MTH320 Midterm Exam

Mathematics, MSU

10:20am-12:20am, July 28, 2017

1. Show $\sqrt[3]{5-\sqrt{3}}$ is not a rational number. [10 points]

2. Assuming $0 \neq 1$, show $0 < 1$. Refer to the following properties exactly in your proof. [10 points]

(A1) $a+(b+c)=(a+b)+c$. (A2) $a+b=b+a$. (A3) $\forall a$, $a+0=a$. (A4) $\forall a$, $\exists -a$, s.t. $a+(-a)=0$.
(M1) $a(bc)=(ab)c$. (M2) $ab=ba$. (M3) $\forall a$, $a\cdot 1=a$. (M4) $\forall a\neq 0$, $\exists a^{-1}$, s.t. $aa^{-1}=1$. (DL) $a(b+c)=ab+ac$.
(O1) Either $a\leq b$ or $b\leq a$. (O2) $a\leq b$ and $b\leq a \Rightarrow a=b$. (O3) $a\leq b$ and $b\leq c \Rightarrow a\leq c$. (O4) $a\leq b \Rightarrow a+c\leq b+c$. (O5) $a\leq b$ and $0\leq c\Rightarrow ac\leq bc$.
(P1) $a+c=b+c\Rightarrow a=b$. (P2) $\forall a$, $a\cdot 0=0$. (P3) $(-a)b=-ab$. (P4) (-a)(-b)=ab. (P5) $ac=bc$ and $c\neq 0\Rightarrow a=b$. (P6) $ab=0\Rightarrow a=0$ or $b=0$.
(PO1) $a\leq b\Rightarrow -b\leq -a$. (PO2) $a\leq b$ and $c\leq 0\Rightarrow bc\leq ac$. (PO3) $0\leq a$ and $0\leq b\Rightarrow 0\leq ab$. (PO4) $\forall a$, $0\leq a^2$.

3. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and $A+B=\{a+b:\ a\in A,\ b\in B\}$. Prove that $\inf(A+B)=\inf A+\inf B$, viz., $\inf(A+B)\leq \inf A+\inf B$ and $\inf(A+B)\geq \inf A+\inf B$. [15 points]

4. Prove $\lim\limits_{n\to\infty}\dfrac{4n+3}{7n-5}=\dfrac{4}{7}$ by the rigorous definition of limit. [14 points]

5. Given that $\lim\limits_{n\to\infty}s_n=+\infty$ and $\inf\{t_n:\ n\in\mathbb{N}\}>-\infty$, prove $\lim\limits_{n\to\infty}(s_n+t_n)=+\infty$ by the rigorous definition of limit. [10 points]

6. Let $\{s_n\}$ be a sequence such that $|s_{n+1}-s_n|<2^{-n}$, $\forall n\in\mathbb{N}$. Prove that $\{s_n\}$ is a Cauchy sequence hence convergent. [15 points]

7. Let $\{s_n\}$ be a sequence such that $s_n=n\cos\dfrac{n\pi}{4}$.

   (a) Give the set of subsequential limits of $\{s_n\}$. [6 points]

   (b) Find $\limsup s_n$ and $\liminf s_n$, and state the reason. [6 points]

8. Determine whether the following series converge or diverge, and justify your answers.

   (a) $s_n=\sum\limits_{n=1}^{\infty}\dfrac{n^2}{n!}$ [7 points]

   (b) $s_n=\sum\limits_{n=1}^{\infty}\left[\dfrac{3}{(-1)^n-4}\right]^n$ [7 points]

1, otherwise, $r = \sqrt[3]{5-\sqrt{3}}$ is a rational number.

$r^3 = 5-\sqrt{3} \Rightarrow (r^3-5)^2 = 3 \Rightarrow r^6 - 10r^3 + 22 = 0$ 5

By the rational zeros theorem, $r = \frac{c}{d}$, $c/22$ and $d/1$

$\Rightarrow c = \pm 1, \pm 22, \pm 2, \pm 11$, $d = \pm 1 \Rightarrow r = \pm 1, \pm 2, \pm 11, \pm 22$ 5

None of them is a solution of the equation.

2, By (PO4), $\forall a, 0 \le a^2 \Rightarrow 0 \le 1^2$ 5

By (M3), $1^2 = 1 \cdot 1 = 1$ 5

$$\left.\begin{array}{l} 0 \le 1^2 \\ 1^2 = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 \le 1 \\ 0 \neq 1 \end{array}\right\} \Rightarrow 0 < 1.$$

3、(i) $\forall s \in A+B$, $\exists a \in A$ and $b \in B$, s.t. $s = a+b$

$$\left.\begin{array}{l} a \in A \Rightarrow \inf A \le a \\ b \in B \Rightarrow \inf B \le b \end{array}\right\} \Rightarrow \inf A + \inf B \le a+b = s, \quad \forall s \in A+B$$

$$\Rightarrow \inf A + \inf B \le \inf(A+B)$$ 7

(ii) $\forall \epsilon > 0$, $\exists a \in A$, s.t. $\inf A + \frac{\epsilon}{2} > a$ and $\exists b \in B$, s.t. $\inf B + \frac{\epsilon}{2} > b$ $\Rightarrow a+b < \inf A + \inf B + \epsilon$ ①

$a \in A, b \in B \Rightarrow a+b \in A+B \Rightarrow \inf(A+B) \le a+b$ ②

Combining ① and ②: $\inf(A+B) < \inf A + \inf B + \epsilon$, $\forall \epsilon > 0$ 8

$$\Rightarrow \inf(A+B) \le \inf A + \inf B$$

Above all, $\inf(A+B) = \inf A + \inf B$.

4. We need: $\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| < \epsilon \iff \left|\frac{41}{7(7n-5)}\right| < \epsilon$.

$n \in \mathbb{N} \Rightarrow 7n-5 > 0$. so $\left|\frac{41}{7(7n-5)}\right| = \frac{41}{7(7n-5)} < \epsilon \iff \frac{41}{7\epsilon} < 7n-5$

$\iff \frac{1}{7}\left(\frac{41}{7\epsilon}+5\right) < n$.

$\forall \epsilon > 0$, $\exists N = \frac{1}{7}\left(\frac{41}{7\epsilon}+5\right) > 0$, s.t. $\forall n > N$, $\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| < \epsilon$

$\Rightarrow \lim_{n\to\infty} \frac{4n+3}{7n-5} = \frac{4}{7}$. 14

5. Denote $m = \inf\{t_n : n \in \mathbb{N}\} > -\infty$. $\forall n \in \mathbb{N}$, $m \leq t_n$. 3

$\forall M > 0$. consider $M_1 = \max\{1, M-m\} > 0$. 3

$\lim_{n\to\infty} s_n = +\infty \Rightarrow \exists N > 0$, $\forall n > N$, $s_n > M_1$

$\Rightarrow s_n + t_n > M_1 + m \geq (M-m) + m = M$ 4

$\Rightarrow \lim_{n\to\infty} (s_n + t_n) = +\infty$.

6. We need: $\forall n > m > N$. $|s_n - s_m| < \epsilon$

$$\Leftrightarrow \quad \epsilon > |s_n - s_m| = \left| \sum_{k=m}^{n-1} (s_{k+1} - s_k) \right| \quad \cdots ①$$

$$\left. \begin{aligned} \left| \sum_{k=m}^{n-1} (s_{k+1} - s_k) \right| &\le \sum_{k=m}^{n-1} |s_{k+1} - s_k| \\ |s_{k+1} - s_k| &< 2^{-k} \end{aligned} \right\} \Rightarrow \left| \sum_{k=m}^{n-1} (s_{k+1} - s_k) \right| < \sum_{k=m}^{n-1} 2^{-k}$$

$$= \frac{2^{-m}\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}{1 - \frac{1}{2}} = 2^{1-m}\left(1 - \left(\frac{1}{2}\right)^{n-m}\right) \le 2^{1-m}$$

So one can let $2^{1-m} < \epsilon$ and it leads to ①.

$$\Rightarrow \log_2(2^{1-m}) < \log_2 \epsilon \Rightarrow 1-m < \log_2 \epsilon \Rightarrow m > 1 - \log_2 \epsilon.$$

Formal proof: $\forall \epsilon > 0$, $\exists N = \max\{1, 1 - \log_2 \epsilon\} > 0$, 3 s.t.

$$\forall n > m > N, \quad |s_n - s_m| = \left| \sum_{k=m}^{n-1} (s_{k+1} - s_k) \right| \le \sum_{k=m}^{n-1} |s_{k+1} - s_k|$$

$$\left. \begin{aligned} &< \sum_{k=m}^{n-1} 2^{-k} = \frac{2^{-m}\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}{1 - \frac{1}{2}} = 2^{1-m}\left(1 - \left(\tfrac{1}{2}\right)^{n-m}\right) \le 2^{1-m} \\ &m > N \ge 1 - \log_2 \epsilon \Rightarrow 1 - m \le \log_2 \epsilon \Rightarrow 2^{1-m} \le 2^{\log_2 \epsilon} = \epsilon \end{aligned} \right\}$$

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$$\Rightarrow |s_n - s_m| < \epsilon.$$

$\Rightarrow \{s_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence hence convergent.

7. (a) $n_k = 8k$, $s_{8k} = 8k \cdot \cos\frac{8k\pi}{4} = 8k\cos 2k\pi = 8k \Rightarrow \lim\limits_{k\to\infty} s_{8k} = +\infty$ 2

$n_k = 8k+2$, $s_{8k+2} = (8k+2)\cos\frac{(8k+2)\pi}{4} = (8k+2)\cos\left(2k\pi + \frac{\pi}{2}\right)$

$= (8k+2)\cos\frac{\pi}{2} = 0 \Rightarrow \lim\limits_{k\to\infty} s_{8k+2} = 0$ 2

$n_k = 8k+4$, $s_{8k+4} = (8k+4)\cos(2k\pi + \pi) = -(8k+4) \Rightarrow \lim\limits_{k\to\infty} s_{8k+4} = -\infty$ 2

$S = \{+\infty, 0, -\infty\}$.

(b) $\limsup s_n = \sup S = +\infty$ 3

$\liminf s_n = \inf S = -\infty$. 3

8. (a) ratio test: $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}}$

$= \lim\limits_{n\to\infty} \dfrac{\frac{n+1}{n!}}{\frac{n^2}{n!}} = \lim\limits_{n\to\infty} \frac{n+1}{n^2} = \lim\limits_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n^2}\right) = 0 < 1$.

$\Rightarrow \sum\limits_{n=1}^{\infty} \frac{n^2}{n!}$ converges. 7

(b) $a_n = \left[\frac{3}{(-1)^n - 4}\right]^n$ and consider $\lim\limits_{n\to\infty} a_n$

$n_k = 2k$, $a_{2k} = \left[\frac{3}{(-1)^{2k}-4}\right]^{2k} = (-1)^{2k} = 1 \Rightarrow \lim\limits_{k\to\infty} a_{2k} = 1$.

$n_k = 2k+1$, $a_{2k+1} = \left[\frac{3}{(-1)^{2k+1}-4}\right]^{2k+1} = \left(-\frac{3}{5}\right)^{2k+1} \Rightarrow \lim\limits_{k\to\infty} a_{2k+1} = 0$

$1 \neq 0 \Rightarrow \lim\limits_{n\to\infty} a_n$ does not exist $\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverges. 7