8.8 (a) $\lim\limits_{n\to\infty}(\sqrt{n^2+1}-n)=0$

$\forall \epsilon>0$, $\exists N=\frac{1}{\epsilon}>0$, $\forall n>N$: $\left|\sqrt{n^2+1}-n\right|=\left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\right|$

$=\frac{1}{\sqrt{n^2+1}+n} < \frac{1}{0+n}=\frac{1}{n}<\frac{1}{N}=\epsilon \Rightarrow \lim\limits_{n\to\infty}(\sqrt{n^2+1}-n)=0$.

9.12 (a) $L=\lim\left|\frac{s_{n+1}}{s_n}\right| \Rightarrow \forall\epsilon>0, \exists N_1>0, \forall n>N_1, \left|\left|\frac{s_{n+1}}{s_n}\right|-L\right|<\epsilon$

$\Rightarrow L-\epsilon<\left|\frac{s_{n+1}}{s_n}\right|<L+\epsilon$. $L<1$, so one can take $0<\epsilon<1-L \Rightarrow 0<L+\epsilon<1$.

Denote $a=L+\epsilon$. $\left|\frac{s_{n+1}}{s_n}\right|<a \Rightarrow |s_{n+1}|<a|s_n| \Rightarrow |s_n|<a^{n-N_1}|s_{N_1}|$, $\forall n>N_1$ ... (*)

Consider $N_2=N_1+\frac{\ln\epsilon-\ln|s_{N_1}|}{\ln a}=N_1+\log_a\frac{\epsilon}{|s_{N_1}|}$, and $N=\max\{N_1,N_2\}\geq N_1>0$.

$\forall n>N$. $n>N_1 \Rightarrow |s_n|<a^{n-N_1}|s_{N_1}|$ ... ①

$n>N_2 \Rightarrow n-N_1>N_2-N_1=\log_a\frac{\epsilon}{|s_{N_1}|} \Rightarrow a^{n-N_1}<a^{N_2-N_1}$ because $0<a<1$.

$\Rightarrow a^{n-N_1}<a^{N_2-N_1}=a^{\log_a\frac{\epsilon}{|s_{N_1}|}}=\frac{\epsilon}{|s_{N_1}|} \Rightarrow a^{n-N_1}|s_{N_1}|<\epsilon$ ... ②

① and ② $\Rightarrow |s_n|<\epsilon$, $\forall n>N \Rightarrow \lim\limits_{n\to\infty}s_n=0$.

r.m.: One can also use the comparison test after (*).

10.7. $\forall n\in\mathbb{N}$, $\epsilon_n=\frac{1}{n}>0 \Rightarrow M_n=\sup S-\epsilon_n<\sup S$

$\Rightarrow \exists s_n\in S$, s.t. $M_n<s_n\leq\sup S \Rightarrow \sup S-\epsilon_n<s_n\leq\sup S$

$\Rightarrow |s_n-\sup S|<\epsilon_n=\frac{1}{n}$, $\forall n\in\mathbb{N}$.

We use such s_n to construct the sequence $\{s_n\}$.

$\forall\epsilon>0$, $\exists N=\frac{1}{\epsilon}>0$, $\forall n>N$, $|s_n-\sup S|<\frac{1}{n}<\frac{1}{N}=\epsilon$

$\Rightarrow \lim\limits_{n\to\infty}s_n=\sup S$.