RESEARCH STATEMENT

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My research focuses on problems in smooth 4-dimensional topology. The primary goal of which is to understand smooth structures on 4-dimensional topological manifolds with and without boundary. This dimension is particularly interesting because it is the only dimension admitting infinitely many smoothings of the same topological manifold. Furthermore, four is “small enough” that I can visualize the spaces in question combinatorially.

The primary tool that I use to understand and distinguish, or identify, such manifolds is 4-dimensional handlebody theory. Any smooth n-manifold can be cut into simple pieces, each diffeomorphic to an n-dimensional ball, $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, glued together along portions of their sphere boundaries. Each ball is known as a handle and the way in which these handles are attached to one another encodes all of the complexities of the given manifold. A collection of handles, together with the necessary attaching data, is known as a handle decomposition for the manifold. Unfortunately, such handle decompositions are far from unique. Therefore given two decompositions, one fundamental problem is how to determine when they specify the same smooth n-manifold.

To aid in answering this question, I use techniques arising from differential geometry. Since we are dealing with smooth manifolds, geometric structures are available to further refine our understanding. Answers to questions like “is the given manifold a complex surface?” or, “can it be imparted with a symplectic structure?” provide considerable insight.

Some of these structures can actually be encoded combinatorially using handle theory. Eliashberg and Gompf [Gom98] show us how to see when a 4-manifold with boundary, presented as a handle decomposition, can be viewed as a Stein domain. This means that the 4-manifold in question can be cut from a holomorphic embedding of an open manifold in $\mathbb{C}^N$ for N large. Manifolds that admit Stein structures are of particular interest because they can always be placed inside smooth 4-manifolds without boundary with very rigid properties [LM97, AO02]. In fact, such domains adhere to strict requirements about embedded surfaces within them. Finding smooth surfaces violating these requirements then precludes the existence of such a Stein structure - which, in turn, obstructs the existence of a diffeomorphism to any other smoothing admitting such a structure (see for example [AM97]).

In particular cases, such geometric information greatly curtails the number of smooth manifolds with given properties. For instance, the diffeomorphism types of 4-manifolds admitting symplectic fillings of lens spaces (certain quotients of the 3-sphere) equipped with specific contact structures are completely enumerated by Lisca [Lis08]. Therefore, if a given 4-manifold with lens space boundary fills the “standard” contact structure on the lens space, then one only has to look to Lisca’s list to determine the manifold in question - a much simpler problem. Using this approach, I prove the following theorem relating two families of rational balls. Each family is defined by a handle decomposition. The first, defined by Yamada [Yam07] and denoted by $A_{m,n}$, and the second, constructed much earlier, by Casson and Harer [CH81] and denoted by $B_{p,q}$.

Theorem 1 ([Wil14]). *For each pair of relatively prime positive integers, (m, n), $A_{m,n}$ carries a Stein structure, $\widetilde{J}_{m,n}$, filling a contact structure contactomorphic to the standard contact structure $\bar{\xi}_{st}$ on the lens space $\partial A_{m,n}$. In particular, each $A_{m,n} \approx B_{p,q}$ if and only if $\partial A_{m,n} \approx \partial B_{p,q}$.*

This theorem is an example of how the differential geometry of a space can be used to determine that space. That said, I do not have to appeal to Lisca's classification result. In fact, by using surgery descriptions of the lens space boundaries to define explicit maps, I prove directly that the two families are diffeomorphic.

Theorem 2 ([Wil14])**.** *Let $(m,n) = A(p-q,q)$ for some $p > q > 0$ relatively prime. Then there exists a diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$ such that f carries the belt sphere, μ_1, of the single 2-handle in $B_{p,q}$ to a knot in $\partial A_{m,n}$ bounding a disk in $A_{m,n}$ (see Figure 2). Moreover, carving $A_{m,n}$ along $f(\mu_1)$ gives $S^1 \times B^3$.*

Corollary 3 ([Wil14])**.** *f extends to a diffeomorphism $\tilde{f} : B_{p,q} \to A_{m,n}$.*

1. Handle Decompositions of Certain Rational Balls

A 4-manifold X is a *rational ball* if its singular homology groups computed with $\mathbb{Q}$-coefficients agree with those of the 4-ball:

$$\widetilde{H}_i(X;\mathbb{Q}) = \widetilde{H}_i(B^4;\mathbb{Q}) = 0.$$

A straightforward way to construct rational balls is to consider Mazur-type manifolds. X is *Mazur-type* if X admits a handle decomposition consisting of a single 0-handle, a single 1-handle and a single 2-handle. Provided the 2-handle geometrically links with the 1-handle, the resulting 4-manifold will necessarily be a rational ball regardless of the framing of the 2-handle. Rational balls always have rational sphere boundaries. When a rational sphere is known to bound more than one 4-manifold, one can interchange these two spaces, and investigate the result.

Fintushel and Stern and later Park [FS97, Par97] define a smooth operation, the rational blow-down, on 4-manifolds containing certain configurations of spheres by removing a neighborhood of these spheres and replacing them by a rational ball with the same boundary, a lens space. In the presence of a symplectic structure, and a symplectic configuration of spheres, both operations can be performed symplectically [Sym98, Sym01]. Under mild assumptions (see [FS97], [Par97] for details), nontrivial solutions to the Seiberg-Witten equations on the original 4-manifold induce nontrivial solutions on the surgered manifold. With this fact in place, exotic 4-manifolds can be constructed (see for instance [SS05]). An understanding of the rational blow-down at the level of handlebodies is important to facilitate work with these examples concretely, as well as to allow handle constructions of new examples.

The basis of both of these operations relies on a construction of Casson and Harer [CH81]. They prove that the lens space $L(p^2, pq-1)$ bounds a Mazur-type rational 4-ball $B_{p,q}$. Their construction relies on the fact that each lens space $L(p,q)$ is a 2-fold cover of S^3 branched over a 2-bridge knot (or link) constructed from the continued fraction expansion of the fraction $-p/q$. In the case of $L(p^2, pq-1)$, they show that the 2-bridge knot associated to $-p^2/(pq-1)$ is smoothly slice. Meaning that there is a proper embedding of the 2-disk in B^4 whose boundary is the aforementioned 2-bridge knot in S^3. Then, $B_{p,q}$ is taken to be the 2-fold cover of B^4 branched over the embedded disk. The authors go on to show that such a cover can be realized as a Mazur-type manifold.

The continued fraction expansion also gives rise to another 4-manifold, $C_{p,q}$, bounding $L(p^2, pq-1)$ - namely a plumbing of disk-bundles over the 2-sphere whose Euler classes are chosen according to the continued fraction expansion. If $C_{p,q} \subset X^4$, then the *rational blow-down of X along $C_{p,q}$* is given by removing $C_{p,q}$ from X and gluing back $B_{p,q}$. The handle description of $C_{p,q}$ is fully understood. However, this leads to a natural question. What is a handle description for $B_{p,q}$? More broadly,

Question 4. Are Mazur-type rational balls bounding $L(p^2, pq-1)$ unique up to diffeomoprhism type?

One possible approach to answering this question (in the negative) is to find knots, $K \subset S^1 \times S^2$, that admit integer surgeries giving the lens space $L(p^2, pq-1)$. Any such knot immediately gives rise to a Mazur-type rational ball bounding $L(p^2, pq-1)$ since we can attach a single 2-handle along K, viewed as being embedded on the boundary of $S^1 \times B^3$ with framing equal to the surgery coefficient realizing the lens space. Here we are using that $S^1 \times B^3$ is the result of attaching a 1-handle to a 0-handle. By assumption, manifolds constructed in this way will have the appropriate boundary and, by construction, they will be Mazur-type rational balls. Once one has such examples, there is still the difficult task of determining whether they represent rational balls which are not diffeomorphic to the examples constructed by Casson and Harer.

There are two known families of such framed knots in $S^1 \times S^2$ each giving rise to rational balls bounding lens spaces. The first family, investigated by Lekili-Maydanskiy in [LM12] is known to give rise to rational balls diffeomorphic to $B_{p,q}$. A proof of this can be found in [LM12]. See the lefthand 4-manifold in Figure 2 for the handle decomposition (and thus framed knot in $S^1 \times S^2$) giving $B_{p,q}$. Yamada finds another family of knots in $S^1 \times S^2$ yielding lens space surgeries [Yam07]. The resulting rational balls, known as $A_{m,n}$, also bound $L(p^2, pq-1)$ when m and n are chosen appropriately. In fact, Yamada defines a symmetric, involutive function A on the set of pairs of relatively prime positive integers so that if $A(p-q, q) = (m, n)$ then $\partial A_{m,n} \approx \partial B_{p,q}$. See the righthand 4-manifold in Figure 2 for the handle decomposition (and thus framed knot in $S^1 \times S^2$) giving $A_{m,n}$.

Given these two families of rational balls bounding lens spaces, a natural question arises (posed in [KY14]).

Question 5. Are $B_{p,q}$ and $A_{m,n}$ diffeomorphic?

Or, is the answer to Question 4 "no"? A major part of my work on these rational balls relates to answering Question 5. In particular, I show that these two families do coincide.

I have two distinct approaches that allow me to conclude this fact. Each relies on fundamentally different techniques to arrive at the same conclusion. The first method is to use the approach that Lekili and Maydanskiy employ to prove that the 4-manifold on the lefthand side of Figure 2 is, in fact, giving $B_{p,q}$. Therein, they use that the handlebody in question admits a Stein structure filling the universally tight contact structure, $\bar{\xi}_{\text{st}}$, on $L(p^2, pq-1)$. This is enough to conclude the result since Lisca proves [Lis08] that $B_{p,q}$ is the only diffeomorphism type of symplectic filling of $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$ with $b_2 = 0$. I show that each $A_{m,n}$ also admits such a Stein structure (Theorem 1) specified by Figure 1.

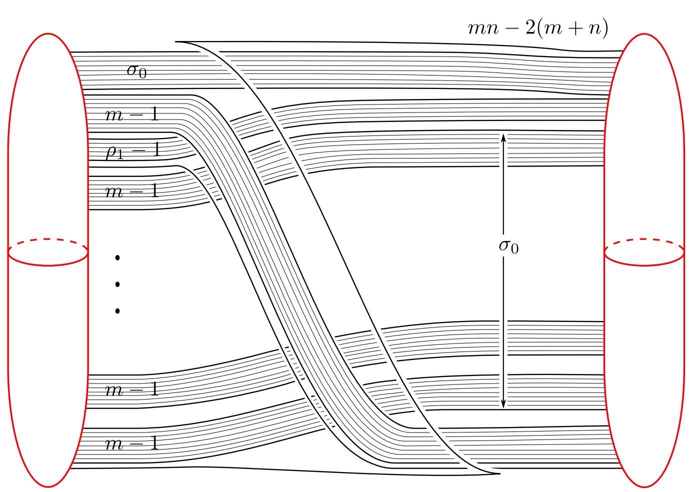

FIGURE 1. $(A_{m,n}, \widetilde{J}_{m,n})$

This result relies on the work of Eliashberg and Gompf outlined in [Gom98] showing that the unique Stein structure on $S^1 \times B^3$ extends across a 2-handle if and only if that 2-handle is attached along a Legendrian knot with Seifert framing (the framing determined by taking a push-off in the direction of an oriented surface bounding the attaching circle) one less than the contact framing (the framing determined by taking a push-off of the attaching circle which is transverse to the contact planes). Figure 1 specifies this Stein structure, $(A_{m,n}, \tilde{J}_{m,n})$, where I am assuming that $m < n$ and that $n = m\sigma_0 + \rho_1$. It is immediate that this determines a Stein domain. What is not immediate is that this handle decomposition still gives $A_{m,n}$ - which it does.

I also construct a diffeomrphism between $B_{p,q}$ and $A_{m,n}$ much more directly. To accomplish this, I make use of the method of "carving" introduced by Akbulut in [Akb77] (see also [Akb14]). This approach relies on building boundary diffeomorphisms which carry the belt spheres of each 2-handle in the domain to slice knots in the target. Given such boundary maps, one can attempt to extend them into the interior of the 4-manifolds in question (i.e. across the co-cores of the 2-handles in the domain). This is successful provided that the boundary diffeomorphism preserves the 0-framings on the belt sphere and the image of the belt sphere induced by the co-core and slice disk respectively. In the case at hand, the existence of such maps allows me to reduce the extension problem from extending a self map of a lens space across a rational ball to a problem of extending a self map on $S^1 \times S^2$ across $S^1 \times B^3$ - a problem which is fully understood [Glu62]. To that end, I prove Theorem 2 which produces a map $f : \partial B_{p,q} \to \partial A_{m,n}$ as in Figure 2. The definition of f is

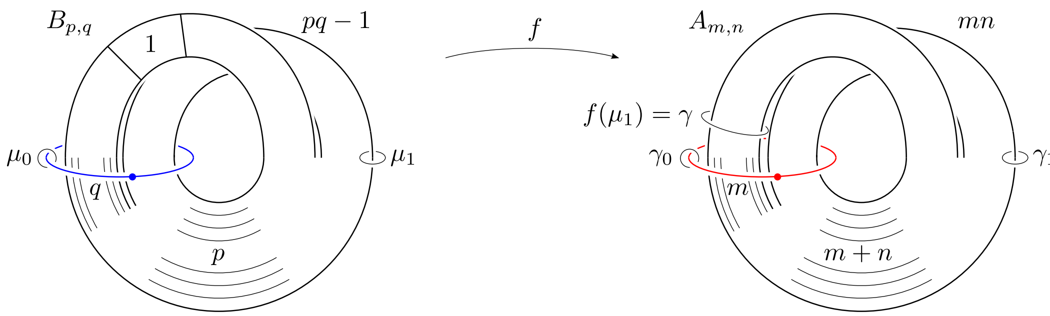

FIGURE 2. The spaces $B_{p,q}$ and $A_{m,n}$.

entirely constructive using explicit handle moves and surgeries [Kir78, FR79]. Furthermore, there is enough freedom in the definition of f so that I can arrange it to be a contactomorphism between the contact structure, $\xi_{J_{p,q}}$, induced by the Stein structure $(B_{p,q}, J_{p,q})$ on $\partial B_{p,q}$ and $\xi_{\tilde{J}_{m,n}}$ induced by $(A_{m,n}, \tilde{J}_{m,n})$ on $\partial A_{m,n}$. To verify this, I show that, for the given f, $\xi_{J_{p,q}}$ and $f^*(\xi_{\tilde{J}_{m,n}})$ are homotopic as 2-plane fields in $\partial B_{p,q}$. [Gom98] provides a complete set of invariants of homotopy classes of 2-plane fields on 3-manifolds as well as the combinatoric means to compute them. [Hon00, Gir00] show that homotopic tight contact structures on lens spaces are isotopic - ensuring that such an f is a contactomorphism.

2. FURTHER PROJECTS

Where my research will go from here splits into two categories. First, there are numerous questions left unanswered from investigating the aforementioned rational balls. Second, there are projects that I'm interested in that will require some of the same tools and techniques but are otherwise unrelated to the project outlined above.

Some of the unanswered questions, resulting from knowing that the rational balls $B_{p,q}$ and $A_{m,n}$ are diffeomorphic whenever their boundaries coincide, can be summarized as follows: Both the method employed to prove Theorem 2 and the method used to prove Corollary 3 give indirect routes to the diffeomorphisms in question. As such, I am left with the following still unanswered:

Question 6. What are the 4-dimensional handle-moves associated with the diffeomorphism between $B_{p,q}$ and $A_{m,n}$ guaranteed in Theorem 1 and Corollary 3?

Such a description is within reach. In fact, given that the boundary diffeomorphism f can be extended through carving, I can use the carving disks as a route to modify the definition of f to give the extension directly. [Akb93] uses this technique successfully to explain the diffeomorphisms first investigated in [Akb77] as well as to generalize the results therein. Theorem 2 is a perfect candidate for this approach.

The answer to Question 6 will also provide a method to explore other families of knots in $S^1 \times S^2$ admitting integral surgeries giving $L(p^2, pq-1)$. Moreover, it could also shed light on the much more subtle question:

Question 7. Are the Stein domains $(B_{p,q}, J_{p,q})$ and $(A_{m,n}, \tilde{J}_{m,n})$ equivalent?

In [LM12], it is shown that lifting $(B_{p,q}, J_{p,q})$ and $(A_{m,n}, \tilde{J}_{m,n})$ to their respective p-fold covers gives rise to equivalent Stein Domains upstairs. That said, it is unknown whether $B_{p,q}$, itself, admits "exotic" Stein structures. In light of Theorem 1, $(A_{m,n}, \tilde{J}_{m,n})$ is a candidate for such a structure.

My research interests extend far beyond questions involving the aforementioned rational balls. There are other projects I am interested in pursuing which my current skill set will aid in answering. One such project has to do with knot surgery and corks. It is known that any pair of homeomorphic non-diffeomorphic closed, simply connected, 4-manifolds are related by a "cork-twist." That is, the diffeomorphism type of one manifold can be changed to the other by locating a contractible manifold, known as a cork, removing it and regluing it by an involution on the boundary which extends as a homeomorphism but not as a diffeomorphism. The first such example was investigated in [Akb91]. General existence proofs can be found in [CFHS96, Kir96, Mat96] and further investigation at the handle level in [AY08]. Therefore, any construction producing homeomorphic non-diffeomorphic closed, simply connected 4-manifolds necessarily arises from a cork twist.

What is often not clear from such examples is how to find that cork. One method to produce such pairs of homeomorphic manifolds is Fintushel and Stern's knot surgery [FS98] whereby a neighborhood of a torus ($T^2 \times D^2$) is removed from a closed 4-manifold and replaced by a homology $T^2 \times D^2$ built from a knot complement in S^3 crossed with a circle. [Akb99] gives a handle description of this process. Even though we know how to perform knot surgery at the handle level, the following remains:

Question 8. Where is the cork in the handle description of knot surgery?

My understanding of handle theory, coupled with the modern techniques I used with success in previous projects, will be invaluable for tackling this question. Initially, I will consider specific cases of knots and closed 4-manifolds. Knowing answers in certain examples will shed considerable light on the knot surgery process. Ultimately, these findings should aid in giving a more complete answer to question 8.

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