

Generalized Faddeev Equations for N -Particle Scattering

G. W. Wei

Department of Computational Science, National University of Singapore, Singapore 117543

R. F. Snider

Department of Chemistry, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1

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A proposal is made for reducing the solution of the N -particle Lippmann-Schwinger equation to that of smaller sets of particles. This consists of first writing the N -particle equation in terms of all possible $N/2$ -particle Lippmann-Schwinger equations. (If N is odd this needs a minor modification.) The second step requires a decoupling of the resolvents for the fewer particle systems so that each can be solved separately. This generalization of the Faddeev approach deals only with connected kernels and the homogeneous solution reproduces the N -particle Schrödinger equation. For four particles the proposed method involves only a 3×3 matrix whereas other approaches typically require the solution of at least a 7×7 matrix equation.

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I. INTRODUCTION

There has been an increased interest in many-body scattering dynamics due to the rapid growth of computer power [1]. The fundamental method of formal scattering theory is the Lippmann-Schwinger equation [2]. This equation is equivalent to the Schrödinger equation for a two-body problem with an appropriate scattering boundary condition. However, it is well recognized that there are problems when solving the Lippmann-Schwinger equation for more than two particles. In particular for three particles, but the same problem occurs when there are more particles, a perturbation expansion in powers of the potential, or better in terms of pair particle T-matrices, allows a third particle to be a spectator and a dangerous delta function to arise in the momentum representation of the Lippmann-Schwinger equation [3]. Faddeev [4] rearranged the equation so as to eliminate the dangerous delta function and in the process he was able to show that the kernel of the rewritten integral equation was compact, essentially by showing that it became connected after one iteration. Lovelace [5] and later, Alt, Grassberger and Sandhas (AGS) [6] gave two different ways of rewriting the Faddeev equations in terms of scattering amplitudes for various collision processes. The purpose of this work is to propose a way of generalizing the Faddeev equation to an arbitrary number of particles.

There has been a large effort in extending Faddeev's work to an N -particle system, in particular to the four-particle system, both for theoretical interest and for practical use in nuclear and chemical reactions [3,6–21,23,24]. Most work puts the emphasis on deriving equations with a connectable kernel, namely, a kernel that is connected after a finite number of iterations. Glöckle [12] showed that a set of generalized Lippmann-Schwinger equations has a unique solution for the three-particle system. He also formulated similar equations for a four-particle scattering system [25]. Special formulations were given by Weinberg [3] and Rosenberg [9], the former emphasized the quasiparticle picture of an interacting N -particle system, the latter provided a set of equations that do not explicitly depend on the scattering potentials, a feature that may be useful for relativistic scattering [26]. A channel array approach was taken by Baer *et al.* [19], to achieve a connected kernel for a three-body system which was later generalized to an N -particle scattering system by Tobocman [20]. Yakubovsky [13] presented a formal generalization of the Faddeev equations to N -particle scattering by appropriately organizing pair-particle interactions. Yakubovsky's equations [13], like those of Faddeev, have considerable advantage in dealing with the bound state problem since they are equivalent to the Schrödinger equation, thus there being less chance of spurious solutions [27,28]. Unfortunately, the complexity and the lack of a simple method of iterating the Yakubovsky equations hinder their practical usage in physical and chemical problems. In a later treatment, Narodetsky and Yakubovsky [14] proposed an alternative two-cluster approach and obtained a new set of equations for the N -particle scattering. The equivalence of these two-cluster equations to the Schrödinger equation has not been proved. Sloan made an important contribution to solving the four-particle problem by allowing only two-body channels in the coupled equations [15] for scattering amplitudes. Bencze [16] obtained generalized Lovelace equations and generalized AGS equations by resumming the two-cluster equations obtained by Narodetsky and Yakubovsky and showed that these equations are equivalent to those of Sloan when $N = 4$. Redish [17] generalized Sloan's equations to an arbitrary number of particles and demonstrated that his results are identical

to those of Bencze. The Sloan-Bencze-Redish equations are connected after one iteration provided that the kernels of the subpartitions are connected.

These previously presented methods appear to emphasize the pair particle transition operator and expand everything in terms of it. This approach of building up everything from pair particle properties involves combining the pair transition operators in more and more elaborate ways, the details of which may often be very tedious. In fact, there are so many terms that they are very cumbersome to implement. Even the comparison of the numerical calculations using different methods appears to raise questions. What is proposed here might be thought of as an approach of pulling the problem apart into smaller pieces. Thus the N -particle system (if N is even) is divided into all possible combinations of $N/2$ -particle subsystems with the coupling of the subsystems carried out in a manner parallel to the method of Faddeev. This is repeated for each $N/2$ -particle subsystem. Eventually, after successive divisions, each subsystem consists of two particles, whose collisional effects are described by a pair transition operator. (A minor modification is required if N is odd.) This gives a structure as to how the N -particle transition operator is dependent on the transition operator of fewer and fewer particles. The calculation of the N -particle transition operator then involves the reverse procedure of successively putting together the transition operators of fewer particles to eventually get an expression for the full transition operator. In going up the chain of partitionings, each step leads to an integral equation of Faddeev type which is connected after one iteration. In this way a set of equations is obtained in which all particles are connected, but the connectedness is organized in sets within sets. A necessary technicality for carrying out this approach is the need to renormalize the potential at each partitioning. Thus the transition operators for smaller sets of particles involve a potential which is generally smaller in magnitude than the true potential. It is believed that the approach presented here has a different basis of approach than those previously presented, has fewer terms and hopefully leads to a more efficient numerical procedure.

This paper is divided into five sections. Section II is devoted to the partitioning of N -particle systems, with emphasis on partitions having two clusters with an equal number of particles. The notation is similar to that of Redish [17] and Yakubovsky [13]. The distinction between a channel and a partition is stressed. A scheme is developed in Section III to generalize the Faddeev equations to a set of equations for N -particles. The derivation given for the generalized Faddeev equations closely parallels Faddeev's original derivation. This emphasizes the simplicity of the new equations. The generalized Faddeev equations for the resolvent operator and the decomposition of the scattering wave functions are given in Section IV. Section V discusses how the proposed procedure would be applied to both 4- and 8-particle scattering. A general discussion in Section VI ends the paper.

II. PROPERTIES OF EQUAL PARTITIONING

Consider a system of N labeled particles $(1, 2, 3, \dots, N)$. There are many topologically distinct ways of partitioning the labeled N particles into sets of clusters of particles. Each particular way of dividing up the N particles will be called a partition, labeled by A, B, C , etc. The detailed presentation given here partitions the N particles into two clusters of nearly equal size (if N is odd, one cluster is to have one more particle than the other), but alternate partitioning schemes are also possible, as mentioned in the Discussion. All partitionings of the same size are to be considered, which differ by particle composition in each cluster, to give a set of partitions \mathcal{C} . The objective of this section is to define the potentials and hamiltonians appropriate for such partitions. As used in this work, a partition only becomes a (rearrangement) channel when the clusters composing the partition are bound, thus distinguishing between the mathematical method of solving the N -particle problem and the physical notion of what asymptotic states arise in a scattering process.

The N -particle hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ consists of a kinetic energy operator \mathbf{H}_0 and an assumed pairwise additive potential operator

$$\mathbf{V} \equiv \sum_{i < j} \mathbf{V}_{ij}, \quad (1)$$

where the sum is over all possible ordered pairs, with $i, j \in \{1, 2, 3, \dots, N\}$. The total hamiltonian is assumed to be self-adjoint and bounded below, and to have spectra that might include a discrete set, associated with the bound eigenstates, as well as the continuum for the scattering states. The objective is to determine the solutions for the Schrödinger equation associated with the total hamiltonian \mathbf{H} ,

$$\mathbf{H} | \Psi \rangle = E | \Psi \rangle. \quad (2)$$

In the present treatment, the N particles are partitioned into two clusters of equal size if at all possible. That is, if N is even, there are $N/2$ particles in each, whereas if N is odd, one cluster in a partition has $(N + 1)/2$ particles and the other $(N - 1)/2$. It follows that the number of partitions in this set \mathcal{C} of partitions is given by

$$\mathcal{N}_N = \begin{cases} \frac{1}{2} \binom{N}{N/2} & \text{for } N \text{ even;} \\ \binom{N}{(N-1)/2} & \text{for } N \text{ odd.} \end{cases} \quad (3)$$

The partition potentials \mathbf{V}_C are to be chosen so that

$$\sum_{C \in \mathcal{C}} \mathbf{V}_C \equiv \mathbf{V}. \quad (4)$$

Thus the potential \mathbf{V}_C of partition C is defined as

$$\mathbf{V}_C = v_N \sum_{\substack{j < k \\ j, k \in C}} \mathbf{V}_{jk}, \quad (5)$$

involving the sum over those pairs of particles jk that appear in the same cluster in the partition C . Here v_N is a renormalization factor and obviously, $v_3 = 1$. For $N > 3$, v_N is chosen as

$$v_N = \begin{cases} \binom{N}{2} / \left[\binom{N/2}{2} \binom{N}{N/2} \right] = \frac{(N/2)!((N/2)-2)!}{(N-2)!} & N \text{ even;} \\ \binom{N}{2} / \left\{ \binom{N}{(N-1)/2} \left[\binom{(N-1)/2}{2} + \binom{(N+1)/2}{2} \right] \right\} = \frac{((N+1)/2)!((N-3)/2)!}{(N-1)!} & N \text{ odd,} \end{cases} \quad (6)$$

so that the total number of potential terms on both sides of Eq. (4) are equal. The special case of $v_4 = 1$ implies that no renormalization is needed for $N = 4$, whereas $v_5 = 1/4$ and $v_6 = 1/4$ demonstrate how the potential for a particular pair of particles gets distributed among the many partitions in the set \mathcal{C} when N is large. Note that other types of two cluster partitions can also be selected (or included), which however, leads to a larger number of coupled equations for a given N -particle system.

A partition hamiltonian is defined as the kinetic energy operator \mathbf{H}_0 plus the mutual potential interactions between those particles in each of its clusters, namely the partition potential, thus

$$\mathbf{H}_C \equiv \mathbf{H}_0 + \mathbf{V}_C. \quad (7)$$

The residual interaction, \mathbf{V}^C , describing how the clusters in partition C interact with each other, is defined as

$$\mathbf{V}^C \equiv \mathbf{V} - \mathbf{V}_C. \quad (8)$$

Thus, for each partition C the total hamiltonian \mathbf{H} can be written as

$$\mathbf{H} = \mathbf{H}_C + \mathbf{V}^C. \quad (9)$$

The resolvent operator $\mathbf{R}(z)$ for the total hamiltonian is defined as

$$\mathbf{R} = \frac{1}{z - \mathbf{H}}, \quad (10)$$

where z is a complex parameter, for scattering theory having a small positive imaginary part. In most equations in this work the dependence on z will be implicitly assumed. The free resolvent operator is defined as

$$\mathbf{R}_0 = \frac{1}{z - \mathbf{H}_0} \quad (11)$$

while the resolvent operator for partition C is

$$\mathbf{R}_C \equiv \frac{1}{z - \mathbf{H}_C}. \quad (12)$$

The object of this paper is to express the total transition operator \mathbf{T} , as determined by the Lippmann-Schwinger equation

$$\mathbf{T} \equiv \mathbf{V} + \mathbf{V} \mathbf{R}_0 \mathbf{T}, \quad (13)$$

and the Schrödinger equation (2) in terms of the clustering discussed above.

III. GENERALIZED FADDEEV EQUATIONS FOR \mathbf{T}

In this section a two-step scheme is proposed for the decomposition of the total transition operator \mathbf{T} in such a manner that its kernel is connected after one iteration. The first step is to decompose \mathbf{T} according to the selected set of *two-cluster* partitions \mathcal{C} , see Sec. II. Conceptually, this is exactly what Faddeev [4] did for the three-body system. However for a system of more than three particles, the resulting expression for a partition transition operator \mathbf{T}_C is governed by a combination of the transition operators for the two clusters. The second step is to identify how this combination is to be carried out, compare Ref. [15]. Finally it is recognized that the separate cluster transition operators are exactly similar to the original N -particle transition operator but with significantly fewer particles, so the whole procedure can be repeated for each cluster. Care must be exercised when reconstructing the partition transition operators from the separate cluster transition operators so that the energies appearing in the resolvents are consistent with the total energy of the system.

A. The N -Particle Transition Operator

The full N -particle transition operator has the equivalent forms

$$\begin{aligned}\mathbf{T} &= \mathbf{V} + \mathbf{V}\mathbf{R}\mathbf{V} \\ &= \mathbf{V} + \mathbf{T}\mathbf{R}_0\mathbf{V} \\ &= \mathbf{V} + \mathbf{V}\mathbf{R}_0\mathbf{T}.\end{aligned}\tag{14}$$

Moreover, it can be decomposed according to

$$\begin{aligned}\mathbf{T} &= \sum_{C \in \mathcal{C}} \mathbf{V}_C + \sum_{C \in \mathcal{C}} \mathbf{V}_C \mathbf{R}_0 \mathbf{T} \\ &= \sum_{C \in \mathcal{C}} \mathbf{T}^C,\end{aligned}\tag{15}$$

where

$$\mathbf{T}^C \equiv \mathbf{V}_C + \mathbf{V}_C \mathbf{R}_0 \mathbf{T}.\tag{16}$$

Subtracting $\mathbf{V}_C \mathbf{R}_0 \mathbf{T}^C$ from both sides of Eq. (16) gives

$$(\mathbf{1} - \mathbf{V}_C \mathbf{R}_0) \mathbf{T}^C = \mathbf{V}_C + \mathbf{V}_C \mathbf{R}_0 (\mathbf{T} - \mathbf{T}^C),\tag{17}$$

which, on multiplying by $(\mathbf{1} - \mathbf{V}_C \mathbf{R}_0)^{-1}$ gives

$$\begin{aligned}\mathbf{T}^C &= (\mathbf{1} - \mathbf{V}_C \mathbf{R}_0)^{-1} \mathbf{V}_C \\ &\quad + (\mathbf{1} - \mathbf{V}_C \mathbf{R}_0)^{-1} \mathbf{V}_C \mathbf{R}_0 (\mathbf{T} - \mathbf{T}^C) \\ &= \mathbf{T}_C + \sum_{B \in \mathcal{C}} \bar{\delta}_{C,B} \mathbf{T}_C \mathbf{R}_0 \mathbf{T}^B.\end{aligned}\tag{18}$$

Here $\bar{\delta}_{C,B} \equiv 1 - \delta_{C,B}$ and the (2-cluster) partition transition operator \mathbf{T}_C is

$$\begin{aligned}\mathbf{T}_C &\equiv (\mathbf{1} - \mathbf{V}_C \mathbf{R}_0)^{-1} \mathbf{V}_C \\ &= \mathbf{V}_C + \mathbf{V}_C \mathbf{R}_0 \mathbf{T}_C = \mathbf{V}_C + \mathbf{T}_C \mathbf{R}_0 \mathbf{V}_C \\ &= \mathbf{V}_C + \mathbf{V}_C \mathbf{R}_C \mathbf{V}_C.\end{aligned}\tag{19}$$

Therefore the whole N -particle transition operator \mathbf{T} can be split into \mathcal{N}_N Faddeev type transition operators \mathbf{T}^C associated with the selected set \mathcal{C} of two-cluster partitions. It is interesting to note that Eq. (18) has the precise structure of the Faddeev equation but is valid for an arbitrary number of particles and with \mathcal{N}_N components. Just like the original Faddeev equation, the kernel in Eq. (18) is connected after one iteration provided the scattering kernels of the subsystems are already connected.

As in the other treatments [15–17] of the N -particle system, 2-cluster transition operators \mathbf{T}_C may be expressed in terms of unconnected subsystems. Thus it is necessary to ensure that each partition transition operator is connected in order for the total transition operator to have a connected kernel. This can be accomplished by assuring that each cluster transition operator is connected. But first, it is necessary to know how to express the partition transition operators in terms of the corresponding single cluster transition operators. This is done in the next subsection.

B. Cluster transition operators

The partition transition operator \mathbf{T}_C can be calculated from the transition operators for the two clusters C_1 and C_2 that constitute the partition C . Two approaches for carrying out calculations of this nature have been presented in the literature. First is the approach of Sloan [15] for a 4-particle system, see his Sec. III. This is essentially similar to the method of the last subsection, dividing up the partition transition operator into two parts as in Eq. (15) and obtaining coupled equations for the two parts, as in Eq. (18). The second approach is to separate the partition resolvent into cluster resolvents by the use of a convolution. This method is useful since the two clusters are dynamically independent, so all operators for one cluster commute with all operators of the other cluster. This property of the resolvent was pointed out by Bianchi and Favella [22] and emphasized for use in parts of the 4-particle problem by Haberkzettel and Sandhas [24]. Quantities that are used by both methods are defined first, then the methods are discussed in turn.

On the basis that the two clusters in a partition are dynamically independent, the partition potential \mathbf{V}_C is a sum of cluster components

$$\mathbf{V}_C = \mathbf{V}_{C_1} + \mathbf{V}_{C_2}. \quad (20)$$

Since the kinetic part, \mathbf{H}_0 , of the hamiltonian also separates into cluster components, it follows that the partition hamiltonian also separates into commuting cluster hamiltonians. That is, these hamiltonians are related according to ($j = 1, 2$)

$$\mathbf{H}_0 = \mathbf{K}_{C_1} + \mathbf{K}_{C_2}, \quad \mathbf{H}_{C_j} = \mathbf{K}_{C_j} + \mathbf{V}_{C_j}, \quad \mathbf{H}_C = \mathbf{H}_{C_1} + \mathbf{H}_{C_2}. \quad (21)$$

In a similar manner, cluster transition operators can also be defined according to

$$\begin{aligned} \mathbf{T}_{C_j}(z') &\equiv \mathbf{V}_{C_j} + \mathbf{V}_{C_j} \frac{1}{z' - \mathbf{K}_{C_j}} \mathbf{T}_{C_j}(z') \\ &= \mathbf{V}_{C_j} + \mathbf{T}_{C_j}(z') \frac{1}{z' - \mathbf{K}_{C_j}} \mathbf{V}_{C_j} \\ &= \mathbf{V}_{C_j} + \mathbf{V}_{C_j} \frac{1}{z' - \mathbf{H}_{C_j}} \mathbf{V}_{C_j}. \end{aligned} \quad (22)$$

What complex parameter z' is to appear in each cluster transition operator depends on how it is to be used, and differs between the two methods.

In the first method, modelled on Sloan's [15] approach, the partition transition operator is written as a sum,

$$\mathbf{T}_C = \mathbf{T}^{C_1} + \mathbf{T}^{C_2}, \quad (23)$$

whose parts are defined as

$$\mathbf{T}^{C_j} \equiv \mathbf{V}_{C_j} + \mathbf{V}_{C_j} \mathbf{R}_0 \mathbf{T}_C. \quad (24)$$

Then repeating a process analogous to deriving Eq. (18), the components of the partition transition operator can be shown to satisfy the coupled equations

$$\begin{pmatrix} \mathbf{T}^{C_1} \\ \mathbf{T}^{C_2} \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{C_1}(z_1) \\ \mathbf{T}_{C_2}(z_2) \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{T}_{C_1}(z_1) \\ \mathbf{T}_{C_2}(z_2) & 0 \end{pmatrix} \mathbf{R}_0 \begin{pmatrix} \mathbf{T}^{C_1} \\ \mathbf{T}^{C_2} \end{pmatrix}, \quad (25)$$

where the parameters

$$z_1 = z - \mathbf{K}_2, \quad z_2 = z - \mathbf{K}_1 \quad (26)$$

have been chosen so that the energy factors are consistent with the properties of the partition transition operator. According to this, the partition transition operator can be expressed as the series

$$\mathbf{T}_C = \mathbf{T}_{C_1}(z_1) + \mathbf{T}_{C_2}(z_2) + \mathbf{T}_{C_1}(z_1) \mathbf{R}_0 \mathbf{T}_{C_2}(z_2) + \mathbf{T}_{C_2}(z_2) \mathbf{R}_0 \mathbf{T}_{C_1}(z_1) + \dots \quad (27)$$

The cluster transition operators $\mathbf{T}_{C_j}(z_j)$ can be evaluated for arbitrary z_j entirely in the respective cluster subspace involving the states of $\frac{N}{2}$ or $\frac{N-1}{2}$ particles in C_j . This can be accomplished in the same manner as that described for

the N particle system since the cluster is dynamically independent. But it must be evaluated with the appropriate z_j parameter when used in calculating the partition transition operator.

In the second method the basic starting point is to express the partition resolvent as the convolution

$$\mathbf{R}_C = \frac{-1}{2\pi i} \int_{\Gamma} \frac{dz'}{(z' - \mathbf{H}_{C_1})(z - z' - \mathbf{H}_{C_2})} \quad (28)$$

of the resolvents of the two clusters. Here the contour Γ is to be the straight line from $-\infty$ to ∞ , but lying above the real axis and below z . Thus z' must be such that $0 < \Im(z') < \Im(z)$. It follows that the partition transition operator is given by

$$\mathbf{T}_C = \mathbf{V}_{C_1} + \mathbf{V}_{C_2} + \frac{-1}{2\pi i} \int_{\Gamma} (\mathbf{V}_{C_1} + \mathbf{V}_{C_2}) \frac{dz'}{(z' - \mathbf{H}_{C_1})(z - z' - \mathbf{H}_{C_2})} (\mathbf{V}_{C_1} + \mathbf{V}_{C_2}). \quad (29)$$

This can be expressed in many different ways. The following emphasizes how the integrand can be expressed in terms of the cluster transition operators, Eq. (22), but leaves the contour integral unchanged.

The integrand consists of four terms, according to the different cluster potentials. The diagonal in cluster potential terms can immediately be recognized as related to the corresponding cluster transition operator, whereas the remainder needs the identity

$$\mathbf{V}_{C_1} \frac{1}{z' - \mathbf{H}_{C_1}} = \mathbf{T}_{C_1}(z') \frac{1}{z' - \mathbf{K}_{C_1}} \quad (30)$$

and its other various combinations. After some calculation, the integrand can be written in the form

$$\begin{aligned} & (\mathbf{V}_{C_1} + \mathbf{V}_{C_2}) \frac{1}{(z' - \mathbf{H}_{C_1})(z - z' - \mathbf{H}_{C_2})} (\mathbf{V}_{C_1} + \mathbf{V}_{C_2}) \\ &= [\mathbf{T}_{C_1}(z') - \mathbf{V}_{C_1}] \frac{1}{z - z' - \mathbf{H}_{C_2}} + [\mathbf{T}_{C_2}(z - z') - \mathbf{V}_{C_2}] \frac{1}{z' - \mathbf{H}_{C_1}} \\ &+ \mathbf{V}_{C_1} \frac{1}{(z' - \mathbf{H}_{C_1})(z - z' - \mathbf{H}_{C_2})} \mathbf{V}_{C_2} + \frac{1}{z' - \mathbf{H}_{C_1}} \mathbf{V}_{C_1} \mathbf{V}_{C_2} \frac{1}{z - z' - \mathbf{H}_{C_2}} \\ &= [\mathbf{T}_{C_1}(z') - \mathbf{V}_{C_1}] \frac{1}{z - z' - \mathbf{K}_{C_2}} + [\mathbf{T}_{C_2}(z - z') - \mathbf{V}_{C_2}] \frac{1}{z' - \mathbf{K}_{C_1}} \\ &+ [\mathbf{T}_{C_1}(z') - \mathbf{V}_{C_1}] \frac{1}{z - z' - \mathbf{K}_{C_2}} \mathbf{T}_{C_2}(z - z') \frac{1}{z - z' - \mathbf{K}_{C_2}} \\ &+ [\mathbf{T}_{C_2}(z - z') - \mathbf{V}_{C_2}] \frac{1}{z' - \mathbf{K}_{C_1}} \mathbf{T}_{C_1}(z') \frac{1}{z' - \mathbf{K}_{C_1}} \\ &+ \mathbf{T}_{C_1}(z') \frac{1}{(z' - \mathbf{K}_{C_1})(z - z' - \mathbf{K}_{C_2})} \mathbf{T}_{C_2}(z - z') \\ &+ \frac{1}{z' - \mathbf{K}_{C_1}} \mathbf{T}_{C_1}(z') \mathbf{T}_{C_2}(z - z') \frac{1}{z - z' - \mathbf{K}_{C_2}}. \end{aligned} \quad (31)$$

Most terms involve the product of the transition operators for the two clusters, weighted with different combinations of the free particle resolvents for the clusters. It is up to the computational method to decide which way these are to be evaluated. But the first two terms each involves only one cluster transition operator. The contour integral of these two terms can easily be done. For the first term the combination $\mathbf{T}_{C_1}(z') - \mathbf{V}_{C_1}$ is analytic in the upper half z' -plane and vanishes for $|z'| \rightarrow \infty$, so closing the contour at $\Im(z') \rightarrow +\infty$ contributes a pole only when $z' = z - \mathbf{K}_{C_2}$. Analogously for the second term, so that the partition transition operator can be written

$$\mathbf{T}_C = \mathbf{T}_{C_1}(z_1) + \mathbf{T}_{C_2}(z_2) + \frac{-1}{2\pi i} \int_{\Gamma} \{ \mathbf{T}_{C_1} \mathbf{T}_{C_2} \text{ terms} \}. \quad (32)$$

While this is only one way of organizing this result there are many other ways in which the cluster transition operator expansion of the partition transition operator could be written and no attempt is made here to catalog all the possibilities. In comparing the two methods, Eq. (27) is an infinite series in cluster transition operators, while Eq. (32) is only quadratic in cluster transition operators, but requires an integration over how the energy is divided up between the two clusters.

In this way the calculation of the partition transition operator has been reduced to the independent computation of the cluster transition operators. The advantage of this decomposition is that the individual cluster transition operators deal with isolated sets of particles whose number is less than N . Such a cluster of particles can be decomposed into partitions as was the original problem, and the whole process repeated. Specifically, for $N = 2^n$ even, the successive problems deal with 2^k particles, $k = n - 1, n - 2, \dots, 1$.

IV. GENERALIZED FADDEEV COMPONENTS

As a function of z , the resolvent operator \mathbf{R} has singularities at the spectrum of the system hamiltonian, thus it is of practical use for finding solutions of the Schrödinger equation [29], for determining the dynamical evolution of the system [30–32], and identifying normalizable resonance states [33]. Faddeev’s approach [4] is followed in order to express the N -particle resolvent operator in terms of selected two-cluster partition transition operators. This representation of the total resolvent operator is then used to decompose a scattering wave function originating from a particular incoming two-cluster partition. The resulting wave functions are here referred to as generalized Faddeev components. The homogeneous system of equations associated with the generalized Faddeev components is shown explicitly to solve the Schrödinger equation. This section first describes the generalized Faddeev equations for the N -particle resolvent operator and subsequently discusses the associated wave functions.

The total resolvent operator \mathbf{R} is related to the total transition operator \mathbf{T} via the Lippmann-Schwinger equation

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}_0 \mathbf{V} \mathbf{R} = \mathbf{R}_0 + \mathbf{R}_0 \mathbf{T} \mathbf{R}_0. \quad (33)$$

On taking over the partition expansion (15) of \mathbf{T} , this can be written as

$$\mathbf{R} = \mathbf{R}_0 + \sum_{C \in \mathcal{C}} \mathbf{R}^C, \quad (34)$$

where the Faddeev type resolvent \mathbf{R}^C is given by

$$\mathbf{R}^C \equiv \mathbf{R}_0 \mathbf{T}^C \mathbf{R}_0. \quad (35)$$

The generalized Faddeev equations (18) lead to the coupled set of equations

$$\begin{aligned} \mathbf{R}^C &= \mathbf{R}_0 \mathbf{T}_C \mathbf{R}_0 + \sum_{B \in \mathcal{C}} \bar{\delta}_{C,B} \mathbf{R}_0 \mathbf{T}_C \mathbf{R}_0 \mathbf{T}^B \mathbf{R}_0 \\ &= \mathbf{R}_C - \mathbf{R}_0 + \sum_{B \in \mathcal{C}} \bar{\delta}_{C,B} \mathbf{R}_0 \mathbf{T}_C \mathbf{R}^B \end{aligned} \quad (36)$$

with partition resolvent operator defined in (12) and related to the partition transition operator by

$$\mathbf{R}_C = \mathbf{R}_0 + \mathbf{R}_0 \mathbf{V}_C \mathbf{R}_C = \mathbf{R}_0 + \mathbf{R}_0 \mathbf{T}_C \mathbf{R}_0. \quad (37)$$

For $N = 3$, Eq. (36) is Faddeev’s decomposition of the total resolvent operator whose kernel is not connected until after one iteration. For $N > 3$, the kernel of Eq. (36) requires one iteration in the same manner providing the kernels of the subsystems are already connected. These relations between the transition operators and the resolvent operators are analogous to the original Faddeev equations for the three-particle resolvent operator. The set of resolvent equations can also be used for the evaluation of statistical mechanical virial coefficients [33,34].

A scattering wave function for the N -particle system is determined by the total resolvent operator \mathbf{R} according to

$$|\Psi_C\rangle = \lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}(E + i\varepsilon) |\phi_C\rangle. \quad (38)$$

This is applied here to the selected set \mathcal{C} of compatible two-cluster partitions, with $|\phi_C\rangle$ a stationary solution of the two-cluster partition hamiltonian \mathbf{H}_C of energy E ,

$$\mathbf{H}_C |\phi_C\rangle = E |\phi_C\rangle, \quad (39)$$

which may be a distorted wave of the two-cluster subsystem.

The detailed structure of the scattering wave function $|\Psi_C\rangle$ of Eq. (38) is now discussed. This is begun by first applying the resolvent expansions (34) and (36) to Eq. (38),

$$|\Psi_C\rangle = \lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}_0(E + i\varepsilon) |\phi_C\rangle + \sum_{B \in \mathcal{C}} \lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}^B(E + i\varepsilon) |\phi_C\rangle. \quad (40)$$

The first term on the right hand side contributes only if C is the channel with all particles free, namely

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}_0(E + i\varepsilon) |\phi_C\rangle = |\phi_C\rangle \delta_{C, C_N}, \quad (41)$$

where C_N is the N -cluster partition. The second term can be written as a sum of generalized Faddeev components

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}^B(E + i\varepsilon) |\phi_C\rangle \equiv |\psi_{BC}\rangle. \quad (42)$$

According to Eq. (36), the generalized Faddeev component $|\psi_{BC}\rangle$ can be expanded as

$$\begin{aligned} |\psi_{BC}\rangle &= \lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}_B |\phi_C\rangle - \lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}_0 |\phi_C\rangle \\ &\quad + \sum_{A \in \mathcal{C}} \bar{\delta}_{A,B} \mathbf{R}_0 \mathbf{T}_B \lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}^A |\phi_C\rangle \\ &= |\phi_C\rangle \delta_{B,C} - |\phi_C\rangle \delta_{C_N,C} + \sum_{A \in \mathcal{C}} \bar{\delta}_{A,B} \mathbf{R}_0 \mathbf{T}_B |\psi_{AC}\rangle \\ &= |\phi_C\rangle \delta_{B,C} - |\phi_C\rangle \delta_{C_N,C} + \sum_{A \in \mathcal{C}} \bar{\delta}_{A,B} \mathbf{R}_B \mathbf{V}_B |\psi_{AC}\rangle. \end{aligned} \quad (43)$$

Here the identities

$$\lim_{\varepsilon \rightarrow 0} i\varepsilon \mathbf{R}_B |\phi_C\rangle = |\phi_C\rangle \delta_{B,C} \quad (44)$$

and

$$\mathbf{R}_0 \mathbf{T}_B = \mathbf{R}_B \mathbf{V}_B \quad (45)$$

have been used as well as the definition in Eq. (42). Thus the scattering wave function, Eq. (40), is expressed in terms of the generalized Faddeev components (43),

$$|\Psi_C\rangle = |\phi_C\rangle \delta_{C, C_N} + \sum_{B \in \mathcal{C}} |\psi_{BC}\rangle. \quad (46)$$

For $N=3$ this expression for the scattering state is exactly the wave function originally derived by Faddeev. Hence Eq. (46) is the generalization of Faddeev's scattering state based on a chosen set \mathcal{C} of two-cluster partitions of N particles.

It is easily shown that Eq. (46) satisfies the Schrödinger equation (2). A special case is the homogeneous analog. The proof that this formally satisfies the Schrödinger equation is as follows:

$$\begin{aligned} |\Psi_C\rangle &= \sum_{B \in \mathcal{C}} |\psi_{BC}\rangle \\ &= \sum_{B \in \mathcal{C}} \left(\sum_{A \in \mathcal{C}} \bar{\delta}_{A,B} \mathbf{R}_B \mathbf{V}_B |\psi_{AC}\rangle \right) \\ &= \sum_{B \in \mathcal{C}} \left(\sum_{A \in \mathcal{C}} \bar{\delta}_{A,B} \mathbf{R}_0 \mathbf{V}_B [\mathbf{1} + \mathbf{R}_B \mathbf{V}_B] |\psi_{AC}\rangle \right) \\ &= \sum_{B \in \mathcal{C}} \mathbf{R}_0 \mathbf{V}_B \left(\sum_{A \in \mathcal{C}} \bar{\delta}_{A,B} |\psi_{AC}\rangle + |\psi_{BC}\rangle \right) \\ &= \sum_{B \in \mathcal{C}} \mathbf{R}_0 \mathbf{V}_B \sum_{A \in \mathcal{C}} |\psi_{AC}\rangle \\ &= \sum_{B \in \mathcal{C}} \mathbf{R}_0 \mathbf{V}_B |\Psi_C\rangle \\ &= \mathbf{R}_0 \mathbf{V} |\Psi_C\rangle. \end{aligned} \quad (47)$$

Of course a homogeneous solution of the Schrödinger equation occurs only for bound states energies, so such a solution plays no role when solving a scattering problem.

In general, a cluster can be either a bound, or an unbound but interacting, set of particles, with a free particle included as a (1-particle) bound state. If both clusters in a partition are bound then the partition is a true asymptotic channel. Whether this is or is not the case, the partition wave function has one of two forms, depending on whether one of the clusters is or is not a free particle, namely

$$|\phi_C\rangle = \begin{cases} |\varphi_{C^1}, \mathbf{q}_C\rangle \\ |\varphi_{C^1}, \varphi_{C^2}, \mathbf{q}_C\rangle. \end{cases} \quad (49)$$

Here $|\varphi_{C^1}\rangle$ is a bound state of the $N - 1$ -particle subsystem and $|\mathbf{q}_C\rangle$ is the momentum generalized eigenstate for the relative motion of the bound and free states, on the basis that the total center of mass momentum of the N -particle system has been removed from discussion. In the second form, the 2nd cluster is now also a bound state, which must be explicitly indicated, while \mathbf{q}_C is again the relative momentum. If a cluster is unbound, but all particles interacting, then the corresponding cluster wavefunction must be replaced by $\varphi_{C^j}^+$, corresponding to the scattering wavefunction from some initial incoming state. It appears intuitively reasonable that this procedure is correct, though its complete mathematical justification may be needed since some limits must already have been taken in order to define (in general, a product of) scattering states with definite energy as an input, with the next step involving a further limit.

V. THE 4- AND 8-PARTICLE SYSTEMS

The treatment so far has emphasized how the N -particle scattering problem can be broken down into a number of problems involving fewer particles. Once a breakdown has been selected, it is then a case of building up the transition operator and scattering wavefunction for the N -particle system. The procedure is illustrated by discussing systems of 4 and 8 particles. For definiteness it is assumed that there are no bound states for any number of particles.

For the specific case of there being four particles, these will be labelled 1, 2, 3 and 4. Then according to Eq. (3) there are 3 two-cluster partitions, which are the three pairs of pair particles 12,34, 13,24 and 14,23. Moreover, Eq. (6) states that no scaling of the potential is required. Thus the procedure is to first find the two-particle transition operator $\mathbf{T}_{12}(z)$ for arbitrary complex z in the upper half plane. On the basis that all particles are the same species, this transition operator is the same for any other pair, except for the labelling. The second step is then to find the partition transition operator for the two-cluster partition 12,34. This is accomplished according to Sec. IIIB, specifically involving either the first approach, Eq. (27), or the second approach, Eq. (29), with its integrand expressed in terms of the pair particle transition operators, Eq. (31). Again this calculation needs to be done only once since a relabelling immediately gives the partition transition operator for the other two partitions. The last step for getting the four-particle transition operator is to solve the 3×3 matrix equations (18) and add the results, Eq. (15). Once the transition operator is known, the various generalized Faddeev components can be calculated, as described in Sec. IV, and the desired scattering amplitude calculated. In this way the four-particle scattering problem is very similar to the three-particle scattering problem, involving only three partitions. The extra complexity is in the added structure of the partition transition operators. In contrast, the method of Sloan [15] involves a 7×7 matrix while the Yakubovsky [13] approach organizes the wavefunction into 18 components. The approach presented here would seem to be both simpler and more efficient.

The 8-particle problem is discussed with the view of clarifying how the proposed approach works for more complicated systems. Equation (3) states that there are 35 two-cluster partitions, each cluster having 4 particles. Moreover, Eq. (6) states that $v_8 = 1/15$, so the potential needs to be scaled by this fraction. This scaling is required when finally solving the 35×35 matrix, Eq. (18), for the components of the 8-particle transition operator, so enters into the calculation of all (the 8-particle, partition, cluster and subpartition and subcluster) transition operators. Now a typical 8-particle partition transition operator is needed, which is a combination of two 4-particle cluster transition operators. The latter must be found by solving the problem discussed in the previous paragraph, but now with the potential scaled by v_8 . Since there is no further scaling required in solving for the 4-particle transition operator, it is only the v_8 scaling that must be applied. Thus the detailed procedure that is to be followed is, in sequence: 1) scale the pair potential by v_8 ; 2) calculate the 4-particle transition operator for this scaled potential as described in the last paragraph; 3) use either of Eqs. (27) or (29) to obtain a two-cluster partition transition operator for the 8-particle system, with the second cluster transition operator obtained from the first 4-particle cluster transition operator by relabelling; 4) obtain the 34 other partition transition operators by relabelling and solve the generalized Faddeev equations (18).

VI. DISCUSSION

A proposal has been made for the generalization of the Faddeev equations [4] to an arbitrary number, N , of particles. Essentially this involves selecting a set of partitionings of the N particles into pairs of clusters and expressing the scattering properties of the total system in terms of the scattering properties of the clusters. Each cluster is treated in the same way so that after successive treatments an individual cluster contains either only a single particle or a pair of particles. Although mathematically, other (combinations of) sets of two-cluster partitions lead to a connected kernel as well in the present method, the detailed presentation of this paper has stressed the choice of the set of two-cluster partitions in which the pair of clusters in any partition is of nearly equal size. This should be the most efficient procedure since then there are fewer types of clusters that need to be treated.

The proposed method requires, in general, a scaling of the potential and expressing the transition operator, and wave function, in terms of partition transition operators which are in turn expressed in terms of the transition operators for the two individual clusters composing the partition. An essential simplifying feature is that the transition operator for an individual cluster can be calculated independently of the presence of the other particles, so is equivalent to solving the scattering of a system of fewer particles, which can in general be treated in the same manner as the original system.

For most N -particle integral equation theories, the starting point is the pair particle interaction or the two-particle transition operator, which is then combined in all possible ways with the transition operators of other pairs of particles. In order to assure that all particles are connected, that is, no disconnected diagrams appear, it is necessary to iterate through a whole sequence of processes. This complicates the description of the N -particle problem as it appears, for example, in the Yakubovsky [13] equations. In contrast, the present method starts by dividing the N -particle system into a selected set of two-cluster partitions. The treatment assures that all clusters are connected to each other. There remains the possibility that there is a disconnectionness within a cluster. This is eliminated by repeating the procedure for each cluster as if it were a separate scattering system. In this way the formulation has a very simple structure which is exactly similar at each stage to the Faddeev equations, in particular reducing to them if $N = 3$.

The total resolvent for the N -particle system is decomposed in a manner analogous to Faddeev's method. This decomposition determines the decomposition of the scattering wave functions, viz. Eq. (46), into what are here called generalized Faddeev components. The sum of the corresponding homogeneous set of generalized Faddeev components is explicitly shown to be completely equivalent to the Schrödinger equation. Such a property was explicitly demonstrated in Faddeev's theory [25] and Yakubovsky's theory [13] but has not been explicitly shown in some other theories. This property is of particular importance for finding the (discrete) eigenvalues for the N -particle system and for avoiding spurious solutions (for bosons and fermions, an appropriate symmetrization of the Faddeev components may be required). In particular, Federbush [35] first found a spurious solution for the Weinberg equation [3] and a systematic study by Glöckle and coworkers [27,28] indicated that most few-body equations admit the existence of discrete spurious solutions. However these spurious solutions were not a problem in finding scattering solutions.

The cluster and partition wavefunctions from all the stages are needed before the total wavefunction can be obtained. In general these are nonphysical in that they represent intermediary results for the final calculation. The same is true even more so for the Faddeev components. In contrast, if both clusters in a partition represent bound states, then this is a valid asymptotic condition and the channel wavefunction does represent a physical state for the corresponding scattering system.

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