1.4, 1.5 Problems

Question 1. (a) Write down the equation of the secant line of $f(x) = x^2 + 1$ crossing points $(1, f(1))$ and $(2, f(2))$.

Solution:

$$\text{slope} = \frac{f(2) - f(1)}{2 - 1} = \frac{5 - 2}{2 - 1} = 3$$

Since we have obtained the slope, all lines with this slope have the form $y = 3x + b$ with some b. To obtain b, plug in one point, say $(1, f(1))$ to $y = 3x + b$, we get

$$2 = 3 + b \Rightarrow b = -1$$

Hence the equation of the secant line is $y = 3x - 1$.

(b) Compute the slope of the secant line joining $(1, f(1))$ and $(h, f(h))$ with the same $f(x)$.

Solution:

$$\text{slope} = \frac{f(h) - f(1)}{h - 1} = \frac{h^2 - 1}{h - 1} = \frac{(h + 1)(h - 1)}{h - 1} = h + 1$$

(c) Use (b) to guess the instantaneous rate of change of f at $x = 1$ (hint: the instantaneous rate of change the slope of the tangent line).

Solution: We use P and Q to denote the two points $(1, f(1))$, $(h, f(h))$ in (b). It is easy to see that as $h \to 1$, $Q \to P$.
Since the tangent line at P is the limit of the secant line (joining P and Q) as Q moves towards P, the slope of the tangent line at P is therefore the limit of the slope of the secant line as $h \to 1$, which means

$$\text{slope of the tangent line } = \lim_{h \to 1} (h + 1) = 2$$

Due to the hint, 2 is the instantaneous rate of change.

Question 2. Consider the graph of the function f below. Evaluate the following

(a) $\lim_{x\to 2^-} f(x) = 2$

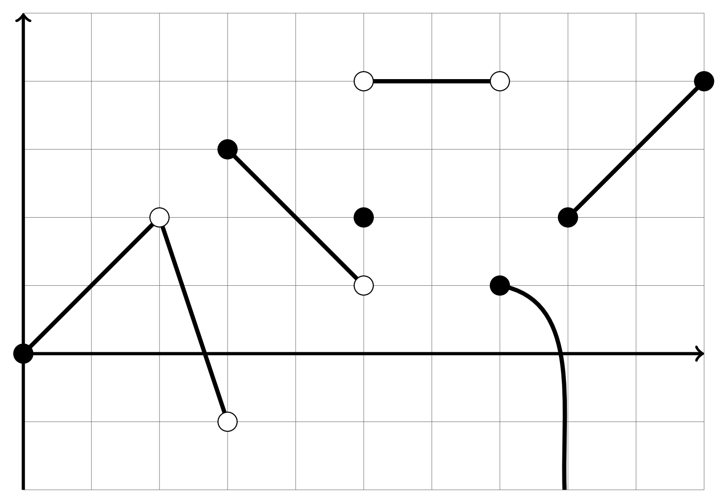
(b) $\lim_{x\to 2^+} f(x) = 2$

(c) $\lim_{x\to 2} f(x) = 2$

(d) $\lim_{x\to 3^-} f(x) = -1$

(e) $\lim_{x\to 3^+} f(x) = 3$

(f) $\lim_{x\to 3} f(x) = D.N.E.$



(g) $\lim_{x\to 5^-} f(x) = 1$

(h) $\lim_{x\to 5^+} f(x) = 4$

(i) $\lim_{x\to 5} f(x) = D.N.E.$

(j) $\lim_{x\to 7^-} f(x) = 4$

(k) $\lim_{x\to 7^+} f(x) = 1$

(l) $\lim_{x\to 7} f(x) = D.N.E.$

(m) $\lim_{x\to 8^-} f(x) = -\infty$

(n) $\lim_{x\to 8^+} f(x) = 2$

(o) $\lim_{x\to 8} f(x) = D.N.E.$

Question 3 (@ home bonus fun)**.** Try to write down the function's equation from the graph in Question 3. (Hint: It should be piecewise defined)

Question 4. Determine the infinite limits. Your final answer should be one of: $\infty, -\infty$, or DNE.

(a) $\displaystyle\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$

(b) $\displaystyle\lim_{x\to 1^+} \frac{-2}{(x-1)^3} = -\infty$

(c) $\displaystyle\lim_{x\to 2^-} \frac{3(x+4)}{x^2+2x-8} = -\infty$

(d) $\displaystyle\lim_{x\to (\pi/10)^+} x^5 \tan(5x) = -\infty$

Question 5. Given the functions below

$$f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1 \\ 2-x & \text{if } x \ge 1 \end{cases} \quad g(x) = \begin{cases} 1+\sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi \\ \sin x & \text{if } x > \pi \end{cases} \quad h(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x\to 1^+} f(x) = 1$

(b) $\lim_{x\to 1} f(x) = 1$

(c) $\lim_{x\to 0^-} g(x) = 1$

(d) $\lim_{x\to 0} g(x) = 1.$

(e) $\lim_{x\to \pi^-} g(x) = -1$

(f) $\lim_{x\to 0} h(x) = D.N.E.$