

## Mathematical analysis of the wavelet method of chaos control

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In this paper, we provide mathematical analysis for the controllability of chaos in wavelet subspaces. We prove that depending on the scale of the wavelet operation and the number of the coupled oscillators, the critical coupling strength for the occurrence of chaos synchronization becomes many times smaller if the original coupling matrix is appropriately treated with the wavelet transform. Moreover, we obtain rigorous relations connecting the critical values and the wavelet subspace operations. Our mathematical results are completely consistent with early numerical simulations. © 2006 American Institute of Physics. [DOI: 10.1063/1.2203229]

### I. INTRODUCTION

Chaos is ubiquitous in nature. Controlling chaos is of both theoretical and of practical importance.<sup>1-5</sup> Recently, a new paradigm of chaos control via wavelet transform has been introduced by Wei, Zhan, and Lai in Ref. 6 (also see Ref. 7). It is found that the transverse stability of the synchronous manifold of a chaotic system could be dramatically enhanced by the means of modifying a tiny fraction of the wavelet subspaces of a coupling matrix. Nevertheless, rigorous mathematical analysis of the aforementioned control has not been reported in the literature. Our objective in the present work is to present detailed mathematical analyses of the wavelet approach.<sup>6</sup>

To be more precise, let  $du/dt=f(u)$  be a given chaotic oscillator. Consider a coupled nonlinear dynamical system of  $N$  chaotic oscillators,

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) + \epsilon\mathbf{A}\mathbf{u}, \quad \mathbf{u} = (u_1, u_2, \dots, u_N)^T, \quad (1.1)$$

where  $(\mathbf{F}(\mathbf{u}))_i=f(u_i)$  is a nonlinear function of the  $i$ th oscillator, which has a state function  $u_i \in [0, \infty) \times \mathbb{R}^n$ ,  $\epsilon$  is a coupling strength, and  $\mathbf{A}$  is a coupling matrix having the periodic structure at the boundaries.

The synchronous manifold of the chaotic system, as a subspace of the original coupled system, Eq. (1.1), can be studied by setting  $u_1(t)=u_2(t)=\dots=u_N(t)=s(t)$ , where the chaotic solution  $s(t)$  satisfies the single oscillator equation  $ds/dt=f(s(t))$ . The stability property of the synchronous manifold can be studied in the space of difference variables  $\delta u_i(t)=u_i(t)-s(t)$ , which are governed by

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$$\frac{d\delta\mathbf{u}}{dt} = (D\mathbf{F}(\mathbf{S}(t)) + \epsilon\mathbf{A})\delta\mathbf{u}, \quad \delta\mathbf{u} = (\delta u_1, \delta u_2, \dots, \delta u_N)^T, \quad (1.2)$$

where  $D\mathbf{F}(\mathbf{u}) = \text{diag}(f'(u_1), f'(u_2), \dots, f'(u_N))$  and  $\mathbf{S}(t)^T = (s(t), s(t), \dots, s(t))_{1 \times N}$ . The second largest eigenvalue  $\lambda_2$  of the matrix  $\mathbf{A}$  plays a dominant role in controlling the stability of chaotic synchronization.<sup>8-10</sup> A critical coupling strength  $\epsilon_c$  can be determined in terms of  $\lambda_2$ ,

$$\epsilon_c = \frac{L_{\max}}{-\lambda_2}, \quad (1.3)$$

where  $L_{\max} > 0$  is the largest Lyapunov exponent of a single chaotic oscillator.

For the nearest neighbor coupling case, the eigenvalue spectrum of an appropriately normalized  $\mathbf{A}$  is given by

$$-4 \sin^2 \frac{\pi(i-1)}{N}, \quad i = 1, 2, \dots, N. \quad (1.4)$$

In general, a wider coupling width gives a smaller  $\lambda_2$ , while a larger number of oscillators requires a larger  $\epsilon_c$ . In controlling a given system, it is desirable to reduce the critical coupling strength  $\epsilon_c$ .

Denote the two dimensional wavelet decomposition and its inverse with periodic boundary condition by  $\mathbf{W}$  and  $\mathbf{W}^{-1}$ , respectively. For a given matrix  $\mathbf{A}$ , the wavelet decomposition allows a perfect reconstruction, by which there is nothing to gain:  $\mathbf{A} = \mathbf{W}^{-1}(\mathbf{W}(\mathbf{A}))$ . In Ref. 6, a simple operation is used to attain a desirable coupling matrix:

$$\tilde{\mathbf{A}} = \mathbf{W}^{-1}(\mathbf{O}_K(\mathbf{W}(\mathbf{A}))), \quad (1.5)$$

where  $\mathbf{O}_K$  is limited to be the multiplication of a scalar factor  $K$  on the elements of subspaces  $\mathbf{LL}_1$ , which corresponds to the lowest resolution subspace in both the horizontal and vertical directions in a two-dimensional multiscale wavelet decomposition. A numerical simulation of a coupled system of 512 Lorenz oscillators in Ref. 6 shows that for the nearest neighbor coupling case, the critical coupling strength  $\epsilon_c$  decreases linearly with respect to the increase of  $K$  up to a critical value  $K_c$ . The smallest  $\epsilon_c$  is about 6, which is about  $10^3$  times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.

In this paper we will provide a rigid mathematical analysis of the above wavelet scheme. For simplicity and without loss of generality, we only consider the wavelet transformation and the reverse transformation based on the Daubechies wavelet, *db1*.<sup>11</sup>

By an  $i$ -scale wavelet operation  $\mathbf{W} = \mathbf{W}(i)$  based on the *db1* wavelet, a  $2^k \times 2^k$  matrix  $\mathbf{A}$  is transformed into another  $2^k \times 2^k$  matrix  $\mathbf{W}(\mathbf{A})$ . Moreover, the subspaces  $\mathbf{LL}_1$  or equally the  $2^{k-i} \times 2^{k-i}$  up-left block of  $\mathbf{W}(\mathbf{A})$  equals  $\mathbf{\Omega A \Omega}^T$ , where

$$\mathbf{\Omega} = \frac{1}{2^{2i}} \begin{bmatrix} \mathbf{e} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{e} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e} \end{bmatrix}_{2^{k-i} \times 2^k},$$

with  $\mathbf{e} = (1, 1, \dots, 1)_{1 \times 2^i}$  and  $\mathbf{0} = (0, 0, \dots, 0)_{1 \times 2^i}$ . In other words, every entry of  $\mathbf{\Omega A \Omega}^T$  is the average value of all items of some  $2^i \times 2^i$  block of  $\mathbf{A}$ . With the transformation  $\mathbf{O}_K$ ,  $\mathbf{\Omega A \Omega}^T$  is transformed into  $K\mathbf{\Omega A \Omega}^T$ . Since  $\mathbf{A} = \mathbf{W}^{-1}(\mathbf{W}(\mathbf{A}))$ , we have that

$$\tilde{\mathbf{A}} = \mathbf{W}^{-1}(\mathbf{O}_K(\mathbf{W}(\mathbf{A}))) = \mathbf{A} + (K-1)\mathbf{\Omega A \Omega}^T. \quad (1.6)$$

Before stating the main theorem, we introduce some notations. Define series  $\{\rho_i\}_{i=1}^{\infty}$  by

$$\rho_i = 2 \cos \frac{\pi}{2^i} - 2, \quad i = 1, 2, 3, \dots \quad (1.7)$$

Moreover, we denote

$$K_c(i, N) = \frac{-2^i \rho_i}{4 \sin^2 \frac{2^i \pi}{N}} + 1, \quad i = 1, 2, 3, \dots \quad (1.8)$$

**Theorem 1:** For any  $i \in \mathcal{N}$ , assume  $\rho_i$  is defined as in (1.7) and the nature numbers  $N$  satisfy  $N/2^{i+2} \in \mathcal{N}$ . Let  $\mathbf{A}$  be the nearest neighbor-coupling matrix of order  $N \times N$  defined as in Eq. (1.1). Suppose by  $i$ -scale wavelet operation (1.5) with the scalar factor  $K \geq 1$  that the coupling matrix is transformed from  $\mathbf{A}$  to  $\tilde{\mathbf{A}}$ . Then for any such  $K$ , all the eigenvalues of  $\tilde{\mathbf{A}}$  are nonpositive; Moreover, it holds that  $\rho_i$  is an eigenvalue of the matrix  $\tilde{\mathbf{A}}$ .

**Theorem 2:** Let  $i, \rho_i, N, \mathbf{A}, \tilde{\mathbf{A}}, K$  be defined as in Theorem 1. Assume  $K_c(i, N)$  are defined by (1.8). Then the second largest eigenvalue of  $\tilde{\mathbf{A}}$  is a decreasing function of  $K$ . Moreover, the second largest eigenvalue of  $\tilde{\mathbf{A}}$  is equal to  $\rho_i$  for any  $K \geq K_c(i, N)$  and is strictly larger than  $\rho_i$  (thus its absolute value is strictly less than  $-\rho_i$ ) for any  $1 \leq K < K_c(i, N)$ .

Since the critical coupling strength is determined in terms of the second largest eigenvalue of the coupling matrix, by Theorem 2 and (1.4) we have the following result.

**Corollary 1.1:** The  $i$ -scale wavelet control method with  $K_c(i, N)$  as the scalar factor can enhance the stability of a synchronous manifold of an  $N$  coupled system by reducing the critical coupling strength as much as  $\rho_i/4 \sin^2(\pi/N)$  time.

In the following, we denote a “block circulant matrix” with blocks  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$  by

$$\text{bcirc}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots & \cdots & \mathbf{A}_n \\ \mathbf{A}_n & \mathbf{A}_1 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathbf{A}_4 & \ddots & \ddots & \ddots & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{A}_3 & \mathbf{A}_4 & \ddots & \ddots & \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 & \cdots & \mathbf{A}_n & \mathbf{A}_1 \end{bmatrix}.$$

Moreover, we denote average values of all elements of a matrix  $\mathbf{A} = (a_{ij})_{n \times n}$  by  $\text{aver}(\mathbf{A})$ , that is,  $\text{aver}(\mathbf{A}) = (1/n^2) \sum_{i,j=1}^n a_{ij}$ .

## II. PROOF OF THE MAIN THEOREMS

In this section, we will prove Theorem 1 and Theorem 2. First, we introduce some useful lemmas.

### A. Preliminaries

*Lemma 2.1:* Assume matrix  $\mathbf{A}$  is of order  $N \times N$  and  $\tilde{\mathbf{A}}$  is defined by  $i$ -scale wavelet operation (1.5) with the scalar factor  $K \geq 1$ . Then it holds that

$$\tilde{\mathbf{A}} - \mathbf{A} = \frac{K-1}{2^{2i}} \mathbf{A}_{N_i} \otimes \mathbf{B}_i, \quad (2.1)$$

with

$$\mathbf{A}_{N_i} = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}_{N_i \times N_i}$$

and

$$\mathbf{B}_i = \mathbf{e}^\top \mathbf{e},$$

where  $N_i = N/2^i$ ,  $\mathbf{e}$  is defined as in the introduction, and  $\otimes$  represents tensor product of matrices.

*Proof:* First note that the nearest neighbor matrix  $\mathbf{A}$  can be written in the form

$$\text{bcirc}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{N_i}), \quad N_i = \frac{N}{2^i},$$

where matrix  $\mathbf{A}_j$  of order  $2^i \times 2^i$  ( $j=1, 2, \dots, N_i$ ) satisfies

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 1 & \dots & 0 \\ 1 & -2 & \ddots & \vdots \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & & 1 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \mathbf{A}_{N_i}^\top = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \end{bmatrix}, \tag{2.2}$$

$$\mathbf{A}_l = \mathbf{0}, \quad 3 \leq l \leq \frac{N}{2^i} - 1. \tag{2.3}$$

It is obvious to see that  $\text{aver}(\mathbf{A}_1) = -1/2^{2i-1}$  and  $\text{aver}(\mathbf{A}_2) = \text{aver}(\mathbf{A}_{N_i}) = 1/2^{2i}$ .

From (1.6), we obtain that  $\tilde{\mathbf{A}}$  is of the form

$$\tilde{\mathbf{A}} = \text{bcirc}(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2, \dots, \tilde{\mathbf{A}}_{N_i}), \tag{2.4}$$

with

$$\tilde{\mathbf{A}}_1 = -\frac{K-1}{2^{2i-1}} \mathbf{B}_i + \mathbf{A}_1, \quad \tilde{\mathbf{A}}_2 = \frac{K-1}{2^{2i}} \mathbf{B}_i + \mathbf{A}_2, \quad \tilde{\mathbf{A}}_{N_i} = \frac{K-1}{2^{2i}} \mathbf{B}_i + \mathbf{A}_{N_i},$$

$$\tilde{\mathbf{A}}_j = \mathbf{0}, \quad 3 \leq j \leq N_i - 1. \tag{2.5}$$

Thus the lemma can be obtained easily. □

*Proposition 2.1:* Let  $\mathbf{A} = \text{bcirc}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)$  be a real symmetric block circulant matrix. Denote  $\mathbf{C}_k = \sum_{j=1}^n e^{i[k(j-1)/n]2\pi} \mathbf{A}_j$ ,  $k=0, 1, \dots, n-1$ ,  $\iota = \sqrt{-1}$ . Then eigenvalues of  $\mathbf{A}$  consist of eigenvalues of  $\mathbf{C}_k$ ,  $k=0, 1, \dots, n-1$ .

*Proof:* See p. 211 in Ref. 12. □

### B. Proof of Theorem 1

*Proof of nonpositivity:* First, we prove that for any  $K \geq 1$ , all the eigenvalues of  $\tilde{\mathbf{A}}$  are non-positive. From Lemma 2.1, we have that

$$\tilde{\mathbf{A}} = \mathbf{A} + \frac{K-1}{2^{2^i}} \mathbf{A}_{N_i} \otimes \mathbf{B}_i. \tag{2.6}$$

We will prove that  $[(K-1)/2^{2^i}] \mathbf{A}_{N_i} \otimes \mathbf{B}_i$  is seminegative definite. Then by the seminegative definiteness of  $\mathbf{A}$  and the fact that the sum of two seminegative definite matrices is seminegative definite, we obtain that  $\tilde{\mathbf{A}}$  is also seminegative definite. Thus all its eigenvalues are nonpositive. Since  $[(K-1)/2^{2^i}] \mathbf{A}_{N_i} \otimes \mathbf{B}_i$  is a real symmetric block circulant, by Proposition 2.1, it is sufficient to prove the nonpositivity of eigenvalues of matrices,

$$-2\mathbf{B}_i + e^{i(k2^i/N)2\pi} \mathbf{B}_i + e^{i[k(N-2^i)/N]2\pi} \mathbf{B}_i, \quad k = 0, 1, \dots, \frac{N}{2^i} - 1, \tag{2.7}$$

or equally,

$$\left( -2 + 2 \cos \frac{k2^i}{N} 2\pi \right) \mathbf{B}_i, \quad k = 0, 1, \dots, \frac{N}{2^i} - 1. \tag{2.8}$$

Obviously,  $\mathbf{B}_i$  has 0 and 1 as its eigenvalues and the term in the parentheses is less than or equal to 0. Thus, each eigenvalue of the matrices defined in (2.7) and (2.8) is nonpositive. Then from the symmetry, we obtain that  $(K-1)/2^{2^j} \mathbf{A}_{N_i} \otimes \mathbf{B}_i$  is seminegative definite. Thus, the proof is completed.  $\square$

*Remark 2.1:* From the fact that  $\mathbf{A}$  has only one zero eigenvalue, we conclude that  $\tilde{\mathbf{A}}$  has at most one zero eigenvalue. On the other hand, since the sum of elements in every column of  $\tilde{\mathbf{A}}$  is zero, we obtain that  $\tilde{\mathbf{A}}$  has at least one zero eigenvalue. In a word, it has and only has one zero eigenvalue.

Now we prove the second part of Theorem 1, that is, for any  $K \geq 1$ ,  $\rho_i$  is an eigenvalue of the matrix  $\tilde{\mathbf{A}}$ .

Combining (2.4), (2.5) and Proposition 2.1, we have that the set of eigenvalues of  $\tilde{\mathbf{A}}$  is equal to the collection of the eigenvalue of the  $2^i \times 2^i$  matrix,

$$\mathbf{D}_1(l) + d_l \mathbf{D}_2(l) := \begin{bmatrix} -2 & 1 & & & e^{-i(2\pi l/n)} \\ 1 & -2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ e^{i(2\pi l/n)} & & & 1 & -2 \end{bmatrix} + d_l \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [1 \dots 1],$$

$$l = 0, 1, \dots, n - 1,$$

with  $n = N/2^i$ ,  $d_l = (c/2^{2^i})[2 \cos(2\pi l/n) - 2]$ ,  $c = K - 1$ .

The following two propositions are useful for the proof of Theorem 1 and Theorem 2.

*Proposition 2.2* (rank one update for Hermitian matrices): *Let  $\mathbf{A}$  be an  $N \times N$  Hermitian matrix and  $\mathbf{u} \in \mathbb{C}^N$ . Suppose  $(\lambda_k, \mathbf{u}_k)$  and  $(\tilde{\lambda}_k(\alpha), \tilde{\mathbf{u}}_k(\alpha))$ ,  $k = 1, \dots, N$ , be, respectively, the eigenpairs for  $\mathbf{A}$  and  $\mathbf{A} + \alpha \mathbf{u} \mathbf{u}^H$  with the order  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$  and  $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_N$ .*

(i) *Assume  $\alpha > 0$ . If  $\mathbf{u}^H \mathbf{u}_k \neq 0$  and  $\mathbf{u}^H \mathbf{u}_{k'} \neq 0$  for some  $k' > k$ ; then  $\lambda_k \leq \tilde{\lambda}_k(\alpha) \leq \lambda_{k'}$ .*

(ii) *If  $\mathbf{u}^H \mathbf{u}_{k''} = 0$  for some  $k''$ ; then  $\tilde{\lambda}_{k''}(\alpha) = \lambda_{k''}$  for all  $\alpha$ .*

*Proof:* See Ref. 13.  $\square$

*Proposition 2.3:* Denote  $(\lambda_k, \mathbf{u}_k)$ ,  $k=0, \dots, m-1$ , the eigenpair of the matrix

$$\mathbf{G} = \begin{bmatrix} -2 & 1 & & & e^{-\alpha} \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{\alpha} & & & 1 & -2 \end{bmatrix}_{m \times m}$$

with  $\alpha$  any real number. Then

$$\lambda_k = 2 \cos \theta_k - 2$$

and

$$\mathbf{u}_k = (u_{k,j})_{j=1}^m,$$

where

$$\theta_k = \frac{2k\pi}{m} + \frac{\alpha}{m}, \quad k = 0, \dots, m-1,$$

and

$$u_{k,j} = \begin{cases} 1, & \text{if } k^2 + \alpha^2 = 0, \\ c_1(\sin j\theta_k + e^{-\alpha} \sin(m+j)\theta_k) + c_2(\sin(m+1+j)\theta_k + e^{\alpha} \sin(j+1)\theta_k), & \text{otherwise.} \end{cases}$$

Here  $c_1$  and  $c_2$  are arbitrary complex numbers.

*Proof:* By directly solving the eigenvalue problem for  $\mathbf{G}$ ,  $\mathbf{G}\mathbf{u} = \lambda\mathbf{u}$ , we obtain the difference equation

$$u_{j+1} - (2 + \lambda)u_j + u_{j-1} = 0, \quad j = 2, \dots, m-1, \quad (2.9)$$

with

$$u_0 = e^{-\alpha}u_m,$$

$$u_{m+1} = e^{\alpha}u_1. \quad (2.10)$$

Equation (2.9) has characteristic equation  $r^2 - (2 + \lambda)r + 1 = 0$ , which yields

$$r_{1,2} = \frac{(2 + \lambda) \pm \sqrt{(2 + \lambda)^2 - 4}}{2} =: e^{\pm i\theta}. \quad (2.11)$$

The second equality holds since eigenvalues of  $\mathbf{G}$  are nonpositive. Now set

$$u_j = Ar_1^j + Br_2^j = Ae^{ij\theta} + Be^{-ij\theta}, \quad (2.12)$$

where  $A$  and  $B$  are constant coefficient to be determined. Substituting (2.12) into (2.10), we obtain

$$A + B = e^{-\alpha}(Ae^{im\theta} + Be^{-im\theta}),$$

$$Ae^{i(m+1)\theta} + Be^{-i(m+1)\theta} = e^{\alpha}(Ae^{i\theta} + Be^{-i\theta}). \quad (2.13)$$

Since  $(A, B)$  is a nonzero solution of Eq. (2.13), it turns out that

$$\begin{vmatrix} 1 - e^{i(m\theta-\alpha)} & 1 - e^{-(im\theta+\alpha)} \\ e^{i(m+1)\theta} - e^{i(\theta+\alpha)} & e^{-i(m+1)\theta} - e^{-i(\theta-\alpha)} \end{vmatrix} = 0,$$

or, equivalently,

$$\cos m\theta = \cos \alpha.$$

Thus,  $\theta_k = 2k\pi/m + \alpha/m$ ,  $k=0, \dots, m-1$ , and, hence, by using (2.11), we have  $\lambda_k = 2 \cos \theta_k - 2$ . This gives the proof of the first assertion.

To see the eigenvector of  $\mathbf{G}$  corresponding to eigenvalue  $\lambda_k$ , we first consider that for  $\alpha=0$  and  $k=0$ , i.e.,  $\lambda_0=0$ . It is easy to see that the vector  $(1, \dots, 1)^T$  is the eigenvector of  $\mathbf{G}$  corresponding to 0. Now, assume  $\alpha \neq 0$  and  $k \neq 0$ . For convenience, we write  $\theta = \theta_k$  and  $u_j = u_{k,j}$ . Choose  $A_1 = 1 - e^{i(m\theta-\alpha)}$ ,  $B_1 = -1 + e^{-i(m\theta+\alpha)}$ ,  $A_2 = e^{i(m+1)\theta} - e^{i(\theta+\alpha)}$ , and  $B_2 = -e^{-i(m+1)\theta} + e^{-i(\theta-\alpha)}$  with  $\theta = \theta_k$  for some  $k$ . Note that  $(A_1, B_1)$  and  $(A_2, B_2)$  are solutions of (2.13). Set  $u_j = \frac{1}{2}(c_1 A_1 + c_2 A_2)e^{ij\theta} + \frac{1}{2}(c_1 B_1 + c_2 B_2)e^{-ij\theta} = c_1(\sin j\theta_k + e^{-i\alpha} \sin(m+j)\theta_k) + c_2(\sin(m+1+j)\theta_k + e^{i\alpha} \sin(j+1)\theta_k)$  and the second assertion follows.  $\square$

*Proof of the second part of Theorem 1:* Let  $\alpha = \pi$  and  $m = 2^i$ . From Proposition 2.3, it follows that  $\rho_i$  is an eigenvalue of  $\mathbf{D}_1(n/2)$ . To see that  $\rho_i$  is also an eigenvalue of  $\mathbf{D}_1(n/2) + d_{n/2}\mathbf{D}_2(n/2)$ , we choose  $c_1 = 1$  and  $c_2 = 0$ . From Proposition 2.3 again,  $\mathbf{D}_1(n/2)$  has an eigenvector  $\mathbf{u} = [\sin(j\pi/m)]_{j=1}^m$  corresponding to the eigenvalue  $\rho_i$ . Since  $\mathbf{D}_2 = \mathbf{e}\mathbf{e}^T$  and  $\mathbf{e}^T \mathbf{u} = \sum_{j=1}^m \sin(j\pi/m) = 0$ , applying (ii) of Proposition 2.2, we see that  $\rho_i$  is an eigenvalue of  $\mathbf{D}_1(n/2) + d_{n/2}\mathbf{D}_2(n/2)$ .  $\square$

**C. Proof of Theorem 2**

*Proof of the first part of Theorem 2:* Let  $n = N/2^i$  and  $\mathbf{D}_1(l), \mathbf{D}_2(l), d_l, l=0, 1, \dots, n-1$  be defined as in the last subsection. Since  $d_l \leq 0$ , applying Proposition 2.2 to  $-\mathbf{D}_1(l) - d_l \mathbf{D}_2(l)$ , we have that the eigenvalues of  $\mathbf{D}_1(l) + d_l \mathbf{D}_2(l)$  are increasing functions of  $d_l$ , which, by Proposition 2.1, implies the first part of Theorem 2, that is, the second largest eigenvalue of  $\tilde{\mathbf{A}}$  is a decreasing function of  $K$ .  $\square$

Now we are in a position to prove the second part of Theorem 2.

Denote  $\alpha = 2\pi l/n$ . We have

$$\mathbf{D}_1(l) = \begin{bmatrix} -2 & 1 & & & e^{-i\alpha} \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{i\alpha} & & & 1 & -2 \end{bmatrix}. \tag{2.14}$$

From Proposition 2.3, we have that any eigenvalue of  $\mathbf{D}_1(l)$  can be written in the form

$$\lambda = 2 \cos \theta(k, l) - 2,$$

with  $\theta(k, l) = 2k\pi/2^i + \alpha/2^i$ ,  $k=0, 1, \dots, 2^i - 1$ ,  $l=0, 1, \dots, n-1$ . Obviously, for every such  $l$  and  $k_1, k_2$  with  $k_1 \neq k_2$ , it holds that  $|\theta(k_1, l) - \theta(k_2, l)| \geq 2 \cdot \pi/2^i$ , which implies that there is at most one  $k \in \{0, 1, \dots, 2^i - 1\}$  such that  $|\theta(k, l)| \leq \pi/2^i$ . Then we have that for every matrix  $\mathbf{D}_1(l) + d_l \mathbf{D}_2(l)$  ( $l=0, 1, \dots, n-1$ ), there is at most one nonzero eigenvalue that is larger than or equal to  $\rho_i = 2 \cos(\pi/2^i) - 2$ . It implies that if for some  $d_l = d_l^*$ ,  $\mathbf{D}_1(l) + d_l \mathbf{D}_2(l)$  has eigenvalue  $\rho_i$ , then all its other eigenvalues are less than  $\rho_i$ .

*Remark 2.2:* If such  $d_l^*$  exists for  $\mathbf{D}_1(l) + d_l \mathbf{D}_2(l)$ , then by the first part of Theorem 2, we have that all its eigenvalues will be less than or equal to  $\rho_i$  for all  $d_l < d_l^*$ . Then for the proof of the second part of Theorem 2, it is sufficient to prove that the existence of such  $d_l^*$  for every  $l$ .

The following proposition will help us to complete the proof of Theorem 2.

*Proposition 2.4:* Let  $d=[2 \cos(\pi/m)-2]/m$ . Then the matrix

$$\mathbf{G} = \begin{bmatrix} -2 & 1 & & & e^{-i\alpha} \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{i\alpha} & & & 1 & -2 \end{bmatrix} + d \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} [1, \dots, 1] \in \mathbb{R}^{m \times m},$$

has eigenvalue  $2 \cos(\pi/m)-2$  for and  $\alpha \neq 0, \pm\pi$ .

*Proof:* Consider the eigenproblem of the matrix  $\mathbf{G}$ , which is equivalent to

$$u_{k-1} - (\lambda + 2)u_k + u_{k+1} + d \sum_{j=1}^m u_j = 0, \quad k = 2, \dots, m-1, \tag{2.15}$$

$$u_2 - 2u_1 + e^{-i\alpha}u_m + d \sum_{j=1}^m u_j = 0, \tag{2.16}$$

$$u_{m-1} - 2u_m + e^{i\alpha}u_1 + d \sum_{j=1}^m u_j = 0. \tag{2.17}$$

Denote  $\tilde{u}_k = u_k / \sum_{j=1}^m u_j$ . Then we have

$$\begin{aligned} \tilde{u}_{k-1} - (\lambda + 2)\tilde{u}_k + \tilde{u}_{k+1} + d &= 0, \\ \tilde{u}_2 - 2\tilde{u}_1 + e^{-i\alpha}\tilde{u}_m + d &= 0, \\ \tilde{u}_{m-1} - 2\tilde{u}_m + e^{i\alpha}\tilde{u}_1 + d &= 0. \end{aligned} \tag{2.18}$$

In the following, for the sake of notation without leading confusion, we use the notation  $u_k$  with the restriction  $\sum_{j=1}^m u_j = 1$  to replace the notation  $\tilde{u}_k$ .

Obviously, (2.18) has a special solution of the form  $u_k = d/\lambda, k=1, \dots, m$ . Thus the general solution of (2.18) is of the form

$$u_k = C_1 e^{ik\theta} + C_2 e^{-ik\theta} + \frac{d}{\lambda}, \tag{2.19}$$

with the boundary condition

$$u_0 = e^{-i\alpha}u_m, \quad u_{m+1} = e^{i\alpha}u_1, \tag{2.20}$$

where  $\cos \theta = (\lambda + 2)/2$ .

Now we determine  $C_1$  and  $C_2$  with the boundary condition

$$C_1 + C_2 + \frac{d}{\lambda} = C_1 e^{i(-\alpha+m\theta)} + C_2 e^{-i(\alpha+m\theta)} + e^{-i\alpha} \frac{d}{\lambda} \tag{2.21}$$

and

$$C_1 e^{i(m+1)\theta} + C_2 e^{-i(m+1)\theta} + \frac{d}{\lambda} = C_1 e^{i(\theta+\alpha)} + C_2 e^{i(-\theta+\alpha)} + e^{i\alpha} \frac{d}{\lambda}. \tag{2.22}$$

Thus we have

$$\begin{bmatrix} 1 & 1 \\ -e^{i(\theta+\alpha)} & -e^{i(-\theta+\alpha)} \end{bmatrix} \begin{bmatrix} 1 - e^{i(-\alpha+m\theta)} & 0 \\ 0 & 1 - e^{-i(\alpha+m\theta)} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{d}{\lambda} \begin{bmatrix} e^{-i\alpha} - 1 \\ e^{i\alpha} - 1 \end{bmatrix}.$$

Then we have

$$C_1 = \frac{-(e^{-i\alpha} - 1)e^{-i(\theta-\alpha)} - (e^{i\alpha} - 1)d}{(1 - e^{i(-\alpha+m\theta)})(e^{i(\theta+\alpha)} - e^{i(-\theta+\alpha)})\lambda} \quad (2.23)$$

and

$$C_2 = \frac{e^{i\alpha} - 1 + (e^{-i\alpha} - 1)e^{i(\theta+\alpha)}d}{(1 - e^{-i(\alpha+m\theta)})(e^{i(\theta+\alpha)} - e^{i(-\theta+\alpha)})\lambda}. \quad (2.24)$$

From  $\sum_{j=1}^m u_j = 1$ , we obtain that

$$C_1 \frac{e^{i\theta}(1 - e^{im\theta})}{1 - e^{i\theta}} + C_2 \frac{e^{-i\theta}(1 - e^{-im\theta})}{1 - e^{-i\theta}} + m \frac{d}{\lambda} = 1. \quad (2.25)$$

Combining (2.23)–(2.25), we have

$$\frac{e^{i\alpha} - 1}{e^{i\theta} - e^{-i\theta}} \left( \frac{1 - e^{im\theta}}{e^{i\alpha} - e^{im\theta}} - \frac{1 - e^{-im\theta}}{e^{i\alpha} - e^{-im\theta}} \right) + m = \frac{\lambda}{d}. \quad (2.26)$$

We note that  $\theta = \arccos[(\lambda + 2)/2] = \pi/m$ . Substituting  $d = [2 \cos(\pi/m) - 2]/m$  into (2.26) and by direct computation, we obtain that  $2 \cos(\pi/m) - 2$  is a root of (2.26) thus an eigenvalue of  $\mathbf{G}$ .  $\square$

*Proof of the second part of Theorem 2:* Denote  $\lambda_1^{(l)}(K)$  and  $\lambda_2^{(l)}(K)$ , respectively, the largest and second largest eigenvalue of  $\mathbf{D}_1(l) + d_l \mathbf{D}_2(l)$ . From Proposition 2.3, it follows that

$$\lambda_1^{(l)}(1) = \begin{cases} 2 \cos\left(\frac{2\pi l}{mn}\right) - 2, & 0 < l < \frac{n}{2}, \\ 2 \cos\frac{\pi}{m} - 2 = \rho_i, & l = \frac{n}{2}, \\ 2 \cos\left(\frac{2(m-1)\pi}{m} + \frac{2\pi l}{mn}\right) - 2, & \frac{n}{2} + 1 < l < n - 1, \end{cases}$$

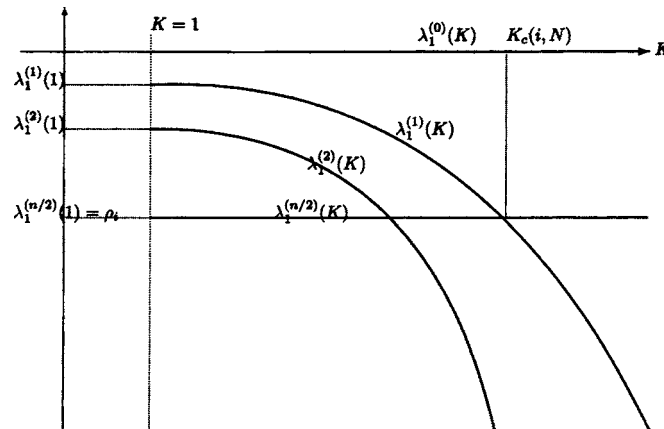
and

$$\lambda_2^{(l)}(1) = \begin{cases} 2 \cos\left(\frac{2(m-1)\pi}{m} + \frac{2\pi l}{mn}\right) - 2, & 0 < l < \frac{n}{2}, \\ 2 \cos\frac{\pi}{m} - 2 = \rho_i, & l = \frac{n}{2}, \\ 2 \cos\left(\frac{2\pi l}{mn}\right) - 2, & \frac{n}{2} + 1 < l < n - 1, \end{cases}$$

where  $m = 2^i$  and  $n = N/m$ . Here we note that  $\lambda_1^{(l)}(1) > \rho_i > \lambda_2^{(l)}(1)$  for  $l = 0, \dots, n/2 - 1, n/2 + 1, \dots, n - 1$ . Since the largest eigenvalue of  $\tilde{\mathbf{A}}$  equals  $0 = \lambda^{(0)}(K) \equiv \bar{\lambda}_1(K)$ , we see that the second largest eigenvalue of  $\tilde{\mathbf{A}}$  at  $K = 1$  equals  $\max\{\lambda_1^{(l)}(1) | l = 0, \dots, n/2 - 1, n/2 + 1, \dots, n - 1\}$ . On the other hand, applying Proposition 2.4 with  $m = 2^i$ , we have that for  $l \neq 0, n/2$  and  $d_l = \rho_i/m$ , i.e.,

$$K = \frac{m\rho_i}{2 \cos\frac{2\pi l}{n} - 2} + 1 \equiv K_c^{(l)},$$

$\mathbf{D}_1(l) + d_l \mathbf{D}_2(l)$  has eigenvalue  $\rho_i$ . By using Proposition 2.2,  $\lambda_1^{(l)}(K)$  is decreasing in  $K$ . Hence

FIG. 1. An illustration of the eigenvalues curves of  $\tilde{A}$ .

$$\lambda_1^{(l)}(K) < \rho_i,$$

for  $K > K_c^{(l)}$  and  $l \neq 0, n/2$ . Thus, for  $K > \max_{l \neq 0, n/2} \{K_c^{(l)}\} = K_c^{(1)} = K_c(i, N)$ , we have  $\bar{\lambda}_2(K) = \rho_i$ . Also, for  $1 < K < K_c(i, N)$ ,  $\bar{\lambda}_2 = \max_{l \neq 0, n/2} \lambda_1^{(l)}(K)$  (see Fig. 1 for an illustration). This completes the proof.  $\square$

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