iy : a \le x \le b,\ y = \pm c; \quad |y| \le c,\ x = a, b\}$$

that shrink to the interval $[-2, 2]$.

Integrating by parts, on a horizontal line we have

$$\begin{aligned} \left| \int \frac{(\lambda - \lambda_n)\,dx}{(x-z)^2} \right| \le& \frac{\int_{\alpha_1}^{\alpha_2} |\lambda - \lambda_n|\,dx}{c^2} + |\lambda(\alpha_2) - \lambda_n(\alpha_2)| \int_{\alpha_2}^{\infty} \frac{dx}{|x-z|^2} \\ \le& \frac{\int_{\alpha_1}^{\alpha_2} |\lambda - \lambda_n|\,dx}{c^2} + \frac{|\lambda(\alpha_2) - \lambda_n(\alpha_2)|}{c}. \end{aligned}$$

Since the $\lambda_n(x)$ are uniformly bounded and $\lim_{n\to\infty} \lambda_n(x) = \lambda(x)$ for all x, the above estimate shows that for every $\epsilon > 0$ there exists n_0 such that

$$\left| \int \frac{d\lambda_n}{x-z} - \int \frac{d\lambda}{x-z} \right| \le \epsilon,\ n \ge n_0,$$

when z runs on a horizontal line of the contour τ.

Next, let us consider, say, the right vertical line on τ. Assume that b is between of two consequent points $x_{k+1} < x_k$ of the set X. We can even specify $b = (x_{k+1} + x_k)/2$. The point is that starting with a suitable n the interval $[b - \delta/2, b + \delta/2]$ is in a gap of the support of $\lambda - \lambda_n$. Here $\delta := (x_k - x_{k+1})/2$. Put $\tilde\lambda(x) = \lambda(x) - \lambda(b)$ and $\tilde\lambda_n(x) = \lambda_n(x) - \lambda_n(b)$. Doing basically the same as on a horizontal line, we get

$$\begin{aligned} \left| \int_{b+\delta/2}^{\infty} \frac{(\tilde\lambda - \tilde\lambda_n)\,dx}{(x-z)^2} \right| \le& \frac{\int_{b+\delta/2}^{\alpha_2} |\tilde\lambda - \tilde\lambda_n|\,dx}{(\delta/2)^2} + |\tilde\lambda(\alpha_2) - \tilde\lambda_n(\alpha_2)| \int_{\alpha_2}^{\infty} \frac{dx}{|x-z|^2} \\ \le& \frac{\int_{\alpha_1}^{\alpha_2} |\tilde\lambda - \tilde\lambda_n|\,dx}{(\delta/2)^2} + \frac{|\tilde\lambda(\alpha_2) - \tilde\lambda_n(\alpha_2)|}{(\delta/2)}, \end{aligned}$$

and the same estimation for $\int_{-\infty}^{b-\delta/2}$.

In other words the estimation

$$\left| A(z)\sqrt{z^2-4}\log\Delta_n(z) - B_n(z) - \int \frac{d\lambda}{x-z} \right| \le \epsilon \tag{36}$$

holds on the rectangle τ if $n \ge n_0$.

Introduce the holomorphic function $D(z)$ by (16), $z \in \overline{\mathbb{C}} \setminus [-2,2]$, and consider the difference

$$\left| \Delta_n(z) e^{-\frac{B_n(z)}{A(z)\sqrt{z^2-4}}} - D(z) \right| = |D(z)| \left| e^{\frac{A(z)\sqrt{z^2-4}\log\Delta_n(z) - B_n(z) - \int \frac{d\lambda}{x-z}}{A(z)\sqrt{z^2-4}}} - 1 \right|$$

on the contour τ. Due to (36) the difference is uniformly small on the contour and therefore also in the exterior of the rectangle.

Thus we have

$$\zeta^n (p_n P_n(z) - \zeta P_{n-1}(z)) \exp\left(- \frac{B_n(z)}{A(z)\sqrt{z^2-4}} \right) \to D(z) \tag{37}$$

uniformly in the domain $\overline{\mathbb{C}} \setminus [-2,2]$. Let us derive from this an asymptotic for the orthonormal polynomials properly.

First of all due to (33) we have [14]

$$\frac{P_{n-1}(z)}{p_n P_n(z)} \to \zeta$$

uniformly in $\overline{\mathbb{C}} \setminus [-2,2]$. Therefore from (37) we get

$$\zeta^n P_n(z) \exp\left(- \frac{B_n(z)}{A(z)\sqrt{z^2-4}} \right) \to \frac{D(z)}{1-\zeta^2}. \tag{38}$$

Next we will adjust a bit the polynomials B_n in (38).

Let $\tilde{J}(n)$ be $n \times n$ matrix with coefficients p_k, q_k, respectively $\tilde{J}_0(n)$ is n by n matrix that we obtain cutting the Chebyshev matrix J_0. Recall that

$$P_n(z) = \frac{1}{p_1 \dots p_n} \det(z - \tilde{J}(n))$$

in particular

$$\det(z - \tilde{J}_0(n)) = \frac{\zeta^{-n-1} - \zeta^{n+1}}{\zeta^{-1} - \zeta}.$$

That is

$$\frac{1}{p_1 \dots p_n} \frac{\det(z - \tilde{J}(n))}{\det(z - \tilde{J}_0(n))} = (\zeta^{-1} - \zeta) \frac{\zeta^{n+1} P_n(z)}{1 - \zeta^{2n+2}},$$

and hence

$$\begin{aligned} &\log(\zeta^{n+1}\sqrt{z^2-4}P_n(z)) \\ &= -\log(p_1 \dots p_n) - \frac{\operatorname{tr}(\tilde{J}(n) - \tilde{J}_0(n))}{z} - \frac{\operatorname{tr}(\tilde{J}^2(n) - \tilde{J}_0^2(n))}{2z^2} - \dots . \end{aligned}$$

Thus we can substitute $B_n(z)$ by the polynomial $\tilde{B}_n(z)$, which is uniquely defined by

$$\log(\zeta^{n+1}\sqrt{z^2-4}P_n(z)) - \frac{\tilde{B}_n(z)}{A(z)\sqrt{z^2-4}} = \underline{O}\left(\frac{1}{z^{m+2}} \right),$$

since by condition (33) for any fixed k

$$\operatorname{tr}(J^k(n) - J_0^k) - \operatorname{tr}(\tilde{J}^k(n) - \tilde{J}_0^k(n)) \to 0, \ n \to \infty.$$

□

6. Appendix: Simon's conjecture and Laptev–Naboko–Safronov Example

It is more convenient (uniform) to use two sided Jacobi matrices acting in $l^2(\mathbb{Z})$. In particular, then the function $H_A(J)$ is positive [11].

6.1. Positive definite Hankel minus Toeplitz.

Recall that the Chebyshev polynomials of the second kind $U_l(z)$ form an orthogonal system with respect to the weight $\sqrt{4-x^2}$,

$$\frac{1}{\pi}\int_{-2}^{2} U_l(x)U_k(x)\sqrt{4-x^2}\,dx = 2\delta_{k,l}, \tag{39}$$

where

$$U_l(z) := \frac{\zeta^{-l}-\zeta^{l}}{\zeta^{-1}-\zeta}, \quad z = \zeta^{-1}+\zeta. \tag{40}$$

Note also that the following map transforms the polynomials of the second kind into the Chebyshev polynomials of the first kind

$$zU_l(z) - \frac{1}{\pi}\int_{-2}^{2}\frac{U_l(x)-U_l(z)}{x-z}\sqrt{4-x^2}\,dx = T_l(z). \tag{41}$$

Lemma 6.1. *For* $m \neq n$

$$H_{U_mU_n}(J) = \operatorname{tr}\left\{\frac{T_{m+n}}{m+n} - \frac{T_{|m-n|}}{|m-n|}\right\}_{J_0}^{J}, \tag{42}$$

and

$$H_{U_n^2}(J) = \operatorname{tr}\left\{\frac{T_{2n}}{2n} - \sum_i \log p_i^2\right\}_{J_0}^{J} = \operatorname{tr}\left\{\frac{T_n^2}{2n} - \sum_i \log p_i^2\right\}_{J_0}^{J}. \tag{43}$$

Proof. We have

$$\begin{aligned}\Phi'(z) = zU_m(z)U_n(z) - \frac{1}{\pi}\int U_m(x)\frac{U_n(x)-U_n(z)}{x-z}\sqrt{4-x^2}dx\\ -\frac{1}{\pi}\int\frac{U_m(x)-U_m(z)}{x-z}U_n(z)\sqrt{4-x^2}dx.\end{aligned}$$

Using (39), (40), (41) we have for $m > n$

$$\begin{aligned}\Phi'(z) =& zU_m(z)U_n(z) - \frac{1}{\pi}\int\frac{U_m(x)-U_m(z)}{x-z}\sqrt{4-x^2}dx\,U_n(z)\\ =& T_m(z)U_n(z) = U_{m+n}(z) - U_{m-n}(z).\end{aligned}$$

Since $T_k' = kU_k$, $k \geq 1$, we get

$$\Phi(z) = \frac{T_{m+n}(z)}{m+n} - \frac{T_{m-n}(z)}{m-n} + \text{const}.$$

By orthogonality also

$$a = \frac{1}{\pi}\int_{-2}^{2} U_m(x)U_n(x)\sqrt{4-x^2}\,dx = 0.$$

Thus (42) is proved. A proof of (43) requires just a minor modification. □

Proposition 6.2. *Let J be a finite dimensional perturbation of J_0. Define*

$$a_k(J) = \begin{cases} \operatorname{tr}\left\{\frac{T_k}{k}\right\}_{J_0}^{J}, & k \geq 1 \\ \sum_i \log p_i^2 & k = 0 \end{cases} \tag{44}$$

Then the matrix $\{a_{k+l}(J) - a_{|k-l|}(J)\}_{k\geq 1, l\geq 1}$ is positive.

Proof. Put $A = |B|^2$ with $B = \sum_l U_l c_l$. Since $H_A(J) \geq 0$, due to Lemma 6.1, we get

$$\sum_{k\geq 1, l\geq 1} \{a_{k+l}(J) - a_{|k-l|}(J)\} c_k \overline{c_l} \geq 0.$$

$\square$

Note that continuous positive kernels of this kind are a classical object, see e.g. [1].

6.2. **Laptev–Naboko–Safronov example:** $A = U_l^2$**.** This case was considered in [11].

Proposition 6.3. *Let $A(z) = U_l^2(z)$. Then $\Lambda_A(J) < \infty$ if and only if $T_l(J) - T_l(J_0)$ is Hilbert–Schmidt.*

Proof. Due to Lemma 6.1

$$H_A(J) = \operatorname{tr} \frac{T_l^2(J) - T_l^2(J_0)}{2l} - 2\sum \log p_i.$$

Note that a row in the matrix $T_l(J)$ is of the form

$$\langle e_i | T_l(J) = \begin{bmatrix} \dots & 0 & (t_l)_{i-l} & (\tilde{q}_l)_i & (t_l)_i & 0 & \dots \end{bmatrix},$$

where $(t_l)_i = p_{i+1}p_{i+2}...p_{i+l}$ and $(\tilde{q}_l)_i$ is a row–vector of dimension $2l-1$. Therefore

$$H_A(J) = \frac{1}{l}\left\{\sum \frac{(\tilde{q}_l)_i(\tilde{q}_l)_i^*}{2} + \sum((t_l)_i^2 - 1 - \log(t_l)_i^2)\right\}$$

and the condition $H_A(J) < \infty$ is equivalent to $T_l(J) - T_l(J_0)$ is a Hilbert–Schmidt operator. $\square$

It is possible to reformulate the above condition in terms of the coefficient sequences of J. For $n = 2$ Theorem 6.4 was proved in [11]. For $n > 2$ it was proved in [11] under a certain extra assumption, which, as we can see, is superfluous.

Theorem 6.4. *Let $A(z) = U_n^2(z)$. Then $\Lambda_A(J) < \infty$ if and only if*

$$\{\sum_{k=1}^{n} u_{j+k}\} \in l^2, \ \{\sum_{k=1}^{n} q_{j+k}\} \in l^2, \ \{u_j^2\} \in l^2, \ \{q_j^2\} \in l^2, \tag{45}$$

where $u_j = p_j^2 - 1$.

A proof is splitted in several lemmas.

Lemma 6.5. *Let $J = S^{-1}\mathbb{P} + \mathbb{Q} + \mathbb{P}S$ and*

$$T_n(J) = \{... + \Lambda_0(n) + \Lambda_1(n)S + ... + \Lambda_n(n)S^n\},$$

where $\mathbb{Q}, \mathbb{P}, \Lambda_k(n)$ are diagonal matrices. Then

$$\Lambda_n(n) = \mathbb{P}\mathbb{P}^{(-1)}...\mathbb{P}^{(-n+1)} \tag{46}$$

$$\Lambda_{n-1}(n) = \mathbb{P}...\mathbb{P}^{(-n+2)}\{\mathbb{Q} + \mathbb{Q}^{(-1)} + ... + \mathbb{Q}^{(-n+1)}\} \tag{47}$$

and

$$\begin{aligned}\Lambda_{n-2}(n) =&\mathbb{P}...\mathbb{P}^{(-n+3)}\{[(\mathbb{P}^{(1)})^2-I+\mathbb{P}^2-I+...+(\mathbb{P}^{(-n+3)})^2-I]\\ &\begin{array}{lllll} +\mathbb{Q}[\mathbb{Q}+ & \mathbb{Q}^{(-1)}+ & ... & +\mathbb{Q}^{(-n+2)}] & \\ & +\mathbb{Q}^{(-1)}[\mathbb{Q}^{(-1)}+ & ... & +\mathbb{Q}^{(-n+2)}] & \\ & & + & ... & \\ & & & +\mathbb{Q}^{(-n+2)}\mathbb{Q}^{(-n+2)}\}. & \end{array}\end{aligned} \tag{48}$$

Proof. All three formulas can be proved by induction using

$$T_n(J) = JT_{n-1}(J) - T_{n-2}.$$

Let us prove (48). We have

$$\begin{aligned}\Lambda_{n-2}(n) = S^{-1}\mathbb{P}\Lambda_{n-1}(n-1)S+&\mathbb{Q}\Lambda_{n-2}(n-1)+\mathbb{P}S\Lambda_{n-3}(n-1)S^{-1}\\ &-\Lambda_{n-2}(n-2).\end{aligned}$$

Substituting (46) and (47) we get

$$\begin{aligned}\Lambda_{n-2}(n) =&S^{-1}\mathbb{P}\mathbb{P}\mathbb{P}^{(-1)}...\mathbb{P}^{(-n+2)}S\\ &+\mathbb{Q}\mathbb{P}...\mathbb{P}^{(-n+3)}\{\mathbb{Q}+\mathbb{Q}^{(-1)}+...+\mathbb{Q}^{(-n+2)}\}\\ &+\mathbb{P}S\Lambda_{n-3}(n-1)S^{-1}-\mathbb{P}\mathbb{P}^{(-1)}...\mathbb{P}^{(-n+3)}\\ =&\mathbb{P}\mathbb{P}^{(-1)}...\mathbb{P}^{(-n+3)}\{(\mathbb{P}^{(1)})^2-I+\mathbb{Q}[\mathbb{Q}+\mathbb{Q}^{(-1)}+...+\mathbb{Q}^{(-n+2)}]\\ &+\mathbb{P}\Lambda_{n-3}^{(-1)}(n-1)\}.\end{aligned}$$

Iterating the last relation we obtain (48). □

Lemma 6.6. *If* $T_n(J)-T_n(J_0)$ *is Hilbert–Schmidt then relations* (45) *are fulfilled.*

Proof. Since $\Lambda_n(n)-I$, $\Lambda_{n-1}(n)$ and $\Lambda_{n-2}(n)$ are Hilbert–Schmidt operators, using Lemma 6.5, we have

$$\{p_{1+i}...p_{n+i}-1\}\in l^2 \tag{49}$$

$$\{p_{1+i}...p_{n-1+i}(q_i+...+q_{n-1+i})\}\in l^2 \tag{50}$$

and

$$\left\{p_{1+i}...p_{n-2+i}\left[\sum_{k=i}^{i+n-1}(p_k^2-1)+\sum_{k=i}^{i+n-2}q_k^2+\sum_{i\le k<l\le i+n-2}q_kq_l\right]\right\}\in l^2. \tag{51}$$

Having in mind (49) we simplify (50) and (51)

$$\{q_i+...+q_{n-1+i}\}\in l^2$$

and

$$\left\{\sum_{k=i}^{i+n-1}(p_k^2-1)+\frac{1}{2}\sum_{k=i}^{i+n-2}q_k^2+\frac{1}{2}\left(\sum_{k=i}^{i+n-2}q_k\right)^2\right\}\in l^2. \tag{52}$$

Now we wish to separate "p" and "q" conditions in (52). It is evident that $a+b\in l^2$ implies $a\in l^2$ and $b\in l^2$ if only $a_i\ge 0$ and $b_i\ge 0$. Note that (49) implies $\{(p_{1+i}...p_{n+i})^{2/n}-1\}\in l^2$. Thus using this condition and the inequality

$$\frac{p_{1+i}^2+...+p_{n+i}^2-n}{n}\ge (p_{1+i}...p_{n+i})^{2/n}-1$$

we get from (52) $\{q_i^2\} \in l^2$ and $\{\sum_{k=1}^n (p_{i+k}^2 - 1)\} \in l^2$.

Finally we note that

$$\begin{aligned}(p_1-1)^2 + ... + (p_n-1)^2 =& (p_1^2-1) + ... + (p_n^2-1)\\ & -2\{(p_1-1) + ... + (p_n-1)\}.\end{aligned}$$

Since

$$\begin{aligned}2n\{(p_1...p_n)^{1/n} - 1\} \le& 2\{(p_1-1) + ... + (p_n-1)\}\\ \le& (p_1^2-1) + ... + (p_n^2-1)\end{aligned}$$

we have $\{\sum_{k=1}^n (p_{i+k} - 1)\} \in l^2$ and therefore $\{(p_i-1)^2\} \in l^2$.

□

The following lemma can be shown by induction.

Lemma 6.7. *Let* $J = J_0 + dJ$ *then*

$$dT_l(J)e_0 = \sum_{k=0}^{l-1} S^{1-l} S^k [dJ + ... + dJ^{(1-l)}] S^k e_0 = \begin{bmatrix} 0 \\ dp_{-l+1} + ... + dp_0 \\ dq_{-l+1} + ... + dq_0 \\ 2dp_{-l+2} + ... + 2dp_1 \\ dq_{-l+2} + ... + dq_1 \\ 2dp_{-l+3} + ... + 2dp_2 \\ \vdots \\ dp_1 + ... + dp_l \\ 0 \end{bmatrix}. \tag{53}$$

Proof of Theorem 6.4. We only have to show that conditions (45) imply $T(J) - T(J_0)$ is Hilbert–Schmidt. Note that each entry is a polynomial of q_j, u_i with $u_i = p_i - 1$. Moreover, the linear term is described in Lemma 6.7. Note also that the sequences $\{u_i^l q_{i+j}^k\}_i$, $\{u_i^l u_{i+j}^k\}_i$, $\{q_i^l q_{i+j}^k\}_i$ belong to l^2 for $k + l \ge 2$. Thus, having in mind the structure of the matrix $T(J) - T(J_0)$, we get that each diagonal forms an l^2–sequence, as was to be proved. □

6.3. **Simon's conjecture.** Since $H_A(J_0) = 0$ and $H_A(J) \ge 0$ the decomposition of H_A about J_0 begins with a quadratic form, more exactly:

Lemma 6.8. *Let* $J = J_0 + dJ$ *then the decomposition of* H_A *about* J_0 *begins with*

$$H_A(J) = \frac{1}{2}\langle dj|A(J_0)|dj\rangle + \dots \tag{54}$$

where $\langle dj| = \{\dots, 2dp_0, dq_0, 2dp_1, dq_1, \dots\}$.

Proof. We start with the formula

$$dH_A(J) = \operatorname{tr}\{A(J)\operatorname{Re}(Z^{-1} - Z)\, dJ\},$$

where Z is the lower triangle solution of the equation $Z^{-1} + Z = J$. Note that the decomposition of $Z^{-1} - Z$ about J_0 is of the form

$$Z^{-1} - Z = S^{-1} - S + dJ - 2dZ + ...$$

Using

$$dJ = -Z^{-1} dZ Z^{-1} + dZ$$

we get

$$-dZ|_{Z=S} = [ZdJZ + Z(-dZ)Z]_{Z=S} = SdJS + S^2 dJ S^2 + ...$$

Therefore the leading term in the decomposition of $\mathrm{Re}(Z^{-1}-Z)$ is the Hankel operator

$$\Gamma = \ldots + S^{-1}dJS^{-1} + dJ + SdJS + \ldots,$$

and

$$H_A(J) = \frac{1}{2}\mathrm{tr}\{A(J_0)\Gamma\, dJ\} + \ldots\,.$$

Let us mention that $\Gamma e_0 = dj$, thus we can rewrite this Hankel operator into the form

$$\Gamma = \sum S^k |dj\rangle\langle e_0| S^k.$$

Since $A(J_0)$ and S commute and $\Gamma S = S^{-1}\Gamma$ we get

$$\begin{aligned}\mathrm{tr}\{A(J_0)\Gamma\, dJ\} &= \mathrm{tr}\{A(J_0)\Gamma(S^{-1}d\mathbb{P} + d\mathbb{Q} + d\mathbb{P}S)\}\\ &= \mathrm{tr}\{A(J_0)\Gamma(2S^{-1}d\mathbb{P} + d\mathbb{Q})\}.\end{aligned}$$

Substituting Γ we obtain

$$\begin{aligned}\mathrm{tr}\{A(J_0)\Gamma\, dJ\} &= \mathrm{tr}\{A(J_0)(\textstyle\sum S^k|dj\rangle\langle e_0|S^k)(2S^{-1}d\mathbb{P} + d\mathbb{Q})\}\\ &= \mathrm{tr}\{A(J_0)|dj\rangle\langle e_0|\textstyle\sum(2S^{k-1}d\mathbb{P}S^k + S^k d\mathbb{Q}S^k)\}.\end{aligned}$$

But $\langle e_0|\sum(2S^{k-1}d\mathbb{P}S^k + S^k d\mathbb{Q}S^k) = \langle dj|$ and this completes the proof. ☐

We believe that related to this quadratic form condition

$$\langle A(J_0)dj, dj\rangle < \infty, \tag{55}$$

should play an important role in a counterpart of Simon's conjecture formulated for the unit circle in several talks, for example [15]. Specifically, in Laptev–Naboko–Safronov case, where

$$A(J_0) = (I + S^2 + \ldots + S^{2l-2})^*(I + S^2 + \ldots + S^{2l-2}),$$

condition (55) means

$$\begin{aligned}&\{dq_{i+1} + dq_{i+2} + \ldots + dq_{i+l}\} \in l^2(\mathbb{Z}),\\ &\{2dp_{i+1} + 2dp_{i+2} + \ldots + 2dp_{i+l}\} \in l^2(\mathbb{Z}),\end{aligned}$$

compare (45).

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