Conceptual levels. Mathematics solves problems partly with technical tools like the differentiation rules, but its most powerful method is to translate between different levels of meaning, transforming the problems to make them accessible to our tools. Problems often originate at the physical or geometric levels, and we translate to the numerical or algebraic levels to solve them, then we translate the answer back to the original level.

Our key concept so far has been the *derivative*, with the following meanings:

- Physical: For a function $y = f(x)$, the derivative $\frac{dy}{dx} = f'(x)$ is the rate of change of y with respect to x, near a particular value of x. For a a particular input, $f'(a)$ means how fast $f(x)$ changes from $f(a)$ per unit change in x away from a. This is the main importance of derivatives.

- Geometric: For a graph $y = f(x)$, the derivative $f'(a)$ is the slope of the tangent line at the point $(a, f(a))$.

- Numerical: We approximate the derivative by the difference quotient:

$$f'(a) \;\cong\; \frac{\Delta f}{\Delta x} \;=\; \frac{f(a+h)-f(a)}{h}.$$

  The right side is the average rate of change of $f(x)$ from $x = a$ to $x = a+h$. As $\Delta x = h \to 0$, the difference quotient approaches the instantaneous rate of change, the derivative $f'(a)$.

- Algebraic: We can easily compute the derivative of almost any function defined by a formula. Basic Derivatives like $(x^p)' = px^{p-1}$, $\sin'(x) = \cos(x)$, and $\cos'(x) = -\sin(x)$ are combined using the Sum, Product, Quotient, and Chain Rules for Derivatives. Occasionally, we must go back to the definition $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Functions of motion. We consider the basic physical quantities describing motion. These are all functions of time t. (See end of §2.3.)

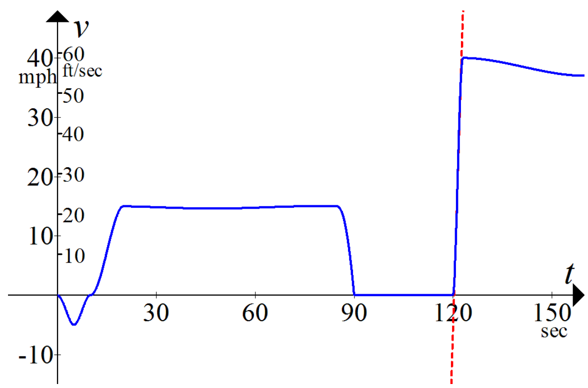
- *Position* or *displacement* s, the distance of an object past a reference point, in feet, at time t seconds.

- *Velocity* $v = \frac{ds}{dt}$, how fast the position is increasing per second (ft/sec); this is negative if position is decreasing. The *speed* is the magnitude $|v|$.

- *Acceleration* $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, how fast the velocity is increasing, the number of ft/sec gained each second (ft/sec^2). Equivalently, this is how fast the object is speeding up (positive) or slowing down (negative).

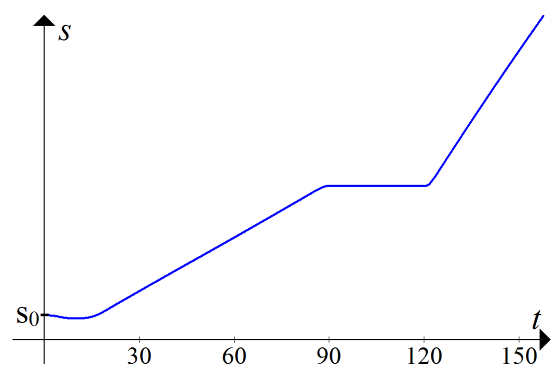
- *Jerk* $j = \frac{da}{dt} = \frac{d^3s}{dt^3}$, the rate of change of acceleration (ft/sec^3).

Driving. An insurance company downloads the following data from a car's speedometer, allowing them to construct the following graph of the car's velocity $v(t)$. What physical story does this graph tell?



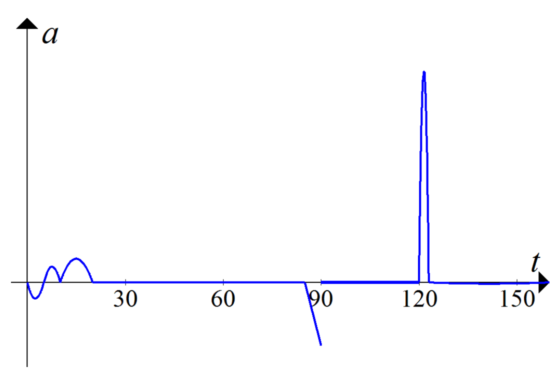
The dip of negative velocity at the beginning is probably the car slowly backing down a driveway. It goes forward a few blocks at a moderate, almost constant speed (positive velocity), stops at an intersection (zero velocity), then continues at higher speed.

From the velocity data, we can reconstruct the odometer data, the graph of the distance function $s(t)$: the *level* of the velocity graph is the *slope* of the distance graph.



The distance starts at some positive odometer reading $s = s_0$ (which we cannot know from the velocity data alone), decreases* a bit (negative slope) because of the negative velocity, increases with constant slope during constant positive velocity, stays at a constant level during zero velocity, then increases with greater slope after the velocity goes up.

Nothing remarkable so far. What about the acceleration, the derivative $a(t) = \frac{dv}{dt}$? The *slope* of the velocity graph is the *level* of the acceleration graph:



*Assuming the odometer runs both ways, like old mechanical odometers used to.

Here $a(t)$ is roughly proportional to the depression of the gas or brake pedal, and it is zero except when the car is speeding up or slowing down. The most prominent feature is the spike after $t = 120$: just how strong an acceleration is this? The tangent line marked on the velocity graph shows a change from 0 to 40 mph in about 3 sec, meaning a slope $a(122) \cong \frac{40}{3} \cong 10.3$ mph/sec. Now,

$$1\,\frac{\text{mi}}{\text{hr}} \;=\; \frac{5280\text{ ft}}{3600\text{ sec}} \;\cong\; 1.47\frac{\text{ft}}{\text{sec}},$$

so we convert 10.3 mph per second $= (10.3)(1.47) \cong 20$ ft/sec per second $= 20$ ft/sec^2. Compare this to the standard acceleration due to gravity: one gee is about 32 ft/sec^2, so this driver feels about 2/3 the force of gravity pushing him into the seat-back. It seems he (I'm pretty sure it's not a she) is flooring the accelerator, roaring ahead from a standstill with tires squealing, then easing up past 40 mph or so. Not responsible driving!

Finally, note the jump in a at $t = 90$, where the car goes from braking deceleration to a standstill. The change in a is not so large, but it happens so fast that it looks instantaneous, and the $a(t)$ graph seems to rise veritically (infinite slope). This means the derivative of acceleration, the jerk $j = \frac{da}{dt}$, is huge at this moment, and the car experiences a lurching stop, another sign of sloppy driving. This driver's insurance rates are going up!

Note that in this analysis, we have translated from the graphical (geometric) to the physical level; and also (for the gee calculation) from the graphical to the numerical to the physical.

Ballistic equation. This is the formula giving the height $s(t)$ for an object launched from initial height s_0, straight upward with initial velocity v_0, under the influence of a constant gravitational acceleration g:

$$s(t) \;=\; s_0 + v_0 t - \tfrac{1}{2}gt^2.$$

To justify this equation, note that the initial height is indeed $s(0) = s_0 + v_0(0) - \frac{1}{2}g(0^2) = s_0$. Also, s_0, v_0, g are constants, so:

$$v(t) = s'(t) = (s_0)' + (v_0 t)' - (\tfrac{1}{2}gt^2)' = v_0 - gt,$$

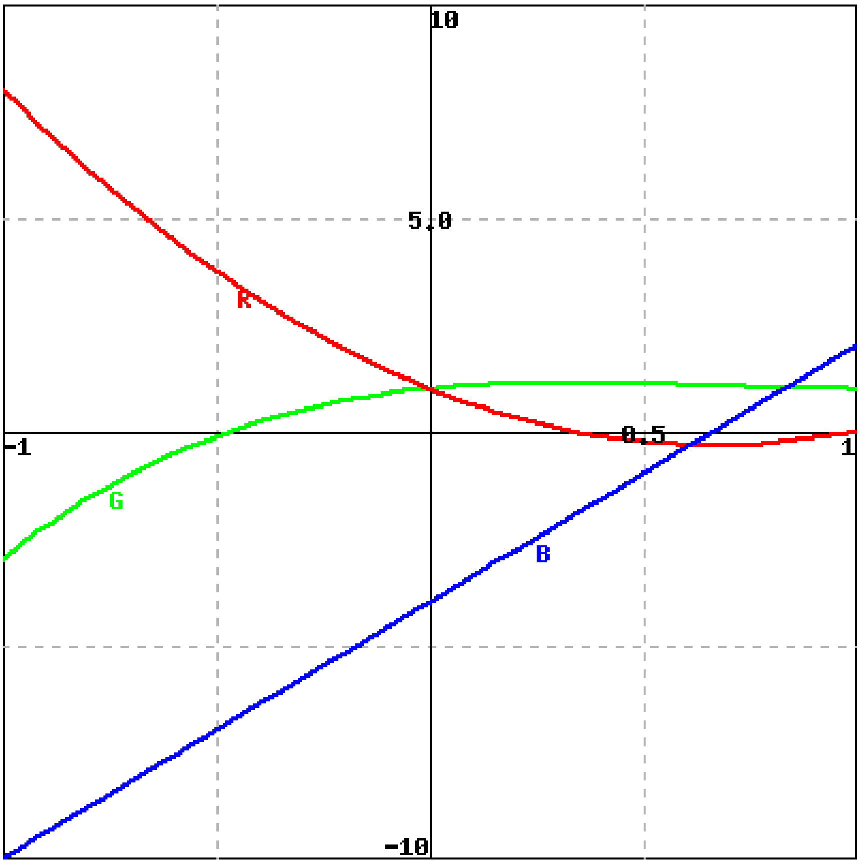
and indeed the initial velocity $v(0) = v_0$. The acceleration is $a(t) = v'(t) = -g$, which is the desired constant in the correct (downward) direction. Finally, the jerk is $j(t) = 0$, which is correct because gravity pulls steadily and never jerks.

EXAMPLE: Given standard gravity of 32 ft/sec^2 and initial height $s_0 = 5$ ft, how fast to throw a ball upward so that it stays airborne for 5 sec? The equation becomes $s(t) = 5 + v_0 t - 16t^2$, with v_0 an unknown constant. Landing at 5 sec means $s(5) = 0$, that is $5 + v_0(5) - 16(5^2) = 0$, and solving, $v_0 = 79$ ft/sec. (This is $79/1.47 \cong 54$ mph!)

How high will the ball go from such a throw? At the instant $t = t_1$ when the ball reaches the top of its arc, its velocity is zero. That is: $v(t_1) = 79 - 32t_1 = 0$, and $t_1 = \frac{79}{32} \cong 2.47$ sec. (This is not quite half the 5 sec interval, because the ball started out at $s_0 = 5$ ft.) The height at this instant is $s(t_1) = 297\frac{35}{64} \cong 297.5$ ft. It would take a baseball pitcher to throw a ball that high.

Note that the graph $s = 5 + 79t + 16t^2$ is a downward-curving parabola, but this is not the trajectory of the ball, which is going straight up and down. For $t < t_1$, the height $s(t)$ is increasing, and the velocity $v(t) = 79 - 32t$ is positive; for $t > t_1$, $s(t)$ is decreasing, and $v(t)$ is negative.

Consider the image below.



Click on the graph to open it in a new window.

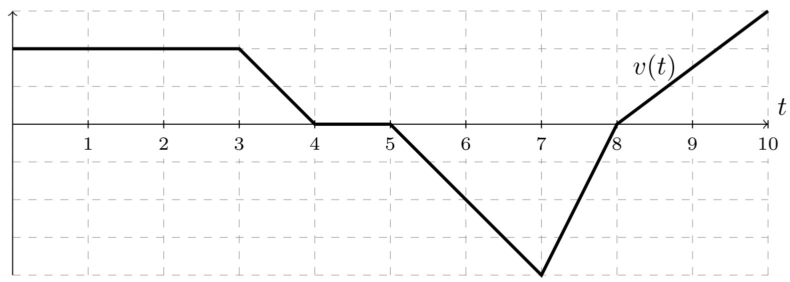
Identify the graphs B (blue), R (red) and G (green) as the graphs of a position function, the corresponding velocity function, and the corresponding acceleration function.

[ ] is the graph of the position function

[ ] is the graph of the velocity function

[ ] is the graph of the acceleration function

3. (3+3+3+3+3+4=19 points) The figure below shows the velocity $v(t)$ of a particle moving on a horizontal coordinate line, for t in a closed interval $[0, 10]$.



Fill in the following blanks. No partial credit available. No work needed. Use interval notation where appropriate. When grading interval notation give full credit for all endpoint selections (i.e., don′ be picky about (, [,),]).

(a) The particle is moving forward during: $(0, 4) \cup (8, 10)$

(b) The particle's speed is increasing during: $(5, 7) \cup (8, 10)$

(c) The particle has positive acceleration during: $(7, 8) \cup (8, 10)$ or $(7, 10)$ is okay.

(d) The particle has zero acceleration during: $(0, 3) \cup (4, 5)$

(e) The particle achieves its greatest speed at: $t = 7$

(f) The particle stands still for more than an instant during: $(4, 5)$

Extra room to work: