Review Problems

Sections 1.4 - 2.8

Example 1. The height in feet of an object thrown directly upward after t seconds is given by $h(t) = -16t^2 + 64t + 5$.

(a) Find the average velocity over the time interval [1,2].

$$\frac{h(2) - h(1)}{2 - 1} = \frac{64 - 48}{1} = 16 \text{ ft/s}$$

(b) Find the instantaneous velocity at time t=1.

$$h'(t) = -32t + 64 \quad (1)$$

$$h'(1) = 32 \text{ ft/s}$$

(c) What is the maximum height obtained by the object?

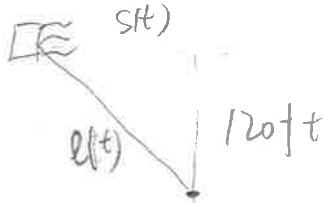
From(1): $h'(t) = 0 \iff t = \frac{64}{32} = 2$

$$h(2) = 69 \text{ ft}$$

Example 2. (a) A girl is flying a kite on a string. The kite is 120 ft. above the ground and the wind is blowing the kite horizontally away from her at 6 ft/sec. At what rate must she let out the string when 130 ft. of the string has been let out?

(b) The radius of a right circular cylinder is increasing at the rate of 4 cm/sec but its total surface area remains constant at 800 cm^2. At what rate is the height changing when the radius is 10 cm?

(a)



$s'(t) = 6$

$l(t_0) = 130$

$s(t_0) = \sqrt{130^2 - 120^2}$

$= 50$

(1) $120^2 + S_{(t)}^2 = l_{(t)}^2$

(2) $0 = 2S \cdot S'_{(t)} - 2l \cdot l'_{(t)}$

At $t = t_0$

$0 = 2S(t_0)\, S'(t_0) - 2l(t_0)\, l'(t_0)$

$= 2 \cdot 50 \cdot 6 - 2 \cdot 130\, l'(t_0)$

$\Rightarrow l'(t_0) = \frac{30}{13}$ ft/s

(b)

(1) $800 = 2\pi r^2 + 2\pi r h \Rightarrow h = \frac{800 - 2\pi r^2}{2\pi r} \Rightarrow h(t_0) = \frac{800 - 200\pi}{20\pi}$

$= \frac{40 - 10\pi}{\pi}$

(2) $0 = 2\pi\, 2r r' + 2\pi(r'h + h'r)$

$r' = 4$

$r(t_0) = 10$

$h'(t_0) = ?$

(3) At $t = t_0$

$0 = \cancel{2\pi} \cdot 2 \cdot 10 \cdot 4 + \cancel{2\pi}(4 \cdot \frac{40 - 10\pi}{\pi} + h' \cdot 10)$

$\Rightarrow h' = -4 - \frac{16}{\pi}$ ft/s

Example 3. Implicit differentiation

(a) Find y' if $x^2 \cos y + \sin 2y = xy$

$$2x\cos y + x^2 \cdot -\sin y \cdot y' + \cos 2y \cdot 2 \cdot y' = y + xy'$$

$$\Leftrightarrow \quad -x^2 \sin y\, y' + 2\cos 2y\, y' - xy' = y - 2x\cos y$$

$$\Leftrightarrow \quad y'(-x^2\sin y + 2\cos 2y - x) = y - 2x\cos y$$

$$\Leftrightarrow \quad y' = \frac{y - 2x\cos y}{-x^2\sin y + 2\cos 2y - x}$$

(b) Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + 4xy + y^2 = 13$ at the point (2,1).

$$2x + 4y + 4xy' + 2yy' = 0$$

$$\Rightarrow y' = -\frac{2x+4y}{4x+2y} = -\frac{x+2y}{2x+y}$$

At (2,1) $\quad y' = -\frac{2+2\cdot 1}{4+1} = -4/5$

$$y - 1 = -\frac{4}{5}(x-2)$$

Example 4. Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{4-x}{3+x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0}\left(\frac{4-(x+h)}{3+x+h} - \frac{4-x}{3+x}\right)\cdot\frac{1}{h}$$

$$= \lim_{h\to 0}\left(\frac{4-(x+h)}{3+x+h}\cdot\frac{3+x}{3+x} - \frac{4-x}{3+x}\cdot\frac{3+x+h}{3+x+h}\right)\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{(4-x)(3+x)} - h(3+x) - \cancel{(4-x)(3+x)} - h(4-x)}{(3+x+h)(3+x)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-7\cancel{h}}{(3+x+h)(3+x)}\,\frac{1}{\cancel{h}} = -\frac{7}{(3+x)^2}$$

Example 5. Use the Intermediate Value Theorem to show that $f(x) = x^4 + x - 1$ has at least one root.

$f(x)$ is a polynomial, so it is continuous on $(-\infty, \infty)$

Note that

$f(-1) = -1 < 0$

$f(1) = 1 > 0$

By IVT, there exist $c \in (-1, 1)$ s.t. $f(c) = 0$

Example 6. True or False. No justification needed.

(a) $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$

False

(b) $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

True

(c) $\dfrac{d}{dx}(\tan^2 x) = \dfrac{d}{dx}(\sec^2 x)$ True