2.1 Problems

A Table and Some Graphs

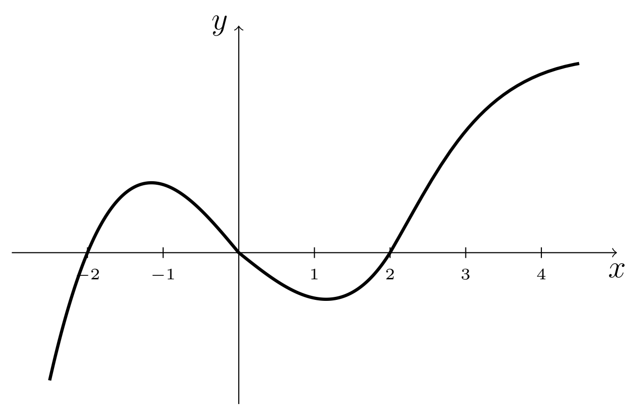
Example 1. Approximate $f'(3)$ using the table to the right:

x	1	2	3	4	5
$f(x)$	5	7	11	4	6

Solution $f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \approx \frac{1}{2}(\frac{f(4)-f(3)}{4-3} + \frac{f(2)-f(3)}{2-3}) = \frac{1}{2}(\frac{4-11}{4-3} + \frac{7-11}{2-3}) = -3/2$

Example 2. For the function f whose graph is given to the right, arrange the following numbers in increasing order and explain your reasoning.

$0 \qquad f'(-2) \qquad f'(0) \qquad f'(2) \qquad f'(4)$



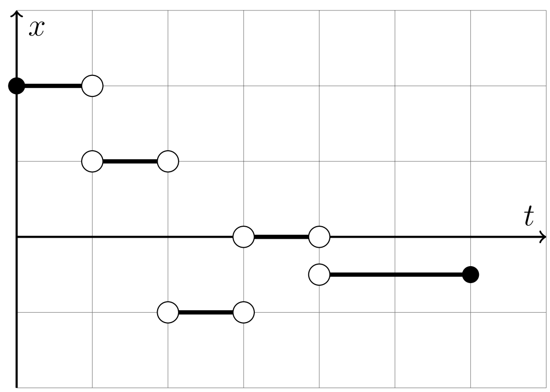
Solution $f'(a) > 0$ mean $f(x)$ is increasing at a, $f'(a) < 0$ means $f(x)$ is decreasing at a. Since $f'(a)$ is the slope of the tangent line, the steeper the tangent line is, the larger the absolute value of $f'(a)$ is. Hence

$$f'(0) < 0 < f'(4) < f'(2) < f'(-2)$$

Example 3. A particle starts by moving to the right along a horizontal line; the graph of its position function is below for $t \in [0, 6]$ seconds.

(a) When is the particle moving to the right? Moving to the left? Standing still?

(b) Draw a graph of the velocity function.



Solution: the particle is moving to the right on $[0, 2)$, to the left on $(2, 3) \cup (4, 6]$, and stand still on $(3, 4)$.

Standard Problems

Example 4. Find $f'(1)$ for $f(x) = 2x^2 - 3x + 5$

Solution: $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \frac{2(1+h)^2-3(1+h)+5-(2-3+5)}{h} = \lim_{h \to 0} \frac{h+2h^2}{h} = \lim_{h \to 0}(1+2h) = 1$

Example 5. Find the equation of the tangent line for $f(t) = \dfrac{2t+1}{t+3}$ at $t = 4$.

Solution: Let us first simplify $f(x)$

$$f(t) = \frac{2t+1}{t+3} = \frac{2t+6-5}{t+3} = \frac{2(t+3)-5}{t+3} = 2 - \frac{5}{t+3}$$

Now we use this simplified f to compute the derivative.

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{2 - \frac{5}{4+h+3} - (2 - \frac{5}{4+3})}{h} = \lim_{h\to 0} \frac{5}{7(7+h)} = \frac{5}{49}$$

The tangent line is

$$y - f(4) = f'(4)(x-4)$$

Evaluate $f(4) = 2 - 5/7 = 9/7$, inserting this and $f'(4) = 5/49$ to the above equation to obtain

$$y - \frac{9}{7} = \frac{5}{49}(x-4)$$

Example 6. Find the equation of the tangent line for $g(x) = \sqrt{5-x}$ at $x = 1$.

Solution: $g'(1) = \lim_{h\to 0} \dfrac{g(1+h)-g(1)}{h} = \lim_{h\to 0} \dfrac{\sqrt{5-(1+h)} - \sqrt{5-1}}{h} = \lim_{h\to 0} \dfrac{\sqrt{4-h}-2}{h}$

$= \lim_{h\to 0} \dfrac{\sqrt{4-h}-2}{h} \cdot \dfrac{\sqrt{4-h}+2}{\sqrt{4-h}+2} = \lim_{h\to 0} \dfrac{4-h-4}{h(\sqrt{4-h}+2)} = \lim_{h\to 0} -\dfrac{1}{\sqrt{4-h}+2} = -1/4.$

And $g(1) = \sqrt{5-1} = 2$. Inserting these into the equation of the tangent line

$$y - g(1) = g'(x)(x-1)$$

we obtain

$$y - 2 = -\frac{1}{4}(x-1).$$

Non-Standard Problems

Example 7. Determine whether $f'(0)$ exists in each case:

(a) $f(x) = \begin{cases} x^2+3x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Solution: By definition $f'(0) = \lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$. We cannot directly evaluate this limit since $f(x)$ has different expressions on the two sides of 0. We need to evaluate the left limit and the right limit separately.

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{0-0}{x-0} = 0$$

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{x^2+3x-0}{x-0} = \lim_{x\to 0+} \frac{x^2+3x}{x} = \lim_{x\to 0+} x+3 = 3$$

Since the limit on the left does not agree with that on the right, the limit $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$ does not exist. Hence $f'(0)$ D.N.E.

(b) $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Solution: By definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x-0} = \lim_{x\to 0} x \sin \frac{1}{x}.$$

Next we use the Squeeze theorem to show that this limit is 0. Multiplying each term in

$$-1 \leq \sin \frac{1}{x} \leq 1$$

by x, if $x \geq 0$, we have

$$-x \leq x \sin \frac{1}{x} \leq x \tag{1}$$

if $x < 0$, we have

$$-x \geq x \sin \frac{1}{x} \geq x \tag{2}$$

Using the squeeze theorem, (1) implies

$$\lim_{x\to 0^+} x \sin \frac{1}{x} = \lim_{x\to 0^+} x = \lim_{x\to 0^+} -x = 0$$

(2) implies

$$\lim_{x\to 0^-} x \sin \frac{1}{x} = \lim_{x\to 0^-} x = \lim_{x\to 0^-} -x = 0$$

Hence

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0$$