MTH320 Final Exam

Mathematics, MSU

10:20am-12:20am, August 18, 2017

1. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$ converges or diverges, and justify your answer. [10 points]

2. Let $\mathbb{R}$ denote the real number set, $\mathbb{Q}$ denote the rational number set, and $h(x) = \begin{cases} x & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.
Prove $h(x)$ is continuous at $x = 0$. [10 points]

3. Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist x,y in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. Hint: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$ and the *Intermediate Value Theorem.* [10 points]

4. Show $f(x) = \sin(x)$ is uniformly continuous on $\mathbb{R}$. [10 points]

5. Find $\lim_{x \to -\infty} \frac{x^3}{|x|}$ and prove it by verifying the $\epsilon - N$ (or $M - N$) definition. [10 points]

6. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n} x^n$. (1) Find the radius of convergence [5 points]; (2) Determine the domain of convergence [5 points].

7. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$. Does $\{f_n(x)\}$ converge uniformly on $[0, 1]$? Prove your assertion. [10 points]

8. Let $f_n(x) = \frac{x^2}{n}$. Show $\{f_n(x)\}$ is uniformly Cauchy on $[-1, 1]$ thus uniformly convergent. [15 points]

9. Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function. [15 points]

1. $\forall n \geq 3$, $0 < \log n < \sqrt{n} \Rightarrow 0 < \frac{1}{\sqrt{n}} < \frac{1}{\log n} \Rightarrow 0 < \frac{1}{n} < \frac{1}{\sqrt{n}\log n}$

$\sum_{n=3}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}\log n}$ diverges by the comparison test

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} = \frac{1}{\sqrt{2}\log 2} + \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}\log n}$ diverges.

10

2. $\forall \epsilon > 0$, $\exists \delta = \epsilon > 0$, s.t. $\forall x \in \mathbb{R}$ and $|x-0| < \delta$, we have:

case 1. $x \in \mathbb{Q}$, $|h(x) - h(0)| = |x-0| < \delta = \epsilon$

case 2. $x \in \mathbb{R} | \mathbb{Q}$, $|h(x) - h(0)| = |0-0| = 0 < \epsilon$

$\Rightarrow h(x)$ is continuous at $x = 0$.

10

3. Consider $g(x) = f(x+1) - f(x)$ on $[0,1]$.

$g(0) = f(1) - f(0)$, $g(1) = f(2) - f(1) \Rightarrow g(0) + g(1) = f(2) - f(0) = 0$

case 1: $g(0) = 0 \Rightarrow f(1) - f(0) = 0$. so $\exists\, x = 0$ and $y = 1$ in $[0,2]$,

$|y-x| = |1-0| = 1$ and $f(1) = f(0)$.

10

case 2: $g(0) \neq 0$. without loss of generality, suppose $g(0) < 0$.

Then $g(1) = -g(0) > 0$. $\Rightarrow 0 \in [g(0), g(1)]$.

$g(x)$ is continuous on $[0,1]$, so by the intermediate value theorem,

$\exists\, r \in [0,1]$, s.t. $g(r) = 0 \Rightarrow f(r+1) - f(r) = 0 \Rightarrow$

$\exists\, x = r$ and $y = r+1$ in $[0,2]$, s.t. $|y-x| = 1$ and $f(x) = f(y)$.

4. $\forall\, x < y \in \mathbb{R}$, $f(x) = \sin x$ is continuous on $[x, y]$ and differentiable on (x, y).

So by the mean value theorem, $\exists\, r \in (x, y)$, s.t. $f'(r) = \dfrac{f(x) - f(y)}{x - y}$

$\Rightarrow |f(y) - f(x)| = |f'(r)| \cdot |y - x| = |-\cos r| \cdot |y - x| \leq |x - y|$. 4

$\forall\, \epsilon > 0$, $\exists\, \delta = \epsilon > 0$, s.t. $\forall\, x, y \in \mathbb{R}$ and $|x - y| < \delta$, we have:

case 1: $x = y \Rightarrow |f(x) - f(y)| = 0 < \epsilon$

Case 2: $x \neq y$, without loss of generality, assume $x < y$, 6

$$|f(x) - f(y)| \leq |x - y| < \delta = \epsilon$$

$\Rightarrow f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

5. $\lim\limits_{x \to -\infty} \dfrac{x^3}{|x|} = \lim\limits_{x \to -\infty} (-x^2) = -\infty$. 2

$\forall\, M > 0$, $\exists\, N = -\sqrt{M} < 0$, s.t. $\forall\, x < N < 0$, we have

$f(x) = \dfrac{x^3}{|x|} = \dfrac{x^3}{-x} = -x^2 < -N^2 = -M \Rightarrow \lim\limits_{x \to -\infty} f(x) = -\infty$ 8

6. (1) $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left| \dfrac{\frac{(-1)^{n+1}}{(n+1) \cdot 4^{n+1}}}{\frac{(-1)^n}{n \cdot 4^n}} \right| = \frac{1}{4} \Rightarrow \beta = \frac{1}{4}$ and $R = \frac{1}{\beta} = 4$ radius of convergence

(2) Firstly, the series converges in $(-4, 4)$ 5

$x = 4$, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n \cdot 4^n} x^n = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ converges by the alternating series theorem

$x = -4$, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n \cdot 4^n} x^n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges

$\Rightarrow$ Domain of convergence: $(-4, 4]$ 5

7. One can expect $\lim_{n\to\infty} f_n(x) = x^2$ by inspection.

Consider $\left|f_n(x) - x^2\right| = \left| -2\frac{x}{n} + \frac{1}{n^2}\right| \le \frac{2x}{n} + \frac{1}{n^2} \le \frac{2}{n} + \frac{1}{n^2}$, $\frac{1}{n^2} \le \frac{1}{n}$ $\Big\} \Rightarrow$

$\left|f_n(x) - x^2\right| \le \frac{2}{n} + \frac{1}{n} = \frac{3}{n}$, $\forall x \in [0,1]$

$\forall \epsilon > 0$, $\exists N = \frac{3}{\epsilon} > 0$, s.t. $\forall n > N$ and $\forall x \in [0,1]$, we have

$\left|f_n(x) - x^2\right| \le \frac{3}{n} < \frac{3}{N} = \epsilon \Rightarrow f_n(x) \to x^2$ uniformly on $[0,1]$.

10

8. $\forall n > m$, consider $\left|f_n(x) - f_m(x)\right| = \left|\frac{x^2}{n} - \frac{x^2}{m}\right| = x^2\left|\frac{m-n}{nm}\right| = x^2 \cdot \frac{n-m}{n \cdot m}$

$0 < n-m < n \Rightarrow 0 < \frac{n-m}{n} < 1 \Rightarrow 0 < \frac{n-m}{n\cdot m} < \frac{1}{m}$ $\Big\}$ 5

$\Rightarrow \left|f_n(x) - f_m(x)\right| \le \frac{x^2}{m} \le \frac{1}{m}$, $\forall x \in [-1,1]$. 5

$\forall \epsilon > 0$, $\exists N = \frac{1}{\epsilon} > 0$, s.t. $\forall n > m > N$ and $\forall x \in [-1,1]$, we have

$\left|f_n(x) - f_m(x)\right| \le \frac{1}{m} < \frac{1}{N} = \epsilon \Rightarrow \{f_n(x)\}$ is uniformly Cauchy on $[-1,1]$. 5

9. $\left|f(x) - f(y)\right| \le (x-y)^2 \Rightarrow \frac{|f(x)-f(y)|}{|x-y|} \le |x-y|$, when $x \ne y$

$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y| \Rightarrow -|x-y| \le \frac{f(x)-f(y)}{x-y} \le |x-y|$, when $x \ne y$. 5

$\lim_{y\to x}\left(-|x-y|\right) = 0 = \lim_{y \to x}\left(|x-y|\right) \Rightarrow \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = 0$ by the sandwich theorem 5

$\Rightarrow f'(x) = 0$, $\forall x \in \mathbb{R} \Rightarrow f(x)$ is a constant function.

5