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A comparison of the spectral and the discrete singular convolution schemes for the KdV-type equations

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Abstract

The discrete singular convolution (DSC) algorithm is applied to the Korteweg-de Vries (KdV) and modified KdV (mKdV) equations with a particular emphasis on a comparison with the Fourier pseudospectral (FPS) method. The errors of the DSC algorithm with respect to the numerical resolution are investigated by using the discrete Fourier analysis. The numerical results indicate that the DSC method can be more accurate than the FPS method for many nonlinear wave problems that are non-bandlimited in nature. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, the discrete singular convolution (DSC) algorithm has been proposed and applied to many practical problems [7–9]. The objective of this letter is to examine the DSC algorithm for solving the Korteweg-de Vries (KdV) and modified KdV equations and to compare the DSC with the Fourier pseudospectral (FPS) method, whose advantage for nonlinear wave equations has been well-documented [1–3,5].

In the DSC approach, the derivative terms are approximated by

$$\left[\frac{\partial^n u}{\partial x^n}\right]_{x=x_l} = \sum_{k=l-W}^{l+W} \delta_{\alpha,\sigma}^{(n)}(x_l - x_k)u_k, \tag{1}$$

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where u_k represents the numerical solution at grid point $x_k = k\Delta$ (Δ is the mesh-size), $2W + 1$ is the computational bandwidth, $\delta^{(n)}_{\alpha,\sigma}(x_l - x_k)$ is a collective symbol for the nth derivative of the regularized Shannon's kernel (RSK)

$$\frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)}\exp[-(x - x_k)^2/2\sigma^2] \tag{2}$$

or the regularized Dirichlet kernel (RDK)

$$\frac{\sin[(\pi/\Delta)(x - x_k)]}{(2M + 1)\sin((\pi/\Delta)(x - x_k)/(2M + 1))}\exp[-(x - x_k)^2/2\sigma^2]. \tag{3}$$

Here, $\exp(-x^2/2\sigma^2)$ is a typical delta regularizer [7]. The RDK has one more parameter M which can be optimized to achieve better results in computations. We select the value of $2M + 1$ to be the grid number N for the periodic problems. Obviously, the RDK converts to the RSK at the limit of $M \to \infty$.

2. Fourier analysis of errors

The use of Fourier analysis to characterize the errors of difference approximations is described extensively in Ref. [6]. The Fourier analysis could provide an effective way to quantify the Fourier resolution of the DSC approximations as well. This quantification may provide a general guide for the optimization of the DSC schemes.

For the purpose of Fourier analysis, the problem is assumed to be periodic over the domain $[0, 2\pi]$, which is discretized by N equidistant points. A function $u(x_j, t)$ is denoted by $u_j(t)$. Then, a set of physical quantities u_j ($j = 0, 1, \ldots, N - 1$) can be transformed into its counterpart in the Fourier space i.e., $\hat{u}_k$ ($k = -N/2, \ldots, N/2 - 1$). The wavenumber part which is larger than the Nyquist frequency π will contribute to the aliasing error. This analysis can also be used for the numerical approximations of derivatives [6].

The exact first derivative gives a set of Fourier coefficients $\hat{u}'_k = \mathrm{i}w\hat{u}_k$. For the DSC scheme, it may be shown that $\hat{u}'_k = \mathrm{i}w'\hat{u}_k$, where the modified wavenumber $w'(w)$ is a real-valued function. The Fourier spectral method provides $w' = w$ for $w \neq \pi$. The plot of the modified wavenumber w' against wavenumber w is presented in Fig. 1 for some DSC and central finite difference (CFD) schemes. Here, CFD $W = 16$ means the CFD scheme with 32 grid points. From Fig. 1, it is obvious that the DSC scheme can stay closer to the exact values over a wide range of wavenumbers than the CFD scheme with the same value of W when an appropriate value of $r = \sigma/\Delta$ is selected. It is seen that the accuracy of the DSC approximation to the derivative is controllable and can be better than the traditional higher-order finite difference approximation. For some practical problems, the Fourier spectrum is broad and the aliasing errors cannot be avoided. If the aliasing error is dominant compared with the differentiation error, the reduction of the resolution near the Nyquist frequency for the DSC schemes actually plays a role of de-aliasing. Thus, the DSC schemes may be expected to perform better than the spectral method. In other words, the Fourier spectral method is exact for bandlimited periodic L^2 functions but produces much larger aliasing errors than the DSC

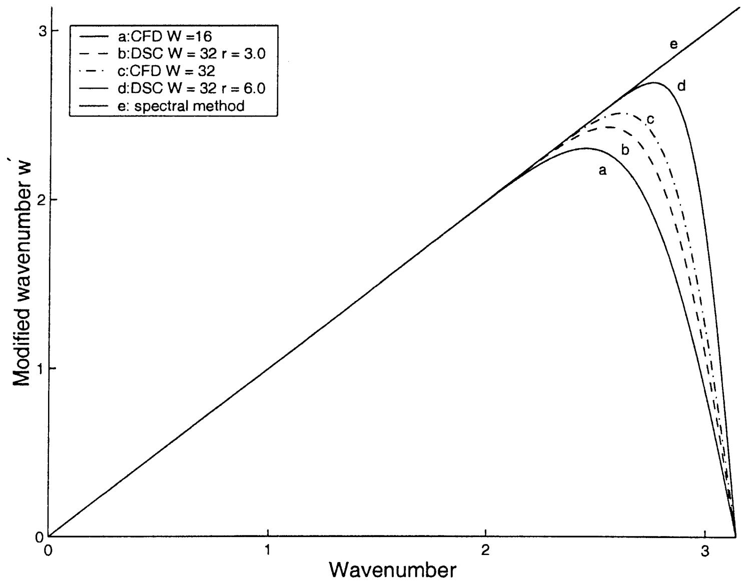

Fig. 1. Plot of the modified wavenumber vs. wavenumber for first derivative approximation.

algorithm for non-bandlimited functions. Such analysis is in good agreement with our numerical experiments.

The error analysis for the higher derivatives approximations proceeds similarly to that of the first derivative approximation. The results show that they share the same characteristics with the first derivative.

3. Numerical experiments

To confirm the analysis, we make a comparison between the FPS and DSC methods by taking the KdV and modified KdV equations as examples. The errors in the L_∞-norm and two of the conservative quantities $I_1 = \int u\,\mathrm{d}x$ and $I_2 = \frac{1}{2}\int u^2\,\mathrm{d}x$ are computed and compared. Here, $L_\infty = \max|\tilde{u}_l - u_l|$, where $\tilde{u}_l$ and u_l represent the numerical and analytical solutions at the grid points $u(l\Delta) = u_l$, respectively. Here, $E_i = |\bar{I}_i - I_i|/|I_i|$, $i = 1,2$, indicate the relative errors of the approximate values in the conservative quantities, and $\bar{I}_1$ and $\bar{I}_2$ stand for the counterparts of I_1 and I_2, respectively. The DSC algorithm is realized by two types of delta kernels, the RSK (2) and the RDK (3), the value of W is fixed to be 32 and the parameter σ is adjusted so that an optimal result is obtained for each test problem. The FPS method described by Sanders et al. [5] is used for the comparison. For both the DSC and FPS methods, the fourth-order Runge–Kutta method (RK4) is used for the time integration. The FFT subroutine DFOUR and the RK4 subroutine DRK4 of the numerical recipes library [4] are used.

Test 1. The KdV equation: It is well known that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{4}$$

admits a one-soliton solution

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2[\kappa(x - 4\kappa^2 - x_0)]. \tag{5}$$

We choose $\kappa = 1.0$ and $x_0 = 20.0$ from Eq. (5) as the initial condition. The KdV equation (4) is numerically solved on the domain $\{x,t\} \in [0,40] \times [0,T]$ with a periodic boundary condition.

Table 1 exhibits the results for the RSK, the RDK, and the FPS, respectively. It is observed from Table 1 that, for $N = 128$, the results of both the RSK and RDK schemes are better than those of the FPS method. However, for $N = 256$, although the two conservative quantities for both the DSC schemes are preserved better than those of the spectral method, as to the L_∞ errors, the RDK and FPS methods have almost the same accuracy, both are better than that of the RSK method. It is obvious that the RDK scheme performs slightly better than the RSK scheme. The stability condition for the DSC schemes is improved when compared with the spectral method. For all the test problems in our study, it is found that the maximum step size permitted for the DSC schemes is about two times larger than that for the FPS method.

Test 2. The modified KdV equation: The modified KdV equation (6)

$$u_t + 3u^2 u_x + u_{xxx} = 0 \tag{6}$$

admits a one-soliton solution

$$u(x,t) = \sqrt{2}\kappa \operatorname{sech}[\kappa(x - \kappa^2 t - x_0)]. \tag{7}$$

In this test problem, the initial condition is taken from Eq. (7) with $\kappa = 2.0$ and $x_0 = 20.0$. The modified KdV equation is numerically solved over the domain $x \in [0,40]$.

We compute the numerical solutions until $t = 40.0$. Table 2 exhibits the errors in the L_∞-norm and the relative errors E_1 and E_2 for two of conservative quantities $I_1 = \int u \, dx$ and $I_2 = \frac{1}{2}\int u^2 \, dx$. It is seen that for either $N = 128$ or 256, the accuracy of the RSK and RDK schemes is better than that of the FPS method. Moreover, for $N = 128$, the RDK scheme performs better than the RSK scheme and for $N = 256$, while both schemes have almost the same accuracy.

4. Conclusions

By focusing on the discrete singular convolution (DSC) kernels of regularized Shannon and Dirichlet, we apply the DSC schemes to the Korteweg-de Vries (KdV) and modified KdV equations. In comparison, the standard Fourier pseudospectral method is also employed to solve the same nonlinear wave equations. The illustrative examples for the nonlinear wave equations in this paper indicate that the DSC algorithm is a robust and reliable numerical method of spectral level accuracy. It has great potential for applications in many science and engineering disciplines.

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Table 1
Comparison of L_∞, E_1 and E_2 errors for the solution of the KdV equation ($k = 1.0$) given by the RSK, RDK and FPS schemes

t	N	Δt	RSK			RDK			FPS		
			L_∞	E_1	E_2	L_∞	E_1	E_2	L_∞	E_1	E_2
10.0	128	0.001	6.5 (−6)	8.5 (−8)	1.6 (−10)	4.8 (−6)	6.9 (−8)	1.7 (−11)	1.1 (−5)	1.1 (−7)	1.7 (−10)
20.0	128	0.001	8.0 (−6)	1.3 (−7)	3.2 (−10)	5.5 (−6)	2.4 (−8)	6.7 (−12)	1.1 (−5)	4.5 (−8)	1.1 (−10)
40.0	128	0.001	1.5 (−5)	2.0 (−7)	4.7 (−10)	6.0 (−6)	2.9 (−8)	7.1 (−12)	1.5 (−5)	2.0 (−7)	4.7 (−10)
10.0	256	0.0002	7.7 (−11)	1.1 (−14)	6.9 (−15)	3.6 (−11)	1.2 (−14)	6.2 (−15)	9.3 (−12)	1.4 (−13)	3.2 (−13)
20.0	256	0.0002	8.0 (−11)	1.4 (−14)	7.7 (−15)	3.6 (−11)	1.0 (−14)	1.1 (−14)	3.0 (−11)	2.7 (−13)	6.4 (−13)
40.0	256	0.0002	8.1 (−11)	9.3 (−15)	4.4 (−15)	3.5 (−11)	8.6 (−15)	2.1 (−14)	1.1 (−10)	5.2 (−13)	1.3 (−12)

Table 2
Comparison of L_∞, E_1 and E_2 errors for the solution of the modified KdV equation ($k = 2.0$) given by the RSK, RDK and FPS schemes

t	N	Δt	RSK			RDK			FPS		
			L_∞	E_1	E_2	L_∞	E_1	E_2	L_∞	E_1	E_2
10.0	128	0.001	6.9 (−2)	3.1 (−4)	3.5 (−4)	3.3 (−2)	4.5 (−4)	3.4 (−4)	5.5 (−1)	2.5 (−4)	4.6 (−3)
20.0	128	0.001	1.4 (−1)	4.5 (−5)	1.4 (−4)	6.0 (−2)	9.1 (−5)	3.4 (−5)	9.7 (−1)	1.7 (−3)	5.2 (−3)
40.0	128	0.001	2.8 (−1)	1.5 (−4)	3.9 (−4)	1.2 (−1)	3.5 (−5)	9.6 (−5)	1.7 (+0)	2.3 (−4)	2.9 (−3)
10.0	256	0.0002	5.1 (−7)	9.4 (−11)	2.8 (−14)	6.1 (−7)	5.9 (−11)	1.8 (−14)	1.5 (−6)	1.1 (−9)	2.2 (−13)
20.0	256	0.0002	5.8 (−7)	3.7 (−10)	5.2 (−14)	5.6 (−7)	4.8 (−10)	1.0 (−13)	1.7 (−6)	6.3 (−10)	5.8 (−13)
40.0	256	0.0002	5.8 (−7)	4.7 (−11)	2.1 (−13)	6.8 (−7)	5.1 (−10)	2.0 (−13)	2.5 (−6)	3.3 (−10)	2.2 (−12)

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