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Local adaptive differential quadrature for free vibration analysis of cylindrical shells with various boundary conditions

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Abstract

This paper presents the formulation and numerical analysis of circular cylindrical shells by the local adaptive differential quadrature method (LaDQM), which employs both localized interpolating basis functions and exterior grid points for boundary treatments. The governing equations of motion are formulated using the Goldenveizer–Novozhilov shell theory. Appropriate management of exterior grid points is presented to couple the discretized boundary conditions with the governing differential equations instead of using the interior points. The use of compactly supported interpolating basis functions leads to banded and well-conditioned matrices, and thus, enables large-scale computations. The treatment of boundary conditions with exterior grid points avoids spurious eigenvalues. Detailed formulations are presented for the treatment of various shell boundary conditions. Convergence and comparison studies against existing solutions in the literature are carried out to examine the efficiency and reliability of the present approach. It is found that accurate natural frequencies can be obtained by using a small number of grid points with exterior points to accommodate the boundary conditions.

Keywords: Local adaptive differential quadrature method; Collocation; Free vibration; Cylindrical shell

1. Introduction

A vast variety of structures or structural components used in civil, mechanical, aerospace and defence engineering can be mathematically modelled as circular cylindrical shells. These shells are often used to store and transport high-pressure gases and liquids for various hydraulic applications. A good understanding of their mechanical behaviour, including vibration, bending and impulse response, is a must for the successful design and application of cylindrical shell structures in engineering practice.

Vibration analysis of cylindrical shells has been studied by many researchers. Leissa gave an excellent collection of early research results on the vibration of shells [1]. In general, methods of vibration analysis can be classified as analytical methods [2–4] and numerical approaches. It is generally agreed that the procedure for analytical/closed-form solutions of such vibration problems is tedious and in most cases, no such solutions might exist at all. This promotes the popularity of the numerical methods in solving this type of problems. One class of numerical approaches is differential quadrature methods (DQMs), which was first introduced by Bellman et al. [5] in 1972 for solving partial differential equations. In fact, the DQM can be formulated via a more general mathematical framework, the method of weighted residual (MWR) [6,7], which advocates the use of polynomial approximations for differential equations. Galerkin, collocation, and subdomain are three typical schemes used in the MWR to determine unknown expansion coefficients for a given differential equation, resulting in a set of desired algebraic

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equations [6,7]. In this sense, the DQM is a collocation scheme, in which the weighted residual is set to zero, or stated differently, the polynomial expansion of the original differential equation is forced to be zero, at a set of collocation points. Consequently, the solution of the differential equation at a given grid point is approximated by a linear combination of function values at a set of collocation points. Each term contributing to the sum is associated with an appropriate weight coefficient, determined by the original equation, the boundary condition, the polynomials used and the collocation points chosen. For a given problem, the performance of the DQM essentially lies on how to choose the polynomials and the collocation points, and how to treat the boundary conditions. Bellman et al. [5] suggested the use of classic polynomials, including the Legendre polynomials, for which a natural choice of collocation points is the set of zeros of the highest degree polynomial used. Most early applications [8–10] of the DQM in engineering usually adopted this approach. However, classic polynomial-based collocation approaches often suffer from severely ill-conditioned matrices, with condition numbers scaling as $O(N^4)$, where N is the number of grid points. Therefore, the number of grid points has to be restricted to less than or equal to 13 in many cases [11]. Fortunately, this problem can be alleviated with appropriate choice of test functions. Shu and Richard [11] presented a generalized differential quadrature (GDQ) method, which utilizes standard Lagrange interpolating functions, which are much more flexible in choosing the collocation points. The resulting GDQ can take appropriate grid distribution to enhance the convergence near a boundary. The GDQ has been successfully applied to analyse fluid flow problems [12,13] and structural problems [14,15].

Apart from Lagrange interpolating functions, other interpolating test functions, such as trigonometric functions, can also be used. The use of trigonometric basis functions leads to a DQM that is exactly the commonly used Fourier pseudospectral method. The resulting Fourier pseudospectral DQM can be employed in both periodic and non-periodic problems, as demonstrated by Shu and Chew [16] and Shu and Xue [17].

Recently, the discrete singular convolution (DSC) algorithm [18–20] has been proposed as a potential numerical approach for numerically handling singular integration. The DSC algorithm is a wavelet collocation scheme that utilizes localized interpolating wavelets or frames as test functions. In particular, these interpolating functions are constructed as approximations to the delta distribution. Consequently, DSC interpolating kernels satisfy the wavelet refinement relation and unify the Galerkin scheme and the collocation scheme [19]. The DSC approach exhibits global methods' accuracy and local methods' flexibility in handling complex geometry and boundary conditions [18,19]. In a general sense, the DSC collocation can also be regarded as a differential quadrature. Nevertheless, the DSC algorithm differs from the classic DQM not only in terms of its test functions, but also in terms of its treatment of boundary conditions. Specifically, the DSC algorithm uses compactly support wavelet interpolating functions which leads to banded differential matrices with well-behaved condition numbers and enable us to construct very large matrices for large-scale computations. Moreover, the DSC utilizes exterior grid points for treating boundary conditions, which results in symmetric or anti-symmetric approximations of derivatives, and is essentially free of spurious eigenvalues. The latter devastate many numerical methods. Early applications of the DSC to structural analysis concern mostly problems with simply supported and clamped boundary edge conditions [20]. More recently, Zhao et al. [21] have proposed an iteratively matched boundary (IMB) method to extend the application of the DSC algorithm to the vibration analysis of beams with free boundary conditions. The IMB method repeatedly uses the given set of boundary conditions to constitute a relationship involving a sufficiently large number of exterior grid points and thus, the DSC kernel can be properly implemented near free edges. Results obtained by this method are in excellent agreement with the exact solutions. With the aid of the IMB technique, the DSC algorithm can be applied to all kinds of problems in structural analysis, in principle.

The DSC algorithm has two distinct features—the use of compactly supported interpolating functions and the use of exterior grid points. The former leads to a banded matrix structure, while the latter avoids spurious eigenvalues. Naturally, these ideas can also be utilized to modify the GDQ approach. In particular, the use of compactly supported Lagrange interpolating polynomials in the collocation scheme has already been speculated by Villadsen and Michelsen, including the derivation of weighting coefficients [7]. Indeed, Wang et al. [22] succeeded in constructing a local adaptive differential quadrature method (LaDQM), which utilizes localized Lagrange interpolating polynomials and adapts the DSC technique of exterior grid points for treating boundary conditions. The resulting LaDQM was employed for solving very high-order differential equations with multiple boundary conditions with considerable success. The concept of local differential quadrature was introduced in the book by Shu [23] for large-scale problems. The LaDQM method is the high order FD scheme discussed by Shu [23].

The primary objective of this paper is to explore the application of the LaDQM to the vibration analysis of cylindrical shells subject to different boundary conditions. A preliminary study on this problem was reported at a conference [24]. The Goldenveizer–Novozhilov shell theory is employed in this study. The convergence and accuracy of the numerical solutions for the vibration analysis of cylindrical shells are examined through a direct comparison of the LaDQM results with corresponding exact solutions obtained by Xiang et al. [25] using the state-space technique.

2. Formulations

2.1. Local adaptive differential quadrature method

In the present study, we shall adopt the LaDQM that was proposed by Wang et al. [22]. First, it should be noted that the DQM approximates the n-order spatial derivative of a sufficiently smooth function with respect to a spatial coordinate at a given grid point as a weighted linear sum of all the functional values at all grid points in the whole domain of the spatial coordinate. The mathematical expression of this basic theorem can be expressed as

$$f_x^{(n)}(x_i) = \sum_{j=1}^{N} c_{i,j}^{n} f(x_j), \quad \text{for } i = 1, 2, \ldots, N; n = 1, 2, \ldots N-1, \tag{1}$$

where N is the total number of grid points in the x direction, x_i is the coordinate of the ith grid point, and $c_{i,j}^{n}$ is the weighting coefficient for the nth derivative approximation of the ith grid point. For generality, the global Lagrange interpolation polynomial is used as the basis polynomial in GDQ [7,11,12]:

$$r_j(x) = \frac{M(x)}{(x - x_j)M^{(1)}(x_j)}, \quad j = 1, 2, \ldots, N-1, N, \tag{2}$$

where

$$M(x) = \prod_{k=1}^{N} (x - x_k), \tag{3}$$

$$M^{(1)}(x_j) = \prod_{k=1, k\neq j}^{N} (x_j - x_k), \tag{4}$$

where $x_1, x_2, \ldots, x_N$ are the coordinates of the grid points whose locations may be chosen arbitrarily. $M^1(x)$ is the first derivative of $M(x)$. The weighting coefficients for the first-order derivative $c_{i,j}^{1}$ [11,12] can be obtained analytically by substituting Eq. (2) into Eq (1).

$$c_{i,j}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \quad \text{for} \quad i,j = 1, 2, \ldots, N, j \neq i, \tag{5}$$

$$c_{i,j}^{(1)} = \sum_{j=1, j\neq 1}^{N} c_{i,j}^{1}, \quad \text{for} \quad \text{i, j} = 1, 2, \ldots, N. \tag{6}$$

The weighting coefficients for second- and higher-order derivatives can be obtained in a similar manner. Following the same procedure as in the discretization of the first-order derivative, a recurrence relationship can be developed to calculate the weighting coefficients for higher-order derivatives [11,12]

$$c_{i,j}^{(m)} = \begin{cases} m\left(c_{i,j}^{(1)} c_{i,i}^{(m-1)} - \frac{c_{i,j}^{(m-1)}}{x_i - x_j}\right) \\ \quad i,j = 1, 2, ..., N, j \neq i, m = 2, 3, \ldots, N-1, \\ -\sum_{j=1, j\neq i}^{N} c_{i,j}^{m} \\ \quad i = 1, 2, ..., N, j = i, m = 2, 3, ..., N-1. \end{cases} \tag{7}$$

It is clear that the weighting coefficients of higher-order derivatives can be calculated from those of the first-order derivative using Eq. (7).

It is possible to use part of the grid points to achieve the approximation rather than all the grid points [7,22]. The use of part grid points can lead to banded matrix, which can significantly improve the efficiency of the method, particularly in large-scale computations. Therefore, the weighting coefficients for the first- and higher-order derivatives can be rewritten as [22]

$$f_x^{(n)}(x_i) = \sum_{j=S_1}^{S_2} c_{i,j}^{n} f(x_j) \quad \text{for } i = 1, 2, \ldots, N; n = 1, 2, \ldots, N-1, \tag{8}$$

$$r_{S_1,S_2,j}(x) = \begin{cases} \frac{M_{S_1,S_2}(x)}{(x - x_j)M_{S_1,S_2}^{(1)}(x_j)}, & S_1 \leqslant j \leqslant S_2, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

$$M_{S_1,S_2}(x) = \begin{cases} \prod_{k=S_1}^{S_2} (x - x_k), & x_{S1} \leqslant x \leqslant x_{S2}, \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

$$M_{S_1,S_2}^{(1)}(x_j) = \begin{cases} \prod_{k=S_1, k\neq j}^{S_2} (x_j - x_k), & S_1 \leqslant j \leqslant S_2 \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

$$c_{i,j}^{(1)} = \begin{cases} \frac{M_{S_1,S_2}^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} & S \leqslant j \leqslant S_2, j \neq i, \\ -\sum_{j=S_1, j\neq i}^{S_2} c_{i,j}^{1} & S_1 \leqslant j \leqslant S_2, j = i, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

$$c_{i,j}^{(n)} = \begin{cases} n\left(c_{i,j}^{(1)} c_{i,i}^{(n-1)} - \frac{c_{i,j}^{(n-1)}}{x_i - x_j}\right) & S_1 \leqslant j \leqslant S_2, j \neq i, \\ -\sum_{j=S_1, j\neq i}^{S_2} c_{i,j}^{n} & S_1 \leqslant j \leqslant S_2, j = i, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

where $S_2 - S_1 + 1$ is the computational bandwidth for a grid point, which is a variable but usually smaller than the whole computational domain.

In the DSC algorithm, the domain of the definition of the system is extended with exterior grid points to ensure that the 'boundary condition' is well defined. In this paper, α and β are denoted as the number of exterior points that extends from the boundary at x_1 and x_N of the cylindrical shell. The exterior points are used to couple the boundary conditions with the governing differential equations instead of the interior points of the shell. Therefore, the boundary conditions of shell can be appropriately accounted. In this study, $\alpha = \beta = 1$ are adopted for all boundary conditions. Two ways for selecting the value of S_1 and S_2 are suggested [22]. It is seen that the value of computational bandwidth is decided by the selection of M.

- DQ(BB):

$$S_1 = \max(i - M, 1 - \alpha), \quad S_2 = \min(S_1 + 2M, N + \beta). \tag{14}$$

- DQ (UB):

$$S_1 = \max(i - M, 1 - \alpha), \quad S_2 = \min(i + M, N + \beta), \tag{15}$$

where N is the number of grid points used to discretize the shell, $1 \leqslant i \leqslant N$ and $2M+1 \leqslant N$. The DQ (BB) and DQ (UB) can produce box- and uniform-banded matrices, respectively, while the GDQ method produce dense (full) differential matrix. Hence, the DQ (BB) and DQ (UB) are more efficient than the GDQ in large-scale computations. In this paper, only the box-banded matrix is considered, which uses $2M+1$ grid points to approximate the derivative at each discrete point.

2.2. Governing differential equations

Consider a typical cylindrical shell as shown is Fig. 1a. R is the mean radius, L the length and h the thickness. The orthogonal coordinate system (x,θ,z) is taken at the shell middle surface. The displacements of the shell in the x, θ and radial directions are denoted by $u(x,\theta,t)$, $v(x,\theta,t)$ and $w(x,\theta,t)$, respectively, where t denotes the time.

The governing differential equations of cylindrical shells based on Goldenveizer–Novazhilov can be expressed as [1,24,25]

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\upsilon)}{2R^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{(1+\upsilon)}{2R}\frac{\partial^2 v}{\partial x \partial \theta} + \frac{\upsilon}{R}\frac{\partial w}{\partial x} = \rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 u}{\partial t^2}, \tag{16}$$

$$\frac{(1+\upsilon)}{2R}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{(1-\upsilon)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{(1+k)}{R^2}\frac{\partial^2 v}{\partial \theta^2} + 2(1-\upsilon)k\frac{\partial^2 v}{\partial x^2} + \frac{1}{R^2}\frac{\partial w}{\partial \theta} - (2-\upsilon)k\frac{\partial^3 w}{\partial x^2 \partial \theta} - \frac{k}{R^2}\frac{\partial^3 w}{\partial \theta^2} = \rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 v}{\partial t^2}, \tag{17}$$

$$\frac{\upsilon}{R}\frac{\partial u}{\partial x} + \frac{1}{R^2}\frac{\partial v}{\partial \theta} + \frac{1}{R^2}w + k \times \left(R^2\frac{\partial^2 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{1}{R^2}\frac{\partial^4 w}{\partial \theta^4}\right) - (2-\upsilon)k\frac{\partial^3 v}{\partial x^2 \partial \theta} - \frac{k}{R^2}\frac{\partial^3 v}{\partial \theta^3} = -\rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 w}{\partial t^2} \tag{18}$$

in which $k = h^2/(12R^2)$.

The displacement fields of a thin circular cylindrical shell with general boundary conditions may be written as

$$u(x,\theta,t) = U(x)\cos m\theta \cos \omega t \tag{19}$$

$$v(x,\theta,t) = V(x)\sin m\theta \cos \omega t, \tag{20}$$

$$w(x,\theta,t) = W(x)\cos m\theta \cos \omega t \tag{21}$$

where $2m$ is the number of circumferential half-waves of a vibration mode, ω is the circular frequency of vibration, and $U(x)$, $V(x)$ and $W(x)$ are unknown functions to be determined.

It is well known that there are a number of different boundary conditions associated with a circular cylindrical shell [1,25]. In this paper three typical boundary conditions are defined as follows:

(1) Simply supported or shear diaphragms (S):

$$w = M_x = N_x = v = 0. \tag{22}$$

(2) Clamped (C):

$$u = v = w = \frac{\partial w}{\partial x} = 0. \tag{23}$$

(3) Free (F):

$$N_x = N_{x\theta} + \frac{M_{x\theta}}{R} = Q_x + \frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta} = M_x = 0. \tag{24}$$

The force and moment resultants are given by Leissa [1,25]

$$N_x = \frac{Eh}{(1-\upsilon^2)}(\varepsilon_x + \upsilon\varepsilon_\theta), \tag{25}$$

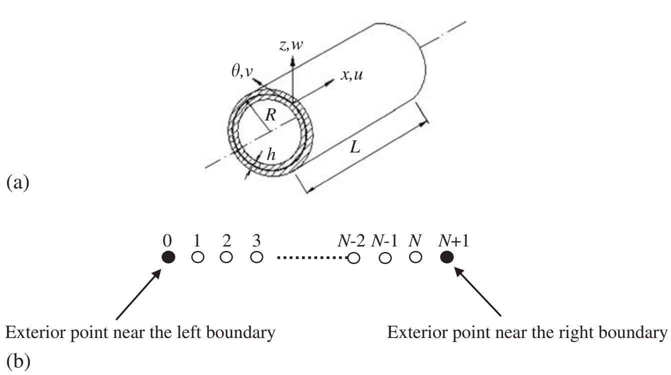

Fig. 1. (a) Geometry of a typical cylindrical shell. (b) Illustration of the exterior points close to the boundary.

$$N_\theta = \frac{Eh}{(1-\upsilon^2)}(\varepsilon_\theta + \upsilon\varepsilon_x), \tag{26}$$

$$N_{x\theta} = \frac{Eh}{2(1+\upsilon)}\left(\varepsilon_{x\theta} + \frac{h^2}{12R}\tau\right), \tag{27}$$

$$M_x = \frac{Eh^3}{12(1-\upsilon^2)}(\kappa_x + \upsilon\kappa_\theta), \tag{28}$$

$$M_{x\theta} = \frac{Eh^3}{24(1+\upsilon)}\tau. \tag{29}$$

The strain, curvature and twist of middle surface terms are related to the displacement fields by

$$\varepsilon_x = \frac{\partial u}{\partial x}, \tag{30}$$

$$\varepsilon_\theta = \frac{1}{R}\left(\frac{\partial v}{\partial\theta} + w\right), \tag{31}$$

$$\varepsilon_{x\theta} = \frac{1}{R}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial x}, \tag{32}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \tag{33}$$

$$\kappa_\theta = \frac{1}{R^2}\left(\frac{\partial v}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2}\right), \tag{34}$$

$$\tau = -\frac{2}{R}\left(\frac{\partial^2 w}{\partial x\partial\theta} - \frac{\partial v}{\partial x}\right), \tag{35}$$

2.3. Implementation of the LaDQM

The shell is first discretized into N uniform grid points ($i, j = 1, 2, \ldots, N$) and two exterior points ($j = 0$ and $N+1$) are then added outside the shell ends (see Fig. 1b). According to the LaDQM, the governing differential equations at the ith grid point ($i = 1, 2, \ldots, N$) can be discretized into the following forms:

$$\sum_{j=0}^{N+1} c_{i,j}^2 U_j + \frac{(1-\upsilon)}{2R^2}(-m^2)U_i + \frac{(1+\upsilon)}{2R} m \sum_{j=0}^{N+1} c_{i,j} V_j + \frac{\upsilon}{R}\sum_{j=0}^{N+1} c_{i,j} W_j = (-\omega^2)\rho\frac{(1-\upsilon^2)}{E}U_i, \tag{36}$$

$$\begin{aligned}&\frac{(1+\upsilon)}{2R}(-m)\sum_{j=0}^{N+1} c_{i,j}U_j + \frac{(1-\upsilon)}{2}\sum_{j=0}^{N+1} c_{i,j}^2 V_j + \frac{(1+k)}{R^2}(-m^2)V_i + 2(1-\upsilon)k\sum_{j=0}^{N+1} c_{i,j}^2 V_j \\ &+ \frac{1}{R^2}(-m)W_i - (2-\upsilon)k(-m)\sum_{j=0}^{N+1} c_{i,j}^2 W_j - \frac{k}{R^2}m^3 W_i = (-\omega^2)\rho\frac{(1-\upsilon^2)}{E}V_i,\end{aligned} \tag{37}$$

$$\begin{aligned}&\frac{\upsilon}{R}\sum_{j=0}^{N+1} c_{i,j}U_j - (2-\upsilon)km\sum_{j=0}^{N+1} c_{i,j}^2 V_j + \frac{m}{R^2}V_i + \frac{k}{R^2}m^3 V_i + kR^2\sum_{j=0}^{N+1} c_{i,j}^4 W_j \\ &+ \frac{1}{R^2}W_i - 2km^2\sum_{j=0}^{N+1} c_{i,j}^2 W_j + \frac{k}{R^2}m^4 W_i = (\omega^2)\rho\frac{(1-\upsilon^2)}{E}W_i,\end{aligned} \tag{38}$$

where $c_{i,j}^m$ is the GDQ weighting coefficient of the mth derivative, and N is the number of the grid points.

Similarly, by using LaDQM, the boundary conditions at the grid points $i = 1$ and N can be discretized as:

1. (SS boundary conditions):

$$V_1 = 0, \quad W_1 = 0, \quad \sum_{j=0}^{N+1} c_{1,j}^1 U_j = 0, \quad \sum_{j=0}^{N+1} c_{1,j}^2 W_j = 0 \quad \text{at } x = 0, \tag{39a–d}$$

$$V_N = 0, \quad W_N = 0, \quad \sum_{j=0}^{N+1} c_{N,j}^1 U_j = 0, \quad \sum_{j=0}^{N+1} c_{N,j}^2 W_j = 0 \quad \text{at } x = L. \tag{40a–d}$$

2. (CC boundary conditions):

$$V_1 = 0, \quad W_1 = 0, \quad U_1 = 0, \quad \sum_{j=0}^{N+1} c_{1,j}^1 W_j = 0 \quad \text{at } x = 0, \tag{41a–d}$$

$$V_N = 0, \quad W_N = 0, \quad U_N = 0, \quad \sum_{j=0}^{N+1} c_{N,j}^1 W_j = 0 \quad \text{at } x = L. \tag{42a–d}$$

3. (FF boundary conditions):

$$\sum_{j=0}^{N+1} c_{1,j}^1 U_j + \frac{\upsilon}{R}mV_1 + \frac{\upsilon}{R}W_1 = 0 \quad \text{at } x = 0, \tag{43a}$$

$$-\sum_{j=0}^{N+1} c_{1,j}^2 W_j + \frac{m\upsilon}{R^2}V_1 + \frac{\upsilon}{R^2}m^2 W_1 = 0 \quad \text{at } x = 0, \tag{43b}$$

$$-\frac{m}{R}U_1 - (1+4q)\sum_{j=0}^{N+1} c_{1,j}V_j + 4qm\sum_{j=0}^{N+1} c_{1,j}W_j = 0 \quad \text{at } x = 0, \tag{43c}$$

$$\left(\frac{\upsilon m}{R^2(1-\upsilon)}+\frac{2m}{R^2}\right)\sum_{j=0}^{N+1}c_{1,j}V_j + \left(\frac{\upsilon m^2}{R^2(1-\upsilon)}+\frac{2m^2}{R^2}\right)\sum_{j=0}^{N+1}c_{1,j}W_j, - \frac{1}{(1-\upsilon)}\sum_{j=0}^{N+1}c^3_{1,j}W_j=0 \quad \text{at } x=0, \tag{43d}$$

$$\sum_{j=0}^{N+1}c^1_{N,j}U_j+\frac{\upsilon}{R}mV_N+\frac{\upsilon}{R}W_N=0 \quad \text{at } x=L, \tag{44a}$$

$$-\sum_{j=0}^{N+1}c^2_{N,j}W_j+\frac{m\upsilon}{R^2}V_N+\frac{\upsilon}{R^2}m^2W_N=0 \quad \text{at } x=L, \tag{44b}$$

$$-\frac{m}{R}U_N-(1+4q)\sum_{j=0}^{N+1}c_{N,j}V_j + 4qm\sum_{j=0}^{N+1}c_{N,j}W_j=0 \quad \text{at } x=L, \tag{44c}$$

$$\left(\frac{\upsilon m}{R^2(1-\upsilon)}+\frac{2m}{R^2}\right)\sum_{j=0}^{N+1}c_{N,j}V_j + \left(\frac{\upsilon m^2}{R^2(1-\upsilon)}+\frac{2m^2}{R^2}\right)\sum_{j=0}^{N+1}c_{N,j}W_j - \frac{1}{(1-\upsilon)}\sum_{j=0}^{N+1}c^3_{N,j}W_j=0 \quad \text{at } x=L. \tag{44d}$$

Other combinations of boundary conditions can be implemented in a similarly way.

When using exterior points together with the shell theory, a difficulty in dealing with boundary conditions arise. The reason is that there are six additional unknowns ($U_0, V_0, W_0, U_{N+1}, V_{N+1}, W_{N+1}$) associated with the two extra exterior points. However, there are eight boundary conditions that need to be implemented at the two ends of the shell. To overcome this difficulty, two boundary conditions at one shell end can be combined into a single expression. For example, for a simply supported left edge (at $x=0$), by substituting Eq. (39b) to Eq. (39d), we can reduce the two conditions into one. Therefore, four boundary conditions at left end are reduced to three boundary conditions as shown in Eq. (45a–c). The processed boundary conditions for the three typical cases are shown below:

1. (SS boundary conditions):

$$V_1=0, \quad \sum_{j=0}^{N+1}c^1_{1,j}U_j=0, \quad \sum_{j=0,j\neq 1}^{N+1}c^2_{1,j}W_j=0 \quad \text{at } x=0, \tag{45a-c}$$

$$V_N=0, \quad \sum_{j=0}^{N+1}c^1_{N,j}U_j=0, \quad \sum_{j=0,j\neq N}^{N+1}c^2_{N,j}W_j=0 \quad \text{at } x=L. \tag{46a-c}$$

2. (CC boundary conditions):

$$V_1=0, \quad U_1=0, \quad \sum_{j=0,j\neq 1}^{N+1}c^1_{1,j}W_j=0 \quad \text{at } x=0, \tag{47a-c}$$

$$V_N=0, \quad U_N=0, \quad \sum_{j=0,j\neq N}^{N+1}c^1_{N,j}W_j=0 \quad \text{at } x=L. \tag{48a-c}$$

3. (FF boundary conditions):

$$\sum_{j=0}^{N+1}c^1_{1,j}U_j+\frac{\upsilon}{R}mV_1+\frac{\upsilon}{R}W_1=0 \quad \text{at } x=0, \tag{49a}$$

$$-\sum_{j=0}^{N+1}c^2_{1,j}W_j+\frac{m\upsilon}{R^2}V_1+\frac{\upsilon}{R^2}m^2W_1=0 \quad \text{at } x=0, \tag{49b}$$

$$\frac{m^2}{R}U_1-\left(\frac{h^2\upsilon m}{12R^2(1-\upsilon)}-m-\frac{h^2m}{12R^2}\right)\sum_{j=0}^{N+1}c_{1,j}V_j + \left(\frac{m^2\upsilon h^2}{12R^2(1-\upsilon)}-\frac{h^2m^2}{12R^2}\right)\sum_{j=0}^{N+1}c_{1,j}W_j-\frac{h^2}{12(1-\upsilon)}\sum_{j=0}^{N+1}c^3_{1,j}W_j=0 \quad \text{at } x=0, \tag{49c}$$

$$\sum_{j=0}^{N+1}c^1_{N,j}U_j+\frac{\upsilon}{R}mV_N+\frac{\upsilon}{R}W_N=0 \quad \text{at } x=L, \tag{50a}$$

$$-\sum_{j=0}^{N+1}c^2_{N,j}W_j+\frac{m\upsilon}{R^2}V_N+\frac{\upsilon}{R^2}m^2W_N=0 \quad \text{at } x=L, \tag{50b}$$

$$\frac{m^2}{R}U_N-\left(\frac{h^2\upsilon m}{12R^2(1-\upsilon)}-m-\frac{h^2m}{12R^2}\right)\sum_{j=0}^{N+1}c_{N,j}V_j + \left(\frac{m^2\upsilon h^2}{12R^2(1-\upsilon)}-\frac{h^2m^2}{12R^2}\right)\sum_{j=0}^{N+1}c_{N,j}W_j-\frac{h^2}{12(1-\upsilon)}\sum_{j=0}^{N+1}c^3_{N,j}W_j=0 \quad \text{at } x=L. \tag{50c}$$

From the six boundary conditions in each case, one can obtain the six unknowns U_0, V_0, W_0, U_{N+1}, V_{N+1} and W_{N+1} in terms of the function values at interior points $i = 1, 2, \ldots, N-1, N$. These solutions are then substituted into the differential equations in Eqs. (36–38) to form an eigenvalue equation:

$$[A]\{X\} = \omega^2\{X\}, \tag{51}$$

where

$$\{X\}^{\mathrm{T}} = \{U_1, U_2 \cdots U_{N-1}, U_N, V_1, V_2 \cdots V_{N-1}, V_N, W_1, W_2 \cdots W_{N-1}, W_N\}^{\mathrm{T}}$$

$[A]$ is a $3N \times 3N$ matrix. The eigenvalues of matrix $[A]$ are the natural frequencies ω^2 of the cylindrical shells, which can be obtained by any standard eigenvalue solver.

3. Results and discussions

The LaDQM is applied, for the first time, to study the free vibration of cylindrical shells with free, simply supported and clamped boundary conditions. For convenience, a two-letter symbol is used to describe shells subjected to different boundary conditions, i.e. the symbol SF denotes a shell having simply supported and free edge conditions at $x = 0$ and $x = L$, respectively. The vibration frequency ω is expressed in terms of a non-dimensionalized frequency parameter $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$. The Poisson ratio υ takes the value 0.3 in this study.

The coordinate distribution of discrete grid points for the shell can be chosen arbitrarily.

- Uniform distribution of discrete points

$$x_i = \frac{(i-1)}{N-1} L \qquad \text{for } i = 1, 2, \ldots, N; \tag{52}$$

- Redistributed Chebyshev–Gauss–Lobatto Grids:

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right] \quad \text{for } i = 1, 2, \ldots, N. \tag{53}$$

It is noted that for Eq. (53), the closer to the ends of the cylindrical shell, the smaller the distance between the grid points becomes and thus the distribution is closer. For simplicity, only uniform grid points are employed to solve the vibration problem in the present study.

In order to verify the accuracy of the LaDQM in solving the vibration of cylindrical shell, a comparison study is conducted. The following expression is defined to be the percentage deviation of LaDQM results from the published solution.

$$\text{Relative difference\%} = \left|\frac{\text{Present result} - \text{Exact result}}{\text{Exact result}}\right| \times 100\%. \tag{54}$$

The convergence and comparison studies of LaDQM must be carried out to ensure the reliability of the results. The normalized frequency parameters λ/λ_{ext} of the first six mode sequences are illustrated in Figs. 2 and 3 with an increasing number of bandwidth parameter and grid points for FF and SF cylindrical shells, respectively. The shell thickness to radius ratio h/R is 0.05 and shell length to radius ratio $L/R = 20$. The values λ_{ext} are the exact solutions obtained by the method in Ref. [25]. It is seen that the convergence of the LaDQM results with increasing the bandwidth parameter and the grid points shows an unsteady trend for low mode sequence, while for the high mode sequence, the frequency parameters obtained by LaDQM demonstrate essentially stable convergence.

Tables 1–3 show typical convergence studies of the first eight frequency parameters λ for cylindrical shells with SS,

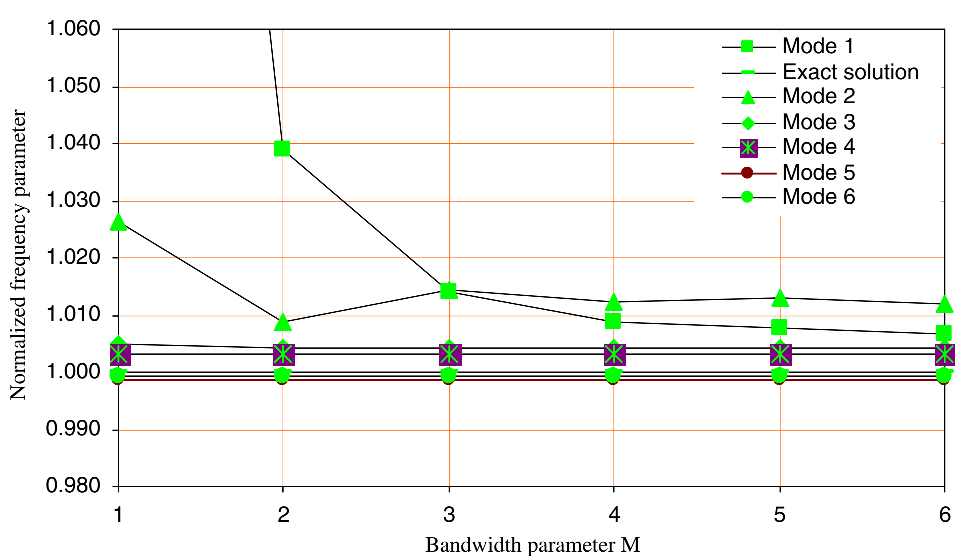

Fig. 2. Convergence and accuracy of the normalized frequency parameter, λ/λ_{ext}, for the first six mode sequences with the bandwidth parameter M for FF boundary conditions.

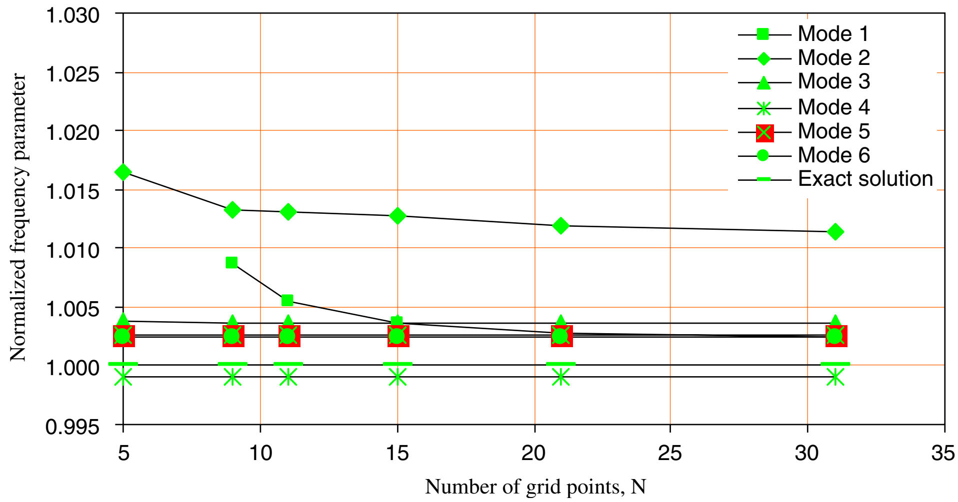

Fig. 3. Convergence and accuracy of the normalized frequency parameter, λ/λ_{ext}, for the first six mode sequences with the grid refinement for SF boundary conditions.

Table 1
Convergence and comparison study of frequency parameters λ for S–S cylindrical shell ($N = 11$, $L/R = 20$, $h/R = 0.05$)

m	*M*						Obtained by the method in Ref. [25]
	1	2	3	4	5	6	
1	0.018811 16.818%	0.016166 0.396%	0.016102 −0.006%	0.016102 −0.002%	0.016103 0.001%	0.016102 0.000%	0.016102
2	0.039788 1.318%	0.039280 0.025%	0.039270 0.000%	0.039271 0.000%	0.039271 0.000%	0.039271 0.000%	0.039271
3	0.109904 0.085%	0.109813 20.001%	0.109811 0.000%	0.109811 0.000%	0.109811 0.000%	0.109811 0.000%	0.109811
4	0.210304 0.013%	0.210277 0.000%	0.210277 0.000%	0.210277 0.000%	0.210277 0.000%	0.210277 0.000%	0.210277
5	0.339866 0.003%	0.339877 40.000%	0.339877 0.000%	0.339877 0.000%	0.339877 0.000%	0.339877 0.000%	0.339877
6	0.498447 0.001%	0.498444 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443
7	0.685919 0.000%	0.685919 0.000%	0.685919 0.000%	0.685919 0.000%	0.685919 0.000%	0.685919 0.000%	0.685919
8	0.902277 0.000%	0.902278 60.000%	0.902278 0.000%	0.902278 0.000%	0.902278 0.000%	0.902278 0.000%	0.902278

CC boundary conditions, and first six frequency parameters for FF cylindrical shells, respectively. The thickness to radius ratio is taken to be $h/R = 0.05$ and the length to radius ratio L/R is set to be 20. The LaDQM results are obtained by using 11 grid points. Analytical solutions obtained by Xiang et al. [25] are included in Table 1. Results generated by using the method in Ref. [25] are given in Tables 2 and 3 as well. It is apparent that for all cases, when increasing the bandwidth parameter M, the accuracy of the LaDQM results improves. When the bandwidth parameter M is taken as 3, the obtained numerical results are very accurate, and the relative differences of all the numerical results are well below 1.421 per cent except for the CC case ($m = 3$). It is observed that the results of SS shells give better accuracy than those of CC and FF shells. The accuracy of the

Table 2
Convergence and comparison study of frequency parameters λ for C–C cylindrical shell ($N = 11$, $L/R = 20$, $h/R = 0.05$)

m	M						Obtained by the method in Ref. [25]
	1	2	3	4	5	6	
1	0.037329 13.598%	0.033157 0.904%	0.032933 0.222%	0.032846 −0.044%	0.032906 0.139%	0.032853 −0.022%	0.032860
2	0.042766 5.223%	0.040628 −0.036%	0.040682 0.096%	0.040620 −0.056%	0.040648 90.014%	0.040638 −0.012%	0.040643
3	0.110402 −4.748%	0.109906 −5.176%	0.109993 −5.100%	0.109965 −5.125%	0.109976 −5.115%	0.109973 −5.118%	0.115905
4	0.210471 0.066%	0.210214 −0.055%	0.210353 0.010%	0.210313 −0.008%	0.210328 −0.001%	0.210324 −0.003%	0.210331
5	0.339963 0.018%	0.339741 −0.047%	0.339926 0.007%	0.339871 −0.009%	0.339893 −0.003%	0.339887 −0.005%	0.339903
6	0.498487 0.006%	0.498260 −0.040%	0.498479 0.004%	0.498413 −0.009%	0.498440 −0.004%	0.498430 −0.006%	0.498459
7	0.685942 0.002%	0.685699 −0.034%	0.685947 0.002%	0.685870 −0.009%	0.685902 −0.004%	0.685887 −0.006%	0.685930
8	0.902290 0.000%	0.902030 −0.028%	0.902300 0.001%	0.902215 −0.008%	0.902250 −0.004%	0.902230 −0.006%	0.902286

Table 3
Convergence and comparison study of frequency parameters λ for F–F cylindrical shell ($N = 11$, $L/R = 20$, $h/R = 0.05$)

m	M						Obtained by the method in Ref. [25]
	1	2	3	4	5	6	
1	0.046007 829.877%	0.036803 3.895%	0.035920 1.402%	0.035734 0.875%	0.035696 0.770%	0.035662 0.673%	0.035424
2	0.039698 2.622%	0.039018 0.867%	0.039233 1.421%	0.039160 1.231%	0.039182 1.290%	0.039149 1.203%	0.038683
3	0.109904 0.492%	0.109813 0.408%	0.109811 0.407%	0.109811 0.407%	0.109811 0.407%	0.109811 0.407%	0.109366
4	0.210308 0.316%	0.210283 0.305%	0.210277 0.302%	0.210281 0.303%	0.210279 0.303%	0.210280 0.303%	0.209644
5	0.339890 −0.129%	0.339882 −0.132%	0.339877 −0.133%	0.339880 −0.132%	0.339879 −0.133%	0.339880 −0.132%	0.340330
6	0.498450 −0.075%	0.498447 −0.075%	0.498444 −0.076%	0.498446 −0.076%	0.498445 −0.076%	0.498446 −0.076%	0.498824

LaDQM results can be improved by using more grid points and a large bandwidth parameter M.

In Table 4, the first eight non-dimensional frequency parameters are presented for the shells with various thicknesses to radius ratios and subject to simply supported boundary condition at both ends. The number of the total grid points N is set to be 11. The bandwidth parameter M is taken to be 3, and the shell length to radius ratio L/R is fixed at 20. It is observed that by using the same number of grid points, the thicker a cylindrical shell is, the more accurate the results by LaDQM. However, the thickness effect becomes less dominant as the number of half waves in the circumferential direction m increases.

Tables 5–7 display the variations of the first six frequency parameters λ against the number of the total

Table 4
Comparison study of frequency parameters λ for S–S cylindrical shells ($N = 11$, $M = 3$, $L/R = 20$)

m	h/R				h/R (obtained by the method in Ref. [25])			
	0.001	0.1	0.2	0.3	0.001	0.1	0.2	0.3
1	0.016100 −0.006%	0.016107 −0.006%	0.016128 −0.004%	0.016164 −0.004%	0.016101	0.016108	0.016129	0.016164
2	0.005282 −0.025%	0.077954 0.000%	0.155136 0.000%	0.231360 0.000%	0.005283	0.077955	0.155136	0.231360
3	0.003307 −0.024%	0.219355 0.000%	0.436860 0.000%	0.650610 0.000%	0.003308	0.219355	0.436860	0.650610
4	0.004441 −0.009%	0.420072 0.000%	0.836226 0.000%	1.243876 0.000%	0.004441	0.420072	0.836226	1.243876
5	0.006862 −0.003%	0.678946 0.000%	1.351051 0.000%	2.007108 0.000%	0.006862	0.678946	1.351051	2.007108
6	0.009993 0.000%	0.995665 0.000%	1.980569 0.000%	2.937449 0.000%	0.009993	0.995665	1.980569	2.937449
7	0.013732 −0.001%	1.370110 0.000%	2.724297 0.000%	4.022449 0.000%	0.013732	1.370110	2.724297	4.022449
8	0.018056 0.000%	1.802227 0.000%	3.581750 0.000%	4.732863 0.000%	0.018056	1.802227	3.581750	4.732863

Table 5
Convergence and comparison study of frequency parameters for S–S cylindrical shells ($L/R = 20$, $h/R = 0.05$)

m	$M = 2$	$M = 4$	$M = 5$	$M = 5$	$M = 5$	$M = 5$	Obtained by the method in Ref. [25]
	$N = 5$	$N = 9$	$N = 11$	$N = 15$	$N = 21$	$N = 31$	
1	0.016825 4.486%	0.016324 1.376%	0.016103 0.001%	0.016102 0.000%	0.016102 0.000%	0.016102 0.000%	0.016102
2	0.039377 0.272%	0.039317 0.118%	0.039271 0.000%	0.039271 0.000%	0.039271 0.000%	0.039271 0.000%	0.039271
3	0.109828 0.016%	0.109819 0.008%	0.109811 0.000%	0.109811 0.000%	0.109811 0.000%	0.109811 0.000%	0.109811
4	0.210282 0.002%	0.210278 0.001%	0.210277 0.000%	0.210277 0.000%	0.210277 0.000%	0.210277 0.000%	0.210277
5	0.339880 0.001%	0.339877 0.000%	0.339877 0.000%	0.339877 0.000%	0.339877 0.000%	0.339877 0.000%	0.339877
6	0.498445 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443 0.000%	0.498443

grid points for the cylindrical shell with SS, SF and CS boundary conditions. The values of M for the first two cases are chosen as the maximum value base on $2M + 1 \leqslant N$, while for the rest are fixed to a constant value $M = 5$. Obviously, by increasing the number of the grid points N, the computed frequency parameter converges rapidly. It is found that when the number of the grid points N is bigger than 11, the LaDQM results are well converged.

The discussion has been focussed on shells with large shell length to radius ratio ($L/R = 20$). The LaDQM method also works for shells with smaller shell length to radius ratios. Table 8 presents the frequency parameters for SS and CC circular cylindrical shells with $h/R = 0.05$ and $L/R = 1$, 5 and 10, respectively. It is observed that, in general, as the shell becomes shorter, the results from the LaDQM method are less accurate as for shells with longer length. However, the maximum error recorded in the

Table 6
Convergence and comparison study of frequency parameter for S–F cylindrical shells ($L/R = 20$, $h/R = 0.05$)

m	$M = 2$	$M = 4$	$M = 5$	$M = 5$	$M = 5$	$M = 5$	Obtained by the method in Ref. [25]
	$N = 5$	$N = 9$	$N = 11$	$N = 15$	$N = 21$	$N = 31$	
1	0.037807 52.740%	0.024968 0.870%	0.024888 0.547%	0.024843 0.365%	0.024822 0.281%	0.024813 0.245%	0.024752
2	0.039354 1.649%	0.039227 1.322%	0.039225 1.316%	0.039207 1.268%	0.039177 1.191%	0.039157 1.140%	0.038716
3	0.109837 0.384%	0.109811 0.360%	0.109811 0.360%	0.109811 0.360%	0.109811 0.360%	0.109811 0.360%	0.109417
4	0.210289 −0.094%	0.210278 −0.099%	0.210278 −0.099%	0.210278 −0.099%	0.210280 −0.098%	0.210286 −0.095%	0.210486
5	0.339884 0.256%	0.339878 0.254%	0.339878 0.254%	0.339878 0.254%	0.339879 0.254%	0.339878 0.254%	0.339017
6	0.498448 0.240%	0.498445 0.240%	0.498444 0.240%	0.498444 0.240%	0.498445 0.240%	0.498436 0.238%	0.497253

Table 7
Convergence and comparison study of frequency parameters for C–S cylindrical shells ($L/R = 20$, $h/R = 0.05$)

m	$M = 2$	$M = 4$	$M = 5$	$M = 5$	$M = 5$	$M = 5$	Obtained by the method in Ref. [25]
	$N = 5$	$N = 9$	$N = 11$	$N = 15$	$N = 21$	$N = 31$	
1	0.024913 3.993%	0.023929 −0.111%	0.023978 0.091%	0.023951 −0.023%	0.023937 −0.081%	0.023934 −0.092%	0.023956
2	0.039908 0.320%	0.039766 −0.037%	0.039784 0.008%	0.039781 0.001%	0.039780 −0.003%	0.039779 −0.005%	0.039781
3	0.109763 −0.108%	0.109873 −0.007%	0.109883 0.001%	0.109882 0.000%	0.109881 0.000%	0.109881 0.000%	0.109881
4	0.210124 −0.085%	0.210286 −0.008%	0.210302 0.000%	0.210302 0.000%	0.210302 0.000%	0.210302 0.000%	0.210302
5	0.339682 −0.061%	0.339861 −0.008%	0.339886 −0.001%	0.339888 0.000%	0.339888 0.000%	0.339888 0.000%	0.339889
6	0.498227 −0.045%	0.498411 −0.008%	0.498443 −0.002%	0.498447 −0.001%	0.498448 −0.001%	0.498448 −0.001%	0.498451

Table 8
Frequency parameters for SS and CC cylindrical shells ($h/R = 0.05$)

Case	m	Present			Obtained by the method in Ref. [25]		
		L/R			L/R		
		1	5	10	1	5	10
SS	1	0.591608 0.000%	0.186391 −0.001%	0.059109 0.000%	0.591608	0.186392	0.059109
	2	0.675800 −0.086%	0.086908 0.000%	0.044535 0.000%	0.676385	0.086908	0.044535
	3	0.537350 −0.574%	0.120170 0.000%	0.111052 0.000%	0.540451	0.120170	0.111052
	4	0.495599 0.130%	0.216081 0.000%	0.211275 0.000%	0.494954	0.216080	0.211275

Table 8 (*continued*)

Case	m	Present			Obtained by the method in Ref. [25]		
		L/R			L/R		
		1	5	10	1	5	10
CC	1	0.896439 −1.610%	0.240509 −0.237%	0.099635 −0.190%	0.911107	0.241081	0.099825
	2	0.742860 −0.192%	0.134597 −0.608%	0.057667 −0.259%	0.744287	0.135420	0.057817
	3	0.637109 −1.475%	0.136136 −0.374%	0.112829 −0.045%	0.646647	0.136647	0.112880
	4	0.618630 −0.697%	0.220506 −0.147%	0.211724 −0.023%	0.622970	0.220830	0.211772

considered cases by the LaDQM method is about 1.6% when comparing with the exact solutions.

4. Conclusion

This paper presents a new numerical approach to investigate the vibration behaviour of circular cylindrical shells. The derivatives in both the governing differential equations and the boundary conditions are discretized by using the local adaptive differential quadrature method (LaDQM). The application of the LaDQM to the new shell problem has been validated by convergent and comparison studies against exact results. It is observed that the boundary condition, shell thickness to radius ratio and mode sequences all have a quite significant effect on the convergence characteristics of the method. It is also found that for shells with simply supported boundary conditions, the present method provides more accurate numerical results than their clamped and free counterparts. Nevertheless, a relatively small number of grid points is required to give accurate results. It can be concluded from extensive numerical tests that the present LaDQM approach is an efficient method for the free vibration of cylindrical shells.

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