Derivative Function

In Notes §2.1, we defined the derivative of a function $f(x)$ at $x = a$, namely the number $f'(a)$. Since this gives an output $f'(a)$ for any input a, the derivative defines a function.

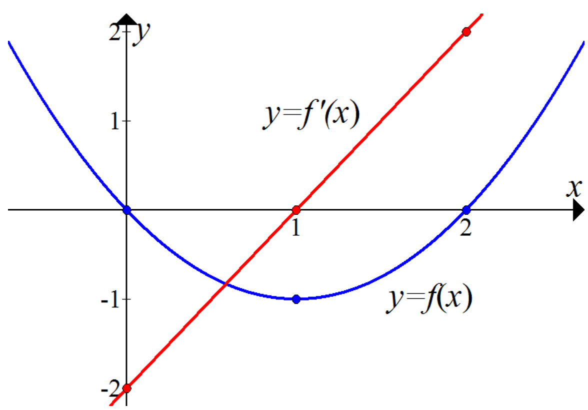
> *Definition:* For a function $f(x)$, we define the *derivative function* $f'(x)$ by:
>
> $$f'(x) \;=\; \lim_{h\to 0}\frac{f(x{+}h)-f(x)}{h} \;=\; \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\,.$$
>
> If the limit $f'(a)$ exists for a given $x = a$, we say $f(x)$ is *differentiable* at a; otherwise $f'(a)$ is undefined, and $f(x)$ is *non-differentiable* or *singular* at a.

This just repeats the definitions in Notes §2.1, except that we think of the derivative as a function of the variable x, rather than as a numerical value at a particular point $x = a$. The choice of letters is meant to suggest different kinds of variables, but they do not have any strict logical meaning: for example, $f(x) = x^2$, $f(a) = a^2$, and $f(t) = t^2$ all define the same function, and $\lim_{x\to a} f(x) = \lim_{t\to a} f(t) = \lim_{z\to a} f(z)$ are all the same limit.

Differentiation. Another name for derivative is *differential.* When we compute $f'(x)$, we *differentiate* $f(x)$. The process of finding derivatives is *differentiation.*

As usual for mathematical objects, we can think of derivatives on four levels of meaning. The <u>real-world</u> meaning of $f'(x)$ is the rate of change of $f(x)$ per unit change in x; for example velocity is the derivative of the position function at time t. At the end of Notes §2.1, we also saw how to compute a <u>numerical</u> approximation of a derivative as the diffference quotient for a small value of h (see also §2.9). In this section, we explore the <u>geometric</u> meaning as the slopes of the graph $y = f(x)$, and <u>algebraic</u> methods for computing the limit $f'(x)$.

EXAMPLE: Let $f(x) = x(x{-}2)$, with graph $y = f(x)$ in blue:



We can sketch the derivative graph $y = f'(x)$ in red, purely from the original graph $y = f(x)$, without any computation. The *slope* of the original graph above a given x-value is the *height* of the derivative graph above that x-value.

At the minimum $x = 1$, the original graph $y = f(x)$ is horizontal and its slope is zero, so $f'(1) = 0$, and we plot the point $(1, 0)$ on the derivative graph $y = f'(x)$. To the right of this point, $y = f(x)$ has positive slope, getting steeper and steeper; so $y = f'(x) > 0$ is above the x-axis, getting higher and higher. Above $x = 2$, the tangent of $y = f(x)$ has slope approximately 2 (considering the relative x and y scales), so we plot $(2, 2)$ on $y = f'(x)$.

As we move left from $x = 1$, the graph $y = f(x)$ has negative slope, getting steeper and steeper, so $y = f'(x) < 0$ is below the x-axis, getting lower and lower. Above $x = 0$, we estimate $y = f(x)$ to have slope -2, and we plot $(0, -2)$ on $y = f'(x)$. Thus, $y = f'(x)$ looks like the red line in the above picture.

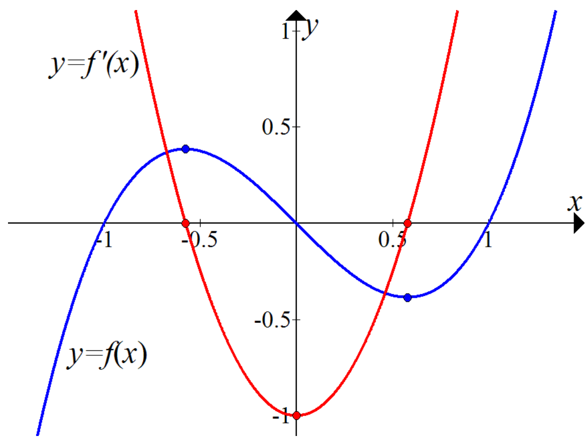
Next we differentiate algebraically. For any value of x:

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \;=\; \lim_{h\to 0} \frac{(x+h)(x+h-2) - x(x-2)}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-2x-2h-x^2+2x}{h} \;=\; \lim_{h\to 0} \frac{2xh-2h+h^2}{h} \;=\; \lim_{h\to 0} 2x-2+h \;=\; 2x-2.$$

That is, $f'(x) = 2x - 2$, which agrees with our sketch of the derivative graph.

EXAMPLE: Let $f(x) = x^3 - x$, with graph in blue:



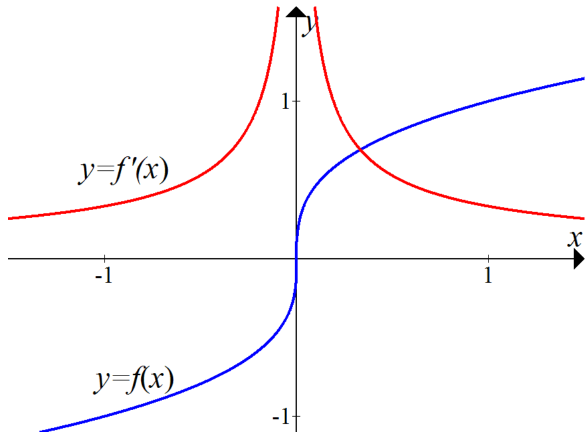
The original graph $y = f(x)$ has a valley with horizontal tangent at $x \cong 0.6$, so the derivative $f'(0.6) \cong 0$, and we plot the approximate point $(0.6, 0)$ on the derivative graph $y = f'(x)$; and similarly the hill on $y = f(x)$ corresponds to the point $(-0.6, 0)$ on $y = f'(x)$. Between these x-values, the slope of $y = f(x)$ is negative, with the slope at $x = 0$ being about -1, so $y = f'(x) < 0$ is below the x-axis, bottoming out at $(0, -1)$.

Algebraically:

$$f'(x) \;=\; \lim_{h\to 0} \frac{((x+h)^3-(x+h)) - (x^3-x)}{h} \;=\; \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3-x-h) - x^3+x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \;=\; \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 \;=\; 3x^2 - 1.$$

EXAMPLE: Let $f(x) = \sqrt[3]{x}$, the cube root function, with graph in blue:



The slopes of the original graph $y = f(x)$ are all positive, with the same slope above a given x and its reflection $-x$. Thus the derivative graph $y = f'(x) > 0$ lies above the x-axis, and it is symmetric across the y-axis (an even function). The slope of $y = f(x)$ gets smaller for large positive or negative x, and it gets steeper and steeper near the origin, with a vertical tangent at $x = 0$. Thus $y = f'(x)$ approaches the x-axis for large x, and shoots up the y-axis on both sides of $x = 0$, with $f'(0)$ undefined.

Algebraically, we have: $f'(x) = \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h}$. We must liberate $\sqrt[3]{x+h}$ from under the $\sqrt[3]{\ }$, so as to be able to cancel $\frac{h}{h}$. In Notes §2.1, we multiplied top and bottom by the conjugate radical, exploiting the identity $(a-b)(a+b) = a^2 - b^2$. Here we have cube roots, so we use the identity: $(a-b)(a^2+ab+b^2) = a^3 - b^3$, taking $a = \sqrt[3]{x+h}$ and $b = \sqrt[3]{x}$:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \cdot \frac{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2}{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{x+h}^3 - \sqrt[3]{x}^3}{h(\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2)} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\,\sqrt[3]{x} + \sqrt[3]{x}^2} = \frac{1}{\sqrt[3]{x+0}^2 + \sqrt[3]{x+0}\,\sqrt[3]{x} + \sqrt[3]{x}^2} = \frac{1}{3\sqrt[3]{x}^2}.$$

In the Notes §2.3, we will develop standard rules for computing derivatives, which let us avoid such complicated limit calculations.

Continuity Theorem. Here is a basic fact relating derivatives and continuity:

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

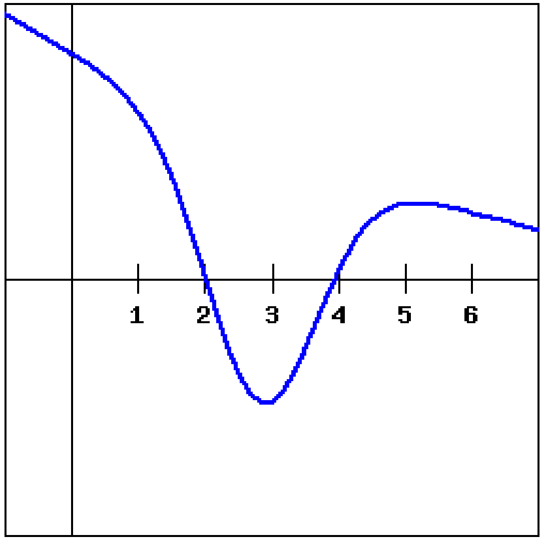
Turing this around, we have the equivalent negative statement (the contrapositive): If $f(x)$ is *not* continuous at $x = a$, then it is *not* differentiable at $x = a$. That is, a discontinuity is also a non-differentiable point (a singularity).

Proof of Theorem: Assume $f(x)$ is differentiable at $x = a$, meaning $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is defined. The Limit Law for Products gives:

$$\lim_{h\to 0}(f(a+h) - f(a)) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot h = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h\to 0} h = f'(a)\cdot 0 = 0.$$

Thus $0 = \lim_{h\to 0}[f(a+h) - f(a)] = [\lim_{h\to 0} f(a+h)] - f(a)$, and $\lim_{h\to 0} f(a+h) = f(a)$, showing that $f(x)$ is continuous at $x = a$.

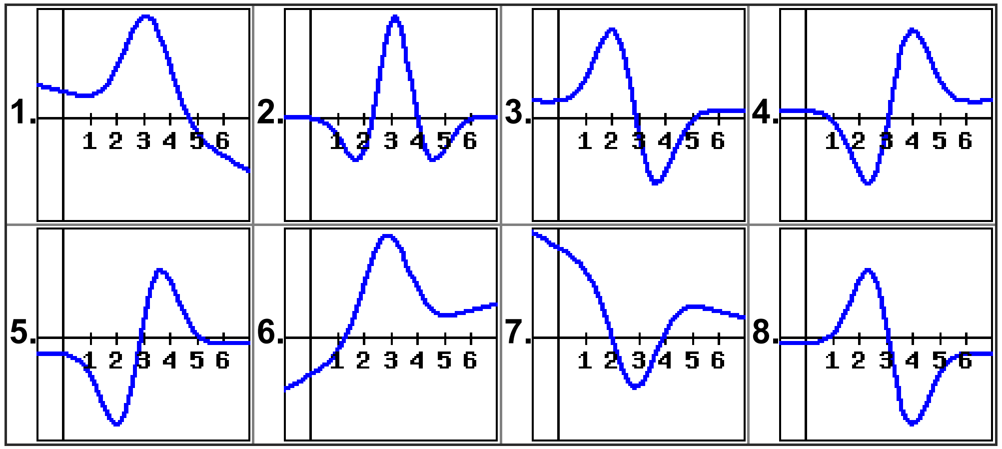
For the function $f(x)$ shown in the graph below, sketch a graph of the derivative. You will then be picking which of the following is the correct derivative gr
but should be sure to first sketch the derivative yourself.

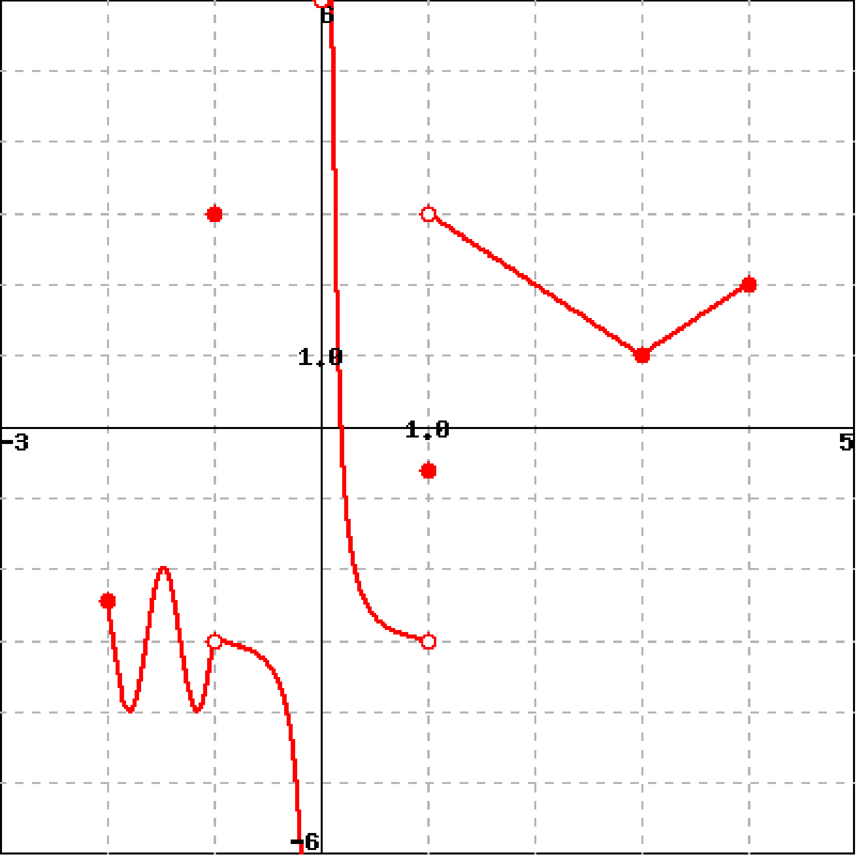


Click on the graph to open it in a new window.

Which of the following graphs is the derivative of $f(x)$? ?

(Click on a graph to view a larger version of it.)




6
1.0
-3
1.0
5
-6