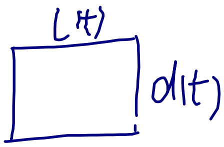
2.8 Problems

Level 1 Problems

Example 1. The length of a rectangle is increasing at a rate of 4 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 5 cm and the width is 6 cm:

(a) how fast is the area of the rectangle increasing?



$\ell' = 4$ $\quad A'(t_0) = ?$

$d' = 3$

$\ell(t_0) = 5$

$d(t_0) = 6$

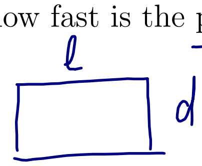
(1) $A = \ell d$

(2) $A' = \ell' d + \ell d'$

(3) $A'(t_0) = \ell'(t_0) d(t_0) + \ell(t_0) d'(t_0)$

$A'(t_0) = 4 \cdot 6 + 5 \cdot 3 = 39 \text{ cm}^2/\text{s}$

(b) how fast is the perimeter of the rectangle increasing?



$\overset{\shortparallel}{P}$

$\ell' = 4$ $\quad P'(t_0)\ ?$

$d' = 3$

$\ell(t_0) = 5$

$d(t_0) = 6$

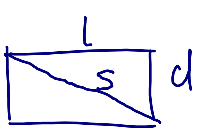
$P = 2\ell + 2d$

$P' = 2\ell' + 2d'$

$P'(t_0) = 2\ell'(t_0) + 2d'(t_0)$

$= 2 \cdot 4 + 2 \cdot 3 = 14 \text{ cm/s}$

(c) how fast is the diagonal of the rectangle increasing?



$s'(t_0)\ ?$

$\ell' = 4$

$d' = 3$

$\ell(t_0) = 5$

$d(t_0) = 6$

(1) $s^2 = d^2 + l^2$

(2) $\cancel{2}s s' = \cancel{2}d d' + \cancel{2}\ell \ell'$

(3) $s(t_0) s'(t_0) = d(t_0) d'(t_0) + \ell(t_0) \ell'(t_0)$

$s(t_0) s'(t_0) = 6 \cdot 3 + 5 \cdot 4$

use (1) to get $s(t_0)$

$s^2(t_0) = d^2(t_0) + \ell^2(t_0)$

$= 5^2 + 6^2$

$\Rightarrow s(t_0) = \sqrt{61}$

$\Rightarrow s'(t_0) = \dfrac{38}{\sqrt{61}} \text{ cm/s}$

Example 2. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$ find dz/dt when $(x, y, z) = (2, 2, 1)$.

(1) $x^2 + y^2 + z^2 = 9$

$\frac{d}{dt}\downarrow \qquad \frac{d}{dt}\downarrow$

(2) $2xx' + 2yy' + 2zz' = 0$

(3) at $t = t_0$

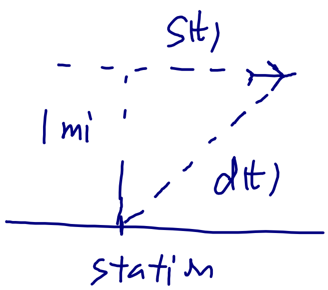
$2x(t_0)\,x'(t_0) + 2y(t_0)\,y'(t_0) + 2z(t_0)\,z'(t_0) = 0$

$\cancel{2}\cdot 2 \cdot 5 + \cancel{2}\cdot 2 \cdot 4 + \cancel{2}\cdot 1\, z'(t_0) = 0$

$\Rightarrow z'(t_0) = -18$

Level 2 Problems

Example 3. A plane flying horizontally at an altitude of 1 mi and a speed of 300 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.



$s'(t) = 300$

$d(t_0) = 2$

$d'(t_0) = ?$

(1) $d^2 = 1 + s^2$

(2) $\cancel{2}dd' = \cancel{2}ss'$

(3) $d(t_0)\,d'(t_0) = s(t_0)\,s'(t_0)$

Use (1) to get $s(t_0)$

$d(t_0)^2 = 1 + s(t_0)^2$

$\Rightarrow s(t_0)^2 = d(t_0)^2 - 1 = 2^2 - 1 = 3$

$\Rightarrow s(t_0) = \sqrt{3}$

$2\, d'(t_0) = \sqrt{3}\cdot 300$

$d'(t_0) = 150\sqrt{3}$ mi/h

Example 4. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the $y-$coordinate is decreasing at a rate of 3 cm/s. How fast is the $x-$coordinate of the point changing at that instant?

$xy = 8$

$(x(t_0), y(t_0)) = (4,2)$

$y'(t_0) = -3$

(1) $xy = 8$

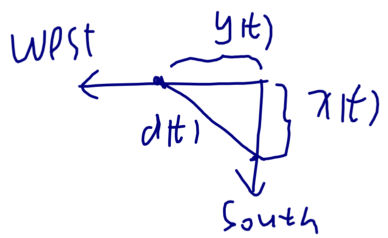
(2) $x'y + y'x = 0$

(3) $x'(t_0)\,y(t_0) + y'(t_0)\,x(t_0) = 0$

$x'(t_0)\cdot 2 + -3\cdot 4 = 0$

$\Rightarrow x'(t_0) = 6$ cm/s

Example 5. Two cars start moving from the same point. One travels south at 20 mi/h and the other travels west at 30 mi/h. At what rate is the distance between the cars increasing two hours later?



$x'(t) = 20$

$y'(t) = 30$

$t_0 = 2$ hours

$d'(t_0) = ?$

(1) $x^2 + y^2 = d^2$

(2) $\cancel{2}xx' + \cancel{2}yy' = \cancel{2}dd'$

(3) $x(t_0)\,x'(t_0) + y(t_0)\,y'(t_0) = d(t_0)\,d'(t_0)$

$40\cdot 20 + 60\cdot 30 = \sqrt{5200}\, d'(t_0)$

Since the cars are travelling at constant speed, the distance they travelled is time × speed so

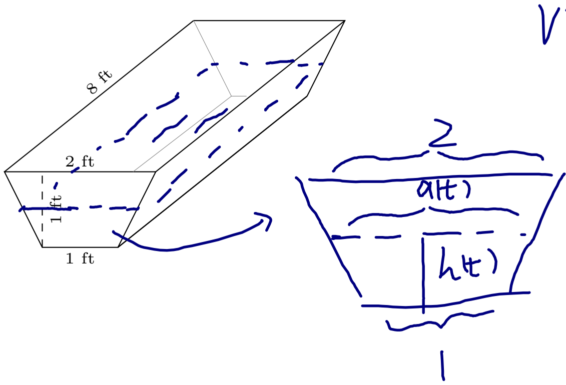
$x(t_0) = \underbrace{x'(t)}_{\text{speed}}\cdot\underbrace{(t_0 - 0)}_{\text{time}} = 20\cdot 2 = 40$

Likewise $y(t_0) = y'(t)\,(t_0 - 0) = 30\cdot 2 = 60$

plug into (1): $d(t_0) = \sqrt{40^2 + 60^2}$

Level 3 Problem

Example 6. A trough is 8 ft long and has a cross section of an isosceles trapezoid with base of 1 ft, height of 1 ft, and top of 2 ft (see the picture below). If the trough is being filled with water at the rate of 3 ft^3/min. how fast is the water level rising when the water is 6 inches deep?



$V'(t) = 3ft \quad h'(t_0) = ? \quad h(t_0) = 6\,in = 0.5\,ft$

Volume of water at t

(1) $V(t)$ = cross-section area $\times$ height

$= \frac{1}{2}(1+a(t))\cdot h(t) \quad \times \quad 8$

height of water at time t

the two base lengths of the cross-section trapezoid at time t

(2) differentiate (1):

$$V'(t) = 4\,a'(t)\cdot h(t) + 4(1+a(t))\,h'(t)$$

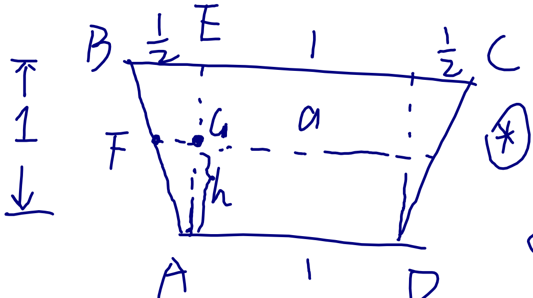
(3) At $t = t_0$:

$$V'(t_0) = 4\,a'(t_0)\,h(t_0) + 4(1+a(t_0))\,h'(t_0)$$

$3 = 4h'(t_0)\frac{1}{2} + 4\cdot 2.5\,h'(t_0)$

$\Rightarrow h'(t_0) = \frac{1}{4}$ ft/min

We need to get $a(t_0)$ and $a'(t_0)$ first. To that end, we need to relate $a(t)$ with $h(t)$ using similar triangle. Look at the graph (*), $\triangle AFG$ and $\triangle ABE$ are similar, so



$$\frac{FG}{BE} = \frac{AG}{AE}$$

$$\Leftrightarrow \frac{\frac{a-1}{2}}{\frac{1}{2}} = \frac{h}{1}$$

$a = 1+h$

$\Downarrow$

$a' = h'$

So $a(t_0) = 1 + h(t_0) = 1.5\,ft$