MEAN CURVATURE FLOW AND LAGRANGIAN EMBEDDINGS

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1. Introduction

In this note we provide examples of compact embedded lagrangians in $\mathbb{C}^n$ for any $n \geq 2$ that under mean curvature flow develop singularities in finite time. When n is odd the lagrangians can be taken to be orientable. By gluing these lagrangians onto a special lagrangian embedding L we provide examples of compact embedded lagrangians in a Calabi-Yau manifold that under mean curvature flow develop arbitrarily many singularities in finite time. These lagrangians look like the special lagrangian L with "filigree" attached at points.

The construction of these examples has been motivated, at least in part, by the desire to obtain a better understanding of a recent conjecture of Thomas-Yau [T-Y]. Given a compact lagrangian submanifold in a Calabi-Yau manifold with zero Maslov index, Thomas and Yau conjecture that mean curvature flow exists for all time and converges to a smooth special lagrangian submanifold. (Actually to eliminate the possibility that in the limit the lagrangian degenerates into a connect sum, Thomas-Yau restrict the range of the "grading" on the lagrangian. For a more detailed discussion of the conjecture see below.) The examples we construct have a strikingly different behavior. However because they are not known to have zero Maslov index they are not counterexamples to the conjecture. On the other hand *if* they did have zero Maslov index then they would be counterexamples.

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2. The conjecture

Let (N, ω) be a Calabi-Yau manifold with ω the Kähler form. Let σ be a unit parallel section of the canonical bundle. If L is a lagrangian submanifold then:

$$\sigma_{|_L} = e^{i\beta}\text{dvol},$$

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where dvol is the volume form on L for the induced metric and β is an S^1 valued function on L, the *lagrangian angle* or the *phase function*. The closed one-form on L, $d\beta$, has two intrepretations: First, if H denotes the mean curvature vector on L then $H \lrcorner\, \omega = d\beta$. Second the homology class $[d\beta] \in H^1(\Sigma, \mathbb{Z})$ is the Maslov class. In particular, if the Maslov class vanishes then β admits a lift from S^1 to $\mathbb{R}$. A *grading* of L is the choice of such a lift. The *phase* of a homology class $[L]$ is defined by noting that the complex number $\int_L \sigma$ and therefore its phase depend only on $[L]$. If L is graded this phase lifts to define a real number $\phi(L)$. A lagrangian submanifold is called *special lagrangian* if $H = 0$, equivalently, if β is constant, equivalently, if after multiplying σ by a suitable phase factor, Im $\sigma_{|_L} = 0$. If L is special lagrangian then $\beta = \phi(L)$. A special lagrangian submanifold minimizes volume in its homology class.

Basic to the discussion of mean curvature and lagrangian submanifolds is that if the ambient Kähler manifold is Kähler-Einstein (in particular, Calabi-Yau) then mean curvature is an infinitesimal symplectic motion. Thus the mean curvature flow preserves the lagrangian constraint. In the Calabi-Yau case, if the Maslov class vanishes then mean curvature is an infinitesimal hamiltonian motion and mean curvature flow preserves the hamiltonian deformation class of the lagrangian. Because the lagrangian angle is a primitive of mean curvature it is not difficult to show that under mean curvature flow:

$$\frac{d}{dt}\beta = -\Delta\beta,$$

$$\frac{d}{dt}\mathrm{dvol}_L = -|d\beta|^2\, \mathrm{dvol}_L.$$

The maximum principle then shows that the osculation of β, $\sup\beta - \inf\beta$ is non-increasing.

Given a graded lagrangian submanifold L in a Calabi-Yau manifold Thomas and Yau conjecture that mean curvature flow exists for all time and converges to a special lagrangian submanifold in the same hamiltonian deformation class as L. Actually Thomas-Yau modify the conjecture to allow for the possibility that in the limit L may decompose into a connect sum $L_1 \# L_2 \sim L$ of two lagrangians L_1 and L_2. Suppose that for all graded connect sums $L_1 \# L_2 \sim L$ we have:

$$[\phi(L_1), \phi(L_2)] \not\subseteq (\inf_L \beta, \sup_L \beta) \tag{2.1}$$

Then since the osculation of β is non-increasing under mean curvature flow if L satisfies (2.1) such a degeneration cannot occur. Requiring, in addition, that L satisfy (2.1) gives the modified conjecture. Note that in case either L_1 or L_2 is null in homology the phase $\phi(L_i)$ is undefined and condition (2.1) is vacuous.

3. Barriers and development of singularities

For the purposes of this paper we define a *k-barrier for mean curvature flow* (simply, a *k-barrier*) to be a smooth oriented hypersurface M of an oriented Riemannian manifold N with the property that the sum of any k principal curvatures is greater than zero, where $1 \le k \le 2n-1$. We say such a hypersurface is a *strong k-barrier* if there is, in addition, a constant $a > 0$ such that the sum of any k principal curvatures is greater than or equal to a. We say a hypersurface is a *weak k-barrier* if the sum of any k principal curvatures is greater than or equal to zero, where $1 \le k \le 2n-1$. Note that the definitions depend on the choice of orientation.

Let Σ be a smooth compact embedded (or immersed) k-dimensional submanifold of $\mathbb{C}^n$ that lies in the "inside" (as determined by the orientation) component of a k-barrier M. Consider the function $d(x)$, for $x \in \Sigma$, the distance to M,

$$d(x) = \inf_{y \in M} d(x, y).$$

Let Σ_t denote the image, at time t, of Σ under mean curvature flow. Set $d(t) = \inf_{x \in \Sigma_t} d(x)$.

Theorem 3.1. *Suppose that Σ_t is a smooth immersion for $0 < t < t_0$. Then the minimum distance to M, $d(t)$, is an increasing function of t, for $t < t_0$. If M is a strong k-barrier then there is a constant $a > 0$ such that $d'(t) \ge a$, for a.e. $t < t_0$.*

Proof. Fix t and let $x_0 \in \Sigma_t$ be a point where the minimum value of $d(x)$ on Σ_t is attained (x_0 need not be unique). Let l be a geodesic in $\mathbb{C}^n$ with an endpoint at y_0 on M and at x_0 that achieves $\min_x d(x)$. Then l is a line segment that meets both M and Σ_t orthogonally. Translate a neighborhood of y_0 in M along l to x_0. Denote the translated neighborhood U. Then U and Σ_t are tangent at x_0 and so the inward pointing unit normal, ν, to M at y_0 is also normal to Σ_t at x_0. Since $d(x)$ has a minimum at x_0, there is a neighborhood $W \subset \Sigma_t$ of x_0 that lies on one side of U. It follows that the sum of the principle curvatures for ν of Σ_t at x_0 is greater than or equal to the sum of any k principle curvatures of M at y_0 and is therefore positive. Thus at x_0 the mean curvature vector to Σ_t has a component that is inward pointing. The remaining components are tangent to U at x_0. Hence under mean curvature flow $d(t)$ is increasing.

The function $d(t)$ need not be differentiable, however, since it is increasing it is differentiable a.e. in t. By the above reasoning if M is a strong k-barrier then $d'(t) \ge a$, for a.e. t. □

Corollary 3.2. *Under the hypotheses of the theorem if $d(t)$ is not increasing on $(0, t_0)$ then for some $t < t_0$, Σ_t is singular.*

Next, let Σ be a smooth compact embedded (or immersed) k-dimensional submanifold of a Riemannian manifold N that lies in the "inside" (as determined by the orientation) component of a weak k-barrier B. Let Σ_t denote the image, at time t, of Σ under mean curvature flow.

Theorem 3.3. *Suppose that Σ_t is a smooth immersion for $0 < t < t_0$. Then Σ_t lies in the closure of the "inside" (as determined by the orientation) component of B for all $0 < t < t_0$.*

Proof. At a point where Σ_t is tangent to B its mean curvature vector is inward pointing or zero. Therefore under mean curvature flow Σ_t cannot cross B, provided it remains regular. □

4. Interior barriers and diameter

Let $S^{2n-1}(\rho)$ denote the standard sphere of radius ρ in $\mathbb{R}^{2n}$. Suppose that Σ is a closed smoothly embedded hypersurface in $\mathbb{R}^{2n}$ that lies in the non-compact component of $\mathbb{R}^{2n} \setminus S^{2n-1}(\rho)$. Suppose that for all time $0 < t < t_0$ the image Σ_t of Σ under mean curvature flow remains immersed. Then it is well known that there is a t_1, $0 < t_1 \leq t_0$, depending only on ρ, such that for all t, $0 < t \leq t_1$, Σ_t lies in the non-compact component of $\mathbb{R}^{2n} \setminus S^{2n-1}(\rho)$. The sphere $S^{2n-1}(\rho)$ is an interior barrier for Σ under mean curvature flow in the sense that it controls the diameter of Σ_t.

This idea can be generalized to higher codimension submanifolds as follows: Consider a linear $(k+1)$-subspace P of $\mathbb{R}^{2n}$ (for example, P can be given by $x_{k+2} = x_{k+3} = \cdots = x_{2n} = 0$). Let $S^k(\rho)$ denote the sphere of radius ρ in this subspace and let $B(\rho)$ denote the hypersurface $S^k(\rho) \times \mathbb{R}^{2n-k-1}$. Suppose that Σ is a closed, smoothly embedded k-submanifold of $\mathbb{R}^{2n}$ that lies in the outside component of $\mathbb{R}^{2n} \setminus B(\rho)$ and that non-trivially links the $(2n-k-1)$-subspace orthogonal to P. We will say that Σ non-trivially links $B(\rho)$. Suppose that for all time $0 < t < t_0$ the image Σ_t of Σ under mean curvature flow remains immersed. We say that Σ_t has $k+1$-*diameter* greater than ρ if Σ_t lies in the outside component of $\mathbb{R}^{2n} \setminus B(\rho/2)$ and non-trivially links $B(\rho/2)$. In particular, the points in the projection of Σ_t onto the $(k+1)$-plane P, denoted $\pi(\Sigma_t)$, are at least $\rho/2$ distant from the origin and $\pi(\Sigma_t)$ links $S^k(\rho/2)$. Therefore, $\operatorname{diam}(\pi(\Sigma_t)) > \rho$.

Theorem 4.1. *There is a t_1, $0 < t_1 < t_0$, depending only on ρ, such that for all t, $0 < t < t_1$, the $k+1$-diameter of Σ_t is greater than ρ.*

Proof. Denote by B_t the image of $B(\rho)$ under mean curvature flow. Note that for any t, $B_t = S^k(r) \times \mathbb{R}^{2n-k-1}$ for some $0 \leq r < \rho$. Let $d(t)$ be the distance between B_t and Σ_t (i.e., $d(t) = \inf\{d(x,y) : x \in B_t, y \in \Sigma_t\}$). Let $x_0 \in B_t$ and $y_0 \in \Sigma_t$ be points where $d(t) = d(x_0, y_0)$. Then $d(t)$ is achieved by a line segment joining x_0 and y_0 that is orthogonal to B_t at x_0 and orthogonal to Σ_t at y_0. The line segment lies in a $(k+1)$-plane parallel to P

and (suitably oriented) points in the direction of the mean curvature vector to B_t at x_0. Translating B_t along the line to y_0 and using the minimizing property of $d(t)$, Σ_t is tangent to B_t and lies on one side of B_t, at least locally. Therefore the component of the mean curvature of Σ_t at y_0 that is parallel to the mean curvature vector of B_t has magnitude less than or equal to that of B_t at x_0. The other components of the mean curvature of Σ_t at y_0 are tangent to B_t. It follows that $d(t)$ is non-decreasing.

It is well known that under mean curvature flow $B(\rho)$ flows to $B(\rho/2)$ in finite time T depending only on ρ. Set $t_1 < \min(T, t_0)$. Since $d(t)$ is non-decreasing, for any $0 < t < t_1$, Σ_t lies in the outside component of $\mathbb{R}^{2n} \setminus B(\rho/2)$. By homotopy invariance, for all $0 < t < t_1$, Σ_t non-trivially links $B(\rho/2)$. □

We will call $B(\rho)$ an *interior barrier* for Σ.

5. **Embedded lagrangrians**

Let $(x_1, \ldots, x_{2n})$ be coordinates on $\mathbb{R}^{2n}$. Set

$$F(x_1, \ldots, x_{2n}) = \sum_{i=1}^{2n-1} x_i^2 - f(x_{2n}),$$

where $f(t)$ is a smooth positive function that will be determined below. Let M be the zero set of F. Then M is a smooth manifold oriented by the choice of unit normal $\nu = -\nabla F/|\nabla F|$. A vector X is tangent to M if $\nabla F \cdot X = 0$. Let X, Y be unit tangent vectors and write $X = \sum_{i=1}^{2n} a_i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^{2n} b_i \frac{\partial}{\partial x_i}$. Then the second fundamental form on X, Y is:

$$\langle \nu, \nabla_X Y \rangle = -\langle \nabla_X \nu, Y \rangle = \frac{1}{|\nabla F|} \sum_{i,j=1}^{2n} a_i b_j F_{ij},$$

where $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$. To compute the second fundamental form of M at any point p it suffices, using the obvious symmetry, to assume that the coordinates of p are $(\sqrt{f(x_{2n})}, 0, \ldots, 0, f(x_{2n}))$. Then at p,

$$\nabla F = (2\sqrt{f(x_{2n})}, 0, \ldots, 0, -f'(x_{2n})).$$

A vector $X = \sum_{i=1}^{2n} a_i \frac{\partial}{\partial x_i}$ is tangent to M at p if:

$$2\sqrt{f(x_{2n})} a_1 - f'(x_{2n}) a_{2n} = 0.$$

Set $Y = f'(x_{2n}) \frac{\partial}{\partial x_1} + 2\sqrt{f(x_{2n})} \frac{\partial}{\partial x_{2n}}$. Then

$$\left\{ X_1 = \frac{Y}{|Y|}, X_2 = \frac{\partial}{\partial x_2}, \ldots, X_{2n-1} = \frac{\partial}{\partial x_{2n-1}} \right\}$$

is an orthonormal frame at p. In terms of this frame the second fundamental form at p is diagonal with diagonal elements:

$$\frac{1}{|\nabla F|}\left(\frac{2(f')^2-4f''f}{((f')^2+4f)},\ 2,\ \ldots,\ 2\right).$$

These are then the principal curvatures at p. It follows that M is a strong n-barrier if there is a scalar $a>0$ such that:

$$2(f')^2-4f''f+2(n-1)((f')^2+4f)\geq a((f')^2+4f)^{\frac{3}{2}}, \tag{5.1}$$

Equivalently, if

$$(n+1)(f')^2+4f((n-1)-f'')+(n-1)((f')^2+4f)\geq a((f')^2+4f)^{\frac{3}{2}},$$

Evidently, if:

$$(f')^2+4f<C,\quad (n+1)(f')^2+4f((n-1)-f'')\geq 0, \tag{5.2}$$

then there is such a scalar $a>0$ depending only on the bound C.

Choose $c\gg 1$ and $0<\varepsilon\ll 1/2$. Define:

$$g_\varepsilon(t)=\begin{cases}\varepsilon^2, & -c\leq t\leq c,\\ \varepsilon^2+\frac{(c-\varepsilon^2)}{c}(|t|-c), & c\leq |t|\leq 2c,\\ c, & |t|\geq 2c.\end{cases}$$

Then g_ε is a piecewise smooth positive function that fails to be smooth only at $\pm c$ and $\pm 2c$. It is possible to smooth these corners such that the smoothed function f_ε equals g_ε, except on intervals of length <1 around $\pm c$ and $\pm 2c$, and such that $f'_\varepsilon\geq 0$ and $f''_\varepsilon\leq n-1$. Then there is a constant C such that (5.2) is satisfied. Note that the scalar $a>0$ is independent of ε. It follows from (5.1) that if M is the zero set of

$$F_\varepsilon(x_1,\ldots,x_{2n})=\sum_{i=1}^{2n-1}x_i^2-f_\varepsilon(x_{2n})$$

then M is a strong n-barrier. We call the portion of M that lies in the set $\{x:-c\leq x_{2n}\leq c\}$, the "tubular portion" of M.

Lemma 5.1. *Suppose that Σ is a smooth n-dimensional embedding in $\mathbb{C}^n$ that passes through the "tubular portion" of M. Let Σ_t denote the image of Σ under mean curvature flow. Suppose that, for all time t for which the mean curvature flow is defined, Σ_t passes through the "tubular portion" of M. Then for some $t\leq t_M=\varepsilon/a$, Σ_t is singular.*

Proof. By Theorem 3.1, $d'(t)\geq a$ for a. e. t. Since the radius of the tubular portion of M is ε, $0\leq d(t)\leq\varepsilon$. The result follows from Corollary 3.2. $\square$

Next we construct a lagrangian embedding lying in the "inside" of the barrier M. We first describe the construction of lagrangian embeddings that have an interior barrier. Let $P\subset\mathbb{R}^{2n}$ denote an $(n+1)$-plane. Consider the standard n-spheres $S^n(\rho)\subset S^n(r)$ of radii $\rho<r$ in P centered at the

origin. Then $S^n(r)$ links the hypersurface $B(\rho) = S^n(\rho) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{2n}$ in the sense defined above. It is not difficult to verify that the embedding $\imath : S^n(r) \to \mathbb{R}^{2n}$ satisfies the topological conditions necessary to homotop it, using the h-principle, to a lagrangian immersion $\ell : S^n(r) \to \mathbb{R}^{2n}$. The lagrangian immersion ℓ can be taken C^0 close to $\imath$ [E-M] and can be assumed to have, at worst, ordinary double points. These double points can then be replaced by lagrangian handles using lagrangian surgery [P]. The resulting submanifold L is a lagrangian embedding. For ρ sufficiently small, L is disjoint from the hypersurface $B(\rho)$ and links it. Thus $B(\rho)$ is an interior barrier for L. Alternately an explicit construction, one not using the h-principle, can be made by observing that the Clifford torus $T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{C}^n$ is a lagrangian embedding that links an $(n-1)$-plane in $\mathbb{C}^n$ and therefore for ρ sufficiently small is disjoint from and links an $S^n(\rho) \times \mathbb{R}^{n-1}$. (For example, introducing complex coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ with $z_j = x_j + iy_j$, the Clifford 2-torus links the line $x_1 = 0, y_1 = t - 1/2, x_2 = t+1, y_2 = t$, for $-\infty < t < \infty$. The Clifford 3-torus links the plane $x_1 = 0, y_1 = t + s - 1/2, x_2 = t + 1, y_2 = t, x_3 = s + 1, y_3 = s$, for $-\infty < t, s < \infty$, etc.)

In terms of the coordinates introduced in the construction of M, put $y_n = x_{2n}$. Consider $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ given by $z_n = 0$. In $\mathbb{C}^{n-1}$ let L_0 be a lagrangian immersion of S^{n-1}. (When $n = 2$ we require that L_0 be a "zero area" immersed curve, that is, we require that the integral of the Liouville form on L_0 vanish.) By perturbation we can assume that L_0 has only ordinary double points. By scaling and translation we can assume that L_0 is contained in the ball of radius $\frac{\varepsilon}{2}$ centered at the origin. Construct a lagrangian cylinder C in $\mathbb{C}^n$ by setting: $C = L_0 \times \{it : -\sigma \le t \le \sigma\} \subset \mathbb{C}^{n-1} \times \mathbb{C}$, where $\sigma > 2c$. The boundary spheres $S_+ = L_0 \times \{i\sigma\}$ and $S_- = L_0 \times \{-i\sigma\}$ bound immersed lagrangian discs $D^n_\pm$ in $\mathbb{C}^n$ [E-M]. Attaching these discs to C at its boundary we construct a lagrangian immersion $\widetilde{C}$ of S^n. Perturbing this immersion we can suppose that it has only ordinary double points. Next attach lagrangian "bubbles" to the ends of $\widetilde{C}$ as follows: In the half spaces $\{z : y_n > \sigma\}$ and $\{z : y_n < -\sigma\}$ consider lagrangian embeddings L_+ and L_-, respectively. We require that both L_+ and L_- are in the "inside" component of $\mathbb{C}^n \setminus M$ and that they have interior barriers $B_+(\rho)$ and $B_-(\rho)$, $\rho > 0$, constructed, as described above, from $(n+1)$-planes P_+ and P_- that are parallel to the line $\{y_n = t : -\infty < t < \infty\}$. By perturbation and translation we can assume that L_+ and L_- intersect $\widetilde{C}$ transversely in a finite number of isolated points. At each of these points we perform a lagrangian surgery [P] and replace the double point with a lagrangian handle. To make the resulting lagrangian immersion an embedding we perform a lagrangian surgery at each remaining ordinary double point. The resulting manifold is a smooth lagrangian embedding Σ that is orientable if n is odd. The surgery can be performed locally so that Σ remains in the "inside" component of $\mathbb{C}^n \setminus M$.

Theorem 5.2. *For ε sufficiently small, under mean curvature flow the lagrangian embedding Σ develops singularities in finite time.*

Proof. The lagrangian embedding Σ has two interior barriers $B_+(\rho)$ and $B_-(\rho)$. Applying Theorem 4.1 to both interior barriers there is a t_1 depending only on ρ such that for $0 < t < t_1$, Σ_t is disjoint from and links both $B_+(\rho/2)$ and $B_-(\rho/2)$. Choose $0 < \varepsilon < \min(\rho/2, t_1 a)$. For all $t < t_1$, Σ_t has interior barriers $B_+(\rho/2)$ and $B_-(\rho/2)$ and therefore, since $\varepsilon < \rho/2$, must pass through the "tubular portion" of M. Since $t_M = \varepsilon/a < t_1$ the result now follows from Lemma 5.1. □

6. Adding filigree to special lagrangrians

In this section we modify the constructions of the previous section to attach "filigree" to an embedded special lagrangian submanifold of a Calabi-Yau n-manifold N. To begin we continue to work in $\mathbb{C}^n$ and we construct two n-barriers.

Choose $c \gg 1$ and $0 < \varepsilon \ll \frac{1}{2}$. Define:

$$g_{0,\varepsilon}(t) = \begin{cases} \varepsilon^2, & -10c \le t \le c, \\ \varepsilon^2 + \frac{(c-\varepsilon^2)}{c}(t-c), & c \le t \le 2c, \\ c, & 2c \le t \le 10c, \\ c - \frac{(c-\varepsilon^2)}{c}(t-10c), & 10c \le t \le 11c, \\ \varepsilon^2, & t \ge 11c, \end{cases}$$

$$g_{1,\varepsilon}(t) = \begin{cases} c, & -10c \le t \le -c, \\ c - \frac{(c-\varepsilon^2)}{c}(t+c), & -c \le t \le 0, \\ \varepsilon^2, & 0 \le t \le c, \\ \varepsilon^2 + \frac{(c-\varepsilon^2)}{c}(t-c), & c \le t \le 2c, \\ c, & 2c \le t \le 10c, \\ c - \frac{(c-\varepsilon^2)}{c}(t-10c), & 10c \le t \le 11c, \\ \varepsilon^2, & t \ge 11c, \end{cases}$$

Then both $g_{0,\varepsilon}$ and $g_{1,\varepsilon}$ are piecewise smooth positive functions that fail to be smooth at finitely many points. As in the previous section it is possible to smooth these corners such that the smoothed functions $f_{0,\varepsilon}$ and $f_{1,\varepsilon}$ equal $g_{0,\varepsilon}$ and $g_{1,\varepsilon}$,respectively, except on intervals of length < 1 around the corners and satisfy $f_{0,\varepsilon} \le f_{1,\varepsilon}$. The functions $f_{0,\varepsilon}$ and $f_{1,\varepsilon}$ are positive and satisfy (5.2) for some constant $C > 0$. Therefore the hypersurface M_0, the zero set of:

$$F_{0,\varepsilon}(x_1, \ldots, x_{2n}) = \sum_{i=1}^{2n-1} x_i^2 - f_{0,\varepsilon}(x_{2n}),$$

and the hypersurface M_1, the zero set of:

$$F_{1,\varepsilon}(x_1, \ldots, x_{2n}) = \sum_{i=1}^{2n-1} x_i^2 - f_{1,\varepsilon}(x_{2n}),$$

are strong n-barriers for a constant $a > 0$ depending only on C.

Next we construct an embedded lagrangian contained in the "inside" region of M_0. As above, consider $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ given by $z_n = 0$. In $\mathbb{C}^{n-1}$ let L_0 be a lagrangian immersion of S^{n-1} with only ordinary double points. By scaling and translation we can assume that L_0 is contained in the ball of radius $\frac{\varepsilon}{2}$ centered at the origin. Construct a lagrangian cylinder C in $\mathbb{C}^n$ by setting: $C = L_0 \times \{it : -10c \leq t \leq 2c\} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. The boundary spheres $S_+ = L_0 \times \{2ci\}$ and $S_- = L_0 \times \{-10ci\}$ bound immersed lagrangian discs $D^n_\pm$ in $\mathbb{C}^n$. Attaching these discs to C at its boundary we construct a lagrangian immersion $\widetilde{C}$ of S^n. Perturbing this immersion we can suppose that it has only ordinary double points. Next attach a lagrangian "bubble" to the "+" end of $\widetilde{C}$ as follows: In the inside region of M_0 with $2c < y_n < 10c$ consider a lagrangian embedding L_+ that has an interior barrier $B_+(\rho)$, $\rho > 0$, constructed, as described in the previous section, from an $(n+1)$-plane P_+ that is parallel to the line $\{y_n = t : -\infty < t < \infty\}$. By perturbation and translation we can assume that L_+ intersects $\widetilde{C}$ transversely in a finite number of isolated points. At each of these points we perform a lagrangian surgery and replace the double point with a lagrangian handle. Finally, to make the resulting lagrangian immersion an embedding we perform a lagrangian surgery at each remaining ordinary double point. Denote the resulting lagrangian embedding by Σ.

Let L be an embedded special lagrangian submanifold of a Calabi-Yau n-manifold N. Note that the homology class $[L]$ is non-trivial. Let B_r denote the normal sphere bundle of L considered as a subset of a tubular neighborhood of $L \subset N$, where r denotes the radius of each fiber sphere $S_x,\ x \in L$. Choose r sufficiently small so that B_r is an embedded submanifold and so that each principal curvature of S_x for all $x \in L$ is greater than the absolute value of every principal curvature of L. Since L is a minimal submanifold it follows that B_r is a weak n-barrier, though not an n-barrier. Let $p \in L$ be a point and U a coordinate neighborhood of $p \in N$. In U construct strong n-barriers M_0 and M_1 and an embedded lagrangian Σ as above. The n-barriers M_0 and M_1 intersect B_r transversely in $(n-2)$-submanifolds that are topological spheres. It is not difficult to smooth these "corners" such that the resulting smoothly embedded hypersurfaces, denoted $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively, are both weak n-barriers. The lagrangian Σ can be constructed to intersect L in the interior region of $\mathcal{M}_0$. By perturbation we can assume that Σ intersects L transversally in a finite number of ordinary double points. Perform lagrangian surgery at these double points to construct an embedded lagrangian that we denote $L\#\Sigma$. These surgeries can be performed so that $L\#\Sigma$ remains in the interior region of $\mathcal{M}_0$.

Note that the Calabi-Yau metric on $\bar{U}$ is uniformly equivalently to the euclidean metric on a compact ball in $\mathbb{C}^n$ and therefore mean curvature flow can be studied using either metric. Under mean curvature flow $\mathcal{M}_0$ is a weak n-barrier and so by Theorem 3.3 $L\#\Sigma$ remains inside $\mathcal{M}_0$ though, a priori, the submanifold may flow through the tubular region into either end. However, $L\#\Sigma$ has an interior barrier $B_+(\rho)$. Therefore, there is a time t_1, depending only on ρ, such that, for $0 < t < t_1$, $(L\#\Sigma)_t$ links $B_+(\rho/2)$. Therefore if $\varepsilon < \rho/2$, for $0 < t < t_1$, the submanifold cannot flow out of the inside region of M_0 between $y_n = c$ and $y_n = 11c$. Since L represents a non-trivial homology class, the entire submanifold cannot flow into the tubular region of M_0 and therefore, for all t with $0 < t < t_1$, the submanifold transverses the region of M_0 between $y_n = -10c$ and $y_n = c$.

Theorem 6.1. *For ε sufficiently small, the lagrangian embedding $L\#\Sigma$ develops a singularity, under mean curvature flow, in finite time.*

Proof. We suppose, by way of contradiction, that for any finite time t, $(L\#\Sigma)_t$ is an immersion. Choose ε sufficiently small so that $\varepsilon < \rho/2$ and such that $t_1 > t_{M_1} = \varepsilon/a$. Then for all t, $0 < t \leq t_{M_1}$, $(L\#\Sigma)_t$ transverses the tubular region of M_0 between $y_n = -10c$ and $y_n = c$. Set $(\Sigma_+)_t = (L\#\Sigma)_t \cap \{z : y_n \geq -5c\}$. Let $d_1(t)$ be the minimum distance between $(\Sigma_+)_t$ and the barrier M_1. On the boundary of $(\Sigma_+)_t$ the distance to M_1 is greater than $\sqrt{c} - \varepsilon > \varepsilon$. For $0 < t < t_1$, since $(\Sigma_+)_t$ transverses the tubular region of M_1, $d_1(t) \leq \varepsilon$. Therefore, for $0 < t < t_1$, $d_1(t)$ assumes its value at points in the interior of $(\Sigma_+)_t$. Thus $d_1(t)$ is a continuous function of t and, as shown in the proof of Theorem 3.1, $d_1'(t) \geq a$ for a.e. t. Hence $(\Sigma_+)_t$ develops a singularity at some $t \leq t_{M_1}$, proving the theorem. □

Let $p_1, \ldots, p_k$ be disjoint points in L with disjoint neighborhoods, $U_1, \ldots, U_k$, in N. Carrying out the previous construction in each neighborhood U_i we construct an embedded lagrangian $L\# \cup_i^k \Sigma_i$ in N that, under mean curvature flow, develops at least k singularities, in finite time. The lagrangians $L\# \cup_i^k \Sigma_i$ look like the special lagrangian L with "filigree" attached at k points, where k can be taken arbitrarily large.

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