The Journal of Geometric Analysis
Volume 3, Number 1, 1993

Pseudo-Holomorphic Maps and Bubble Trees

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ABSTRACT. This paper proves a strong convergence theorem for sequences of pseudo-holomorphic maps from a Riemann surface to a symplectic manifold N with tamed almost complex structure. (These are the objects used by Gromov to define his symplectic invariants.) The paper begins by developing some analytic facts about such maps, including a simple new isoperimetric inequality and a new removable singularity theorem.

The main technique is a general procedure for renormalizing sequences of maps to obtain "bubbles on bubbles." This is a significant step beyond the standard renormalization procedure of Sacks and Uhlenbeck. The renormalized maps give rise to a sequence of maps from a "bubble tree"—a map from a wedge $\Sigma \vee S^2 \vee S^2 \vee \cdots \to N$. The main result is that the images of these renormalized maps converge in $L^{1,2} \cap C^0$ to the image of a limiting pseudo-holomorphic map from the bubble tree. This implies several important properties of the bubble tree. In particular, the images of consecutive bubbles in the bubble tree intersect, and if a sequence of maps represents a homology class then the limiting map represents this class.

While the main focus is on holomorphic maps, the bubble tree construction applies to other conformally invariant problems, including minimal surfaces and Yang–Mills fields.

Introduction

It has long been known that the space of solutions of many nonlinear elliptic differential equations is noncompact in any reasonable topology. This is true, for example, for the space of harmonic maps from the two-sphere to a Riemannian manifold. In 1979 Sacks and Uhlenbeck proved an existence theorem for harmonic maps of two-spheres by exploiting precisely this noncompactness. Their key observation was that the lack of compactness is associated with the concentration of the energy density of solutions at isolated points and that, by using the conformal invariance of the equations, one could renormalize the solutions around these points of concentration to obtain other solutions. Their renormalization procedure is now known as "bubbling."

The bubbling phenomenon has subsequently been recognized and studied in a wide variety of other geometric differential equations, including the Yamabe equation, the Yang–Mills equation, and the J-holomorphic map equation. Each case involves an elliptic equation whose nonlinear terms are borderline for the Sobolev inequalities. It is well known that the basic Sacks–Uhlenbeck procedure applies to each.

Math Subject Classification 53C15.
Key Words and Phrases Bubble tree, Gromov moduli space, pseudo-holomorphic map, symplectic manifold.
T. H. P. research partially supported by NSF Grant DMS-9004836. J. G. W. research partially supported by NSF Grant DMS-8901230.

ISSN 1050-6926

The Sacks–Uhlenbeck renormalization works well for existence theorems, but is flawed as a convergence scheme: the rescaling pushes energy to infinity where it can be lost, so one does not get convergence in the energy norm. This paper develops a new and more systematic renormalization scheme. This scheme applies to any geometric problem where "bubbling" occurs. In the cases where a suitable isoperimetric inequality holds, it gives convergence with no energy loss. This yields a precise convergence result for sequences of solutions to the J-holomorphic map equation.

The main focus of this paper is on J-holomorphic maps. This is in keeping with our original aim of proving and improving Gromov's convergence results that are the crucial underpinning of Gromov's theory of J-holomorphic curves in symplectic manifolds [G]. This is one of the simplest geometric contexts where bubbling occurs. Our renormalization scheme works nicely for it and yields an elegant and complete proof of Gromov's convergence statements.

To fix ideas, let (Σ, j) be a compact Riemann surface with complex structure j and let (N, J) be an almost complex manifold. A map $f : \Sigma \to N$ is called jJ-holomorphic if

$$df \circ j = J \circ df.$$

Given a compact symplectic manifold N, Gromov chooses an almost complex structure J on N compatible with the symplectic structure and considers the moduli space of jJ-holomorphic maps representing a fixed homology class. The compatibility condition implies a uniform energy bound on such maps. The issue of the compactness of the moduli space can then be formulated as a convergence question: Let $f_n : \Sigma \to N$ be a sequence of jJ holomorphic maps with bounded energy. Does there exist a subsequence that converges in C^1 (or in some other topology) to a jJ-holomorphic map $f_0 : \Sigma \to N$?

The analysis begins by showing that there is a subsequence of the $\{f_n\}$ which converges in C^1 to a jJ-holomorphic map $f_0 : \Sigma \to N$. However this convergence is not on all of Σ but rather on Σ with a finite number of points $\{x_i\}$ deleted. The Sacks–Uhlenbeck method proceeds by using a rescaling argument to construct at each x_i a J-holomorphic map f_{x_i} from the two-sphere into M—a bubble. The resulting "limit" of the subsequence is a J-holomorphic map from the disjoint union of Σ and a finite number of two-spheres into N. This "Sacks–Uhlenbeck limit" is the limit that is used in [W] to compactify moduli space. Notice that on the one hand the bubbles f_{x_i} act as obstructions to C^1 convergence and on the other hand their absence implies C^1 convergence. Thus conditions on a moduli space that forbid the decomposition of maps into a sum of J-holomorphic maps imply the C^1 compactness of the moduli space. For this reason the compactification of [W] is adequate for the applications of [G].

The Sacks–Uhlenbeck limit is not, however, the complete limit of the sequence $\{f_n\}$. The occurence of "energy loss," mentioned above, means that the sum of the energies of the limit map f_0 and of all the bubbles may be less than the limit of the energies of the maps f_n of the sequence. Similarly, the homology class of the maps f_n may not be preserved in this limit, and the image curve of the Sachs–Uhlenbeck limit map may not be connected.

In this paper we modify the Sacks–Uhlenbeck construction, revising the renormalization procedure so as to allow iteration. At each bubble point x_i we construct a sequence of J-holomorphic maps from the two-sphere into N with bounded energy. This sequence in turn has a subsequence that converges in C^1 (on the two-sphere with a finite number of points $\{y_k\}$ deleted) to a J-holomorphic map $f_{x_i} : S^2 \longrightarrow N$. We then repeat our procedure at each y_k. Interating, we construct bubbles on bubbles on bubbles, etc. The result is a finite tower of bubbles, consisting of a jJ-holomorphic map $f_0 : \Sigma \longrightarrow N$ and a collection of J-holomorphic maps from the two-sphere to N (intrinsically, the two-spheres are fibers of compactified iterated tangent bundles as described in Section 4). This collection has many beautiful properties which can be elegantly described using the structure of a tree, as in Figure 1 below. In particular, both area and homology are preserved by this limit and the image curve of the limit is connected.

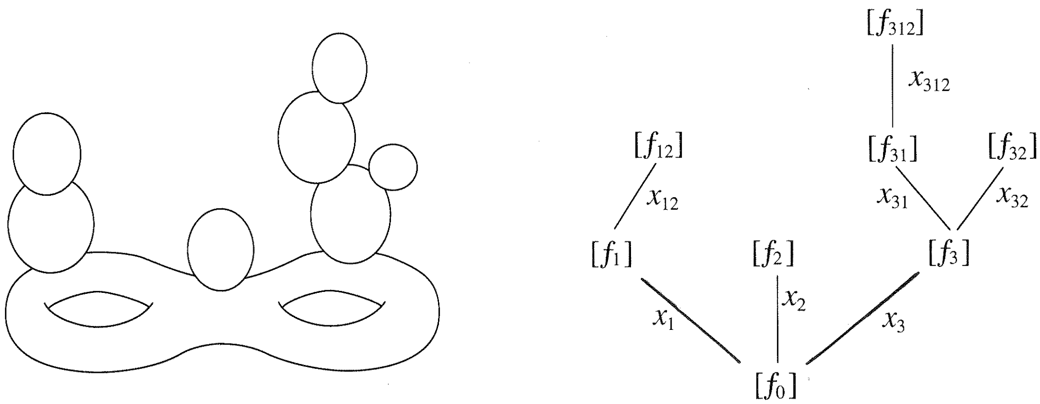

Figure 1.

Bubbling for J-holomorphic maps is a very real geometric phenomenon. For a simple example, take the space X obtained by blowing up $\mathbb{CP}^2$ at two points p and q. Then the projection $\pi : X \longrightarrow \mathbb{CP}^2$ is a holomorphic isomorphism over $\mathbb{CP}^2 - \{p, q\}$, and $H_2(X) \cong \mathbb{Z}^3$ is generated by the standard line $[L]$ and the exceptional curves $E_1 = \pi^{-1}(p)$ and $E_2 = \pi^{-1}(q)$. Now consider a one-parameter family of lines $f_t : S^2 \simeq \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2$ which miss p and q for all $t > 0$, but with f_0 passing through both p and q. For $t > 0$ these lift to holomorphic maps $\tilde{f}_t : S^2 \longrightarrow X$. One can also lift f_0: first remove $x = f_0^{-1}(p)$ and $y = f_0^{-1}(q)$ and lift, obtaining a map $\tilde{f}_0$ defined on $S^2 - \{x, y\}$; this extends to a holomorphic map $\tilde{f}_0 : S^2 \longrightarrow X$ (the "strict transform" of f_0) which intersects each exceptional curve in exactly one point. Clearly, as $t \longrightarrow 0$ we have $\tilde{f}_t \longrightarrow \tilde{f}_0$ in C^k at all points on S^2 away from x and y. However, *this family of maps bubbles at the points x and y*. This can be seen by considering homology classes. For $t > 0$ $\tilde{f}_t$ represents the homology class $[L] \in H_2(X)$. On the other hand, $[\tilde{f}_0]$ projects to $[L] \in H_2(\mathbb{CP}^2)$ and satisfies $[\tilde{f}_0] \cdot [E_i] = 1$ for $i = 1, 2$. It follows (since $[L] \cdot [L] = 1$ and $[E_i] \cdot [E_j] = -\delta_{ij}$) that $[\tilde{f}_0] = [L] - [E_1] - [E_2]$. Thus as $t \longrightarrow 0$ the family $\{\tilde{f}_t\}$ bubbles according to the bubble tree shown in Figure 2, where the $g_i : S^2 \longrightarrow X$ are maps into the exceptional curves. It is easy to elaborate on the example. Using maps of higher degree and blowing up at more points—including

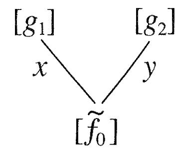

Figure 2.

points on exceptional curves—one obtains examples of complicated bubble trees (see [McD3] for some interesting consequences of this idea).

Heuristically the sequence $\{f_n\}$ "converges" to a map f_0 from the bubble tree T into N. It is difficult to make this convergence rigorous because the domain of the f_n is Σ, not T. To rectify this we use minimal surfaces to define, for each $\epsilon > 0$, a canonical surgery on each f_n. The result is a "prolongation" of f_n to a map $\mathcal{P}_\epsilon(f_n)$ from T to N. Although f_n and $\mathcal{P}_\epsilon(f_n)$ have different domains they are close in the sense that the difference in their energies goes to zero as ϵ goes to zero and their images coincide except on a finite number of ϵ-balls. Our main result, given in Section 6, shows that the $\mathcal{P}_\epsilon(f_n)$ have good convergence properties.

Theorem 6.2. *There is a sequence $\epsilon_n \searrow 0$ such that a subsequence of*

$$\mathcal{P}_{\epsilon_n}(f_n) : T \to N$$

converges in $C^0 \cap L^{1,2}$ to a smooth jJ-holomorphic map $f_0 : T \to N$.

It follows that the areas of the f_n converge to the area of f_0 and the images of the f_n converge pointwise to the image of f_0. This last statement can be reformulated as a compactness result for the space of unparameterized J-holomorphic curves in N [G].

As mentioned earlier, our convergence theorem applies to other important conformally invariant geometric problems, such as harmonic maps, the Yamabe problem, and Yang-Mills. There are some differences between these problems. The renormalization scheme and the contruction of the bubble tree described in Section 4 applies to each of these. The associated bubble trees are slightly more general than the one in Figure 1: one must include nonnegative constants ϵ_i associated with each edge of the tree. This is because with each renormalization there may be some energy caught between the bubbles. As shown in Lemma 5.1, this energy is carried by a solution defined on an annular region. Thus the general bubble tree must include the energies of both the bubbles and of these annular regions.

For sequences of J-holomorphic maps it is proved in Section 5 that these annuli are mapped onto "thin tubes" in the image and such tubes violate the isoperimetric inequality. Hence there is no energy loss in this case (thus simplifying the bubble tree). This is a special feature of this problem; the corresponding fact must be checked separately for each of the other conformally invariant problems. In fact, one can show that energy loss does not occur for Yang–Mills with the consequential simplification in the bubble tree. On the other hand, energy loss does seem to occur for some sequences of harmonic maps. This phenomenon should be studied further.

A number of authors have addressed various of the topics of this paper. The energy loss phenomenon for harmonic maps was recognized and dealt with in previous works by ad hoc methods (cf. [SY]). Recently, R. Ye [Y] has applied a similar technique to J-holomorphic maps and independently obtained some of the results of this work. Taubes [T] has devised a method for dealing with multiple bubbling in the Yang–Mills theory. We are indebted to J. D. Moore, D. McDuff, and R. Ye for pointing out errors in preliminary versions of this paper. Finally we direct the reader's attention to Section 3, where a complete and simple proof of the removable singularity theorem is given.

1. J-holomorphic maps

Let N be a closed symplectic manifold of dimension $2n$ with symplectic form ω. An almost complex structure on N is an endomorphism $J : TN \to TN$ satisfying $J^2 = -Id.$; this is equivalent to a reduction of the $Sp(2n)$ frame bundle to a $U(n)$ bundle. Since $U(n)$ is a deformation retract of $Sp(2n)$, the tangent bundle of (N, ω) admits an almost complex structure and any two such structures are homotopic. Let Σ be a closed Riemann surface of genus g with complex (conformal) structure j. A map $f : \Sigma \to N$ is called *jJ-holomorphic* if

$$df \circ j = J \circ df. \tag{1.1}$$

This is the Cauchy–Riemann equation for the map f; it is a first-order elliptic system. Each solution f represents a class $\alpha \in H_2(N, \mathbb{Z})$. We call f a *jJ-holomorphic α-map* and the image of f a *jJ-holomorphic α-curve*.

Gromov's idea is to obtain invariants of the symplectic manifold (N, ω) by choosing an almost complex structure J and studying the space of holomorphic curves for various (Σ, j). To make this work, of course, we must choose a J that is related to the symplectic form ω. There are two ways of doing this.

An almost complex structure J on (N, ω) is said to be *ω-tamed* if ω is positive on all J complex lines in TN, i.e., if

$$\omega(v, Jv) > 0 \quad \forall v \in TN \quad v \neq 0. \tag{1.2}$$

An almost complex structure J on (N, ω) is called *ω-compatible* if J satisfies (1.2) and

$$\omega(Ju, Jv) = \omega(u, v) \qquad \text{for all } u, v \in TN. \tag{1.3}$$

The spaces of all smooth ω-tamed and ω-compatible almost complex structures will be denoted $\mathcal{J}(\omega)$ and $\mathcal{J}_c(\omega)$ respectively. It is easy to see that both $\mathcal{J}(\omega)$ and $\mathcal{J}_c(\omega)$ are nonempty and connected.

The advantage of imposing the additional condition (1.3) is to make contact with (almost) hermitian geometry. For each $J \in \mathcal{J}_c(\omega)$ there is an associated hermitian metric h on N called the *J-compatible metric* defined

$$h(u,v) = \omega(u, Jv). \tag{1.4}$$

Note h is positive-definite by (1.2) and symmetric by (1.3). The Wirtinger inequality of Kähler geometry applies to the triple (ω, J, h). Consequently, if f is a J-holomorphic α-curve and Area(f) is the area of the image of f with respect to the metric h then

$$\text{Area}(f) = \int_\Sigma dv_{f^*h} = \int_\Sigma f^*\omega = \int_{f(\Sigma)} \omega = \langle [\omega], \alpha \rangle, \tag{1.5}$$

where $[\omega]$ denotes the cohomology class of ω and $\langle\ ,\ \rangle$ is the homology-cohomology pairing. The number $\langle [\omega], \alpha\rangle$ depends only on the symplectic form ω. Thus for $J \in \mathcal{J}_c(\omega)$ (1.5) implies

(i) The area of a J-holomorphic α-curve is a symplectic invariant.

(ii) A J-holomorphic curve is absolutely area minimizing in its homology class, and therefore its image is a minimal surface (for the J-compatible metric).

If $J \in \mathcal{J}(\omega)$ then we proceed as follows. Let η be any hermitian metric on (N, J). Since N is compact (1.2) implies that there is a constants $c, C > 0$ such that

$$c\eta(v,v) \le \omega(v, Jv) \le C\eta(v,v) \qquad \forall v \in TN.$$

It is convenient to work with the hermitian metric $h = c\eta$, which satisfies (after renaming C)

$$h(v,v) \le \omega(v, Jv) \le Ch(v,v) \qquad \forall v \in TN. \tag{1.6}$$

This weakening of (1.4) turns out to be of no consequence in the theory. In particular, as we will show in later sections, many results about minimal surfaces extend to tamed holomorphic maps.

For example, we can apply the above reasoning to (ω, J, h) obtaining

$$\text{Area}_h(f) \le \int_{f(\Sigma)} \omega = \langle [\omega], \alpha\rangle \le C\text{Area}_h(f) \tag{1.7}$$

for a J-holomorphic α-curve f. These elementary considerations yield a fact that will be of fundamental importance in later sections:

Proposition 1.1. *For a tamed almost complex structure J and a hermitian metric h satisfying (1.6):*

(a) *The space of J-holomorphic maps $\Sigma \to N$ representing a class $\alpha \in H_2(N, \mathbb{Z})$ satisfies a uniform h-area bound A depending only on ω and α.*

(b) *There is a constant $B_0 > 0$ depending only on J, ω, and h such that any smooth J-holomorphic map $f : \Sigma \longrightarrow N$, with* $\text{Area}_h(f) < B_0$ *is a map to a point.*

Proof. Part (a) is clear from equation (1.7). Lemma 3.3 below implies that if $\text{Area}_h(f)$ is sufficiently small then the image curve of f lies in a coordinate neighborhood and consequently f represents the trivial element in homology. But then (1.7) shows that $\text{Area}_h(f) = 0$, so f is a map to a point. $\square$

Remark. Theorem 3.3 in [SU] gives a similar statement for harmonic maps: there is a constant B_0 such that any harmonic or α-harmonic map $f : \Sigma \longrightarrow N$ with energy $E(f) < B_0$ is a map to a point. $\square$

For unparameterized curves area is a natural notion, but for parameterized curves (maps) it is more natural and technically more convenient to use energy. To do this, we choose a Riemannian metric μ on Σ in the conformal class of j. The energy of a map $f : \Sigma \longrightarrow N$ is then

$$E(f) = \frac{1}{2} \int_\Sigma |df|^2. \tag{1.8}$$

The integrand in this expression depends on the choice of the metric μ, but $E(f)$ depends only on the conformal class j. Moreover, for $J \in \mathcal{J}(\omega)$ a J-holomorphic map $f : \Sigma \longrightarrow (N, h)$ is conformal in the sense that f^*h is in the conformal class j. Hence if f is a J-holomorphic map then the energy of f for the metrics μ and h equals the area of f measured with the metric h:

$$\text{Area}_h(f) = E(f) = \frac{1}{2} \int_\Sigma |df|^2. \tag{1.9}$$

Consequently, for a metric on Σ in the conformal class j the uniform area bound of Proposition 1.1 gives a uniform energy bound.

We can now consider the space of all holomorphic maps. For this it is most convenient of work with Sobolev spaces. Fix, once and for all, an integer $s \geq 3$. Let $L^{s,p}$ be the Sobolev space whose norm is the sum of the L^p norms of the derivatives through order s. The Sobolev embedding theorem shows that any $L^{s,2}$ function on Σ is C^1, and any $L^{s+n-1,2}$ function on N is C^1. We can then complete the space $\text{Map}(\Sigma, N)$ of all smooth maps $f : \Sigma \longrightarrow N$ in the $L^{s,2}$ norm, obtaining a smooth Hilbert manifold $\text{Map}^s(\Sigma, N)$ (this construction is standard and functorial—see Palais [P]). In the same way we can use the $L^{s+n-1,2}$ norm to complete $\mathcal{J}(\omega)$ and $\mathcal{J}_c(\omega)$ to smooth Hilbert manifolds $\mathcal{J}^s(\omega)$ and $\mathcal{J}_c^s(\omega)$. With these definitions each $f \in \text{Map}^s(\Sigma, N)$ and each $J \in \mathcal{J}^s(\omega)$ are C^1.

For each homology class $\alpha \in H_2(N, \mathbb{Z})$, each conformal structure j on Σ and each $J \in \mathcal{J}^s(\omega)$ we obtain a moduli space of J-holomorphic α-curves

$$\mathcal{M}_{\alpha,j,J} = \Big\{ f \in \text{Map}^s(\Sigma, N) \ : df \circ j = J \circ df \Big\}.$$

By varying the conformal structure over the moduli space $\mathcal{C}_g$ of conformal structures on Σ we obtain a larger moduli space (depending on α and the genus g)

$$\mathcal{M}_{\alpha,g,J} = \left\{ (f,j) \in \mathrm{Map}^s(\Sigma, N) \times \mathcal{C}_g \ : df \circ j = J \circ df \right\}$$

and a fibration $\pi_\alpha : \mathcal{M}_{\alpha,g,j} \to \mathcal{C}_g$ whose fibers are the $\mathcal{M}_{\alpha,j,J}$. (Most of the following discussion carries over to compact families of conformal structures, but additional complications arise as the conformal structure approaches the boundary of $\mathcal{C}_g$). The area bound of Proposition 1.1 implies that the maps $f \in \mathcal{M}_{\alpha,j,J}$ have uniformly bounded area and hence uniformly bounded energy. Our objective in this paper is to study the compactness properties of $\mathcal{M}_{\alpha,j,J}$. In particular we ask:

Question. If $\{f_n\}$ is a sequence of jJ-holomorphic maps $f : \Sigma \to N$ for tamed J, what is the "limit" of $\{f_n\}$?

2. Elliptic estimates

Let Σ be a Riemann surface equipped with a metric μ compatible with j. Let (N, J) be an almost complex manifold equipped with a hermitian metric h. Suppose that $f : \Sigma \to N$ is a jJ-holomorphic map. The differential of f extends to a complex linear map

$$df : T\Sigma \otimes \mathbb{C} \to TN \otimes \mathbb{C}.$$

We can decompose $T\Sigma \otimes \mathbb{C}$ (resp. $TN \otimes \mathbb{C}$) into the $\pm i$ eigenspaces of j (resp. J) , which we denote by $T^{1,0}\Sigma$ and $T^{0,1}\Sigma$ (resp. $T^{1,0}N$ and $T^{0,1}N$). Thus $u \in T\Sigma$ corresponds to

$$\left(\frac{1}{2}(1 - ij)u, \frac{1}{2}(1 + ij)u \right) \in T^{1,0}\Sigma \oplus T^{0,1}\Sigma.$$

Similarly, the differential decomposes as

$$df = \partial f \oplus \overline{\partial} f \tag{2.1}$$

where ∂f and $\overline{\partial} f$ are the restrictions of df to $T^{1,0}\Sigma$ and $T^{0,1}\Sigma$ respectively. The condition (1.1) that f is jJ-holomorphic then says that

$$\begin{aligned} df\left(\frac{1}{2}(1 - ij)u\right) &= \frac{1}{2}(1 - iJ)df(u) \in T^{1,0}N \\ df\left(\frac{1}{2}(1 + ij)u\right) &= \frac{1}{2}(1 + iJ)df(u) \in T^{0,1}N. \end{aligned} \tag{2.2}$$

Thus

$$\begin{aligned} \partial f : T^{1,0}\Sigma &\rightarrow T^{1,0}N \\ \overline{\partial} f : T^{0,1}\Sigma &\rightarrow T^{0,1}N, \end{aligned} \tag{2.3}$$

i.e., holomorphic maps preserve type. Moreover, for $u \in T^{1,0}\Sigma$ we have

$$\overline{\partial} f(\overline{u}) = \overline{\partial f(u)}, \tag{2.4}$$

so ∂f determines $\bar{\partial} f$.

To obtain estimates, we must differentiate these equations. We will do this using invariant notation; the same computations are done in the appendix using moving frames. Choose a hermitian connection ∇ on N . This pulls back to a connection on f^*TN, which we will denote by ∇^f or, when confusion is not possible, by ∇. The skew-symmetrization of the connection is an operator

$$d^\nabla : \Omega^1(f^*TN) \rightarrow \Omega^2(f^*TN). \tag{2.5}$$

The restriction to $\Omega^{1,0}$ decomposes into the sum of a ∂ and a $\bar{\partial}$ operator:

$$d^\nabla = \partial^\nabla + \overline{\partial}^\nabla : \Omega^{1,0}(f^*TN) \longrightarrow \Omega^{2,0}(f^*TN) \oplus \Omega^{1,1}(f^*TN). \tag{2.6}$$

Since Σ is a Riemann surface there are no (2,0) forms, so $\partial^\nabla = 0$. Because $\partial f \in \Omega^{1,0}(f^*TN)$ we have

$$\partial^\nabla(\partial f) = 0 \tag{2.7}$$

and

$$\overline{\partial}^\nabla(\partial f) = d^\nabla(\partial f). \tag{2.8}$$

To compute (2.8) we first consider $d^\nabla(df)$ applied to $X, Y \in T\Sigma$:

$$\begin{aligned} d^\nabla(df)(X,Y) &= \nabla_X f_*(Y) - \nabla_Y f_*(X) - f_*([X,Y]) \\ &= (f^*T)(X,Y) \end{aligned} \tag{2.9}$$

where T is the torsion tensor of ∇. Thus

$$d^\nabla(df) = f^*T. \tag{2.10}$$

Now, for $X \in T\Sigma$

$$\begin{aligned}
\nabla_X(\partial f) &= \nabla_X\left(\frac{1}{2}(1-iJ)df\right) \\
&= \frac{1}{2}\nabla_X(df) - \frac{i}{2}(\nabla_X J)df - \frac{i}{2}J\nabla_X(df) \\
&= \frac{1}{2}(1-iJ)\nabla_X(df) - \frac{i}{2}(\nabla_X J)df.
\end{aligned} \tag{2.11}$$

Combining (2.9) and (2.11) we have

$$\begin{aligned}
\overline{\partial}^{\nabla}(\partial f) = d^{\nabla}(\partial f) &= \frac{1}{2}(1-iJ)f^*T - \frac{i}{2}\nabla J \wedge df \\
&= (f^*T)^{1,0} - \frac{i}{2}\nabla J \wedge df.
\end{aligned} \tag{2.12}$$

Now let $*$ be the Hodge star operator on Σ. If $\xi \in (T^*)^{1,0}$ then $*\xi = -i\xi$. In particular, $*\partial f = -i\partial f$. Thus when we apply the hermitian adjoint $(\partial^{\nabla})^* = - * \bar{\partial}^{\nabla} *$ to ∂f we obtain

$$\begin{aligned}
(\partial^{\nabla})^*(\partial f) = - * \overline{\partial}^{\nabla} * (\partial f) &= i * \overline{\partial}^{\nabla}(\partial f) \\
&= i * \left[(f^*T)^{1,0} - \frac{i}{2}(\nabla J) \wedge df\right].
\end{aligned} \tag{2.13}$$

Note that the right-hand side of (2.13) is quadratic in $df = \partial f + \overline{\partial} f$. Thus we can rewrite equations (2.13) and (2.7) as

$$\begin{aligned}
(\partial^{\nabla})^*\partial f &= q(\partial f, \partial f) \\
\partial^{\nabla}\partial f &= 0
\end{aligned} \tag{2.14}$$

where the coefficients of the quadratic q depend linearly on T and ∇J and therefore linearly on J and ∇J at the image point of f. These equations are an elliptic system for ∂f. We can rewrite them as a single equation by differentiating again and adding:

$$\Delta^f \partial f = (\partial^{\nabla}(\partial^{\nabla})^* + (\partial^{\nabla})^*\partial^{\nabla})\partial f = \partial^{\nabla}(q(\partial f, \partial f)). \tag{2.15}$$

Here Δ^f is the Hodge Laplacian; it depends on f because it involves the pullback connection. Now differentiate the right-hand side of (2.15). As (2.14) is an elliptic system all derivatives of df can be written as quadratic expressions in ∂f. Hence (2.15) becomes

$$\Delta^f \partial f = t(\partial f, \partial f, \partial f) \tag{2.16}$$

where t arises as the pullback of a tensor involving J and ∇J. Now the Hodge Laplacian is half the Laplacian $d^{\nabla}(d^{\nabla})^* + (d^{\nabla})^* d^{\nabla}$ on 1-forms, so has a Bochner–Weitzenböck formula

$$2\Delta^f(\partial f) = \nabla^*\nabla(\partial f) + K(\partial f) + R((\partial f), (\partial f))(\partial f) \tag{2.17}$$

where $\nabla^*\nabla$ is the trace Laplacian, K is (a constant times) the Gauss curvature of Σ and R is the sectional curvature of the hermitian connection on N. Altogether, we have

$$\nabla^*\nabla(\partial f) = -K(\partial f) + s((\partial f), (\partial f), (\partial f)) \tag{2.18}$$

where s involves J, ∇J, and R.

Theorem 2.1. (Regularity Theorem) *Suppose (Σ, j) and (N, ω) are smooth and $J \in \mathcal{J}^{s+2}$. Then any weakly jJ-holomorphic map $f : \Sigma \longrightarrow N$ that is $L^{1,p}$ for some $p > 2$, or is C^α for some $\alpha > 0$, lies in $L^{s,2}$.*

Proof. The Sobolev embedding implies that any $f \in L^{1,p}$ with $p > 2$ is Hölder continuous, so we can suppose that f is a C^α solution of the weak elliptic equation (2.14), i.e.,

$$0 = \int \langle(\partial^\nabla + (\partial^\nabla)^*)\xi, \partial f\rangle - \langle\xi, B(\partial f)\rangle \qquad \forall\xi \in \Omega^0(f^*TN) \oplus \Omega^2(f^*TN) \tag{2.19}$$

where $B = q(\partial f, \cdot)$. The coefficient B is Hölder continuous since $f \in C^\alpha$ and J and ∇J are C^1 tensors on N. Since regularity is local we can assume that ξ has support on a small ball in which we have a smooth moving frame $\{\omega_\alpha\}$. Then ∂^∇ and $(\partial^\nabla)^*$ are components of the covariant derivative, so are the ordinary ∂ and ∂^* plus terms that are smooth functions of f and df (cf. (A.8)). Theorem 1.11.1 in Morrey [M] then implies $f \in L^{2,2} \cap C^\alpha$. But then the right-hand side of the differentiated equation (2.18) lies in L^4. In fact, using (A.8) we can replace the covariant Laplacian with the ordinary Laplacian, moving all connection terms to the right and still have the right-hand side in L^4. Regularity for the Laplacian then means that $df \in L^{2,2} \hookrightarrow C^\alpha$, so $f \in C^{1,\alpha}$. Further regularity then follows by regarding (2.18) as a linear equation whose coefficients are Hölder continuous and applying standard bootstrap arguments, such as [M] Theorem 5.6.3. Our hypothesis that $J \in \mathcal{J}^{s+2}$ means that $s \in L^{s,2}$, so the bootstrap argument stops once we have $f \in L^{s,2}$. $\square$

Lemma 2.2. *Let f be a weak jJ-holomorphic map. Then the energy function $e(f) = |df|^2 = |\partial f|^2 + |\bar\partial f|^2 = 2|\partial f|^2$ satisfies*

$$\Delta e(f) \le C_1 e(f) + C_2 e^2(f) \tag{2.20}$$

weakly. The constants C_1 and C_2 depend only on the hermitian structures of Σ and N.

Proof. Since the connection is hermitian, we have

$$d\langle\partial f, \partial f\rangle = \langle\nabla\partial f, \partial f\rangle \ + \ \langle\partial f, \nabla\partial f\rangle. \tag{2.21}$$

Differentiating again and using (2.18) gives

$$\begin{aligned} d^*de(f) = d^*d\langle \partial f, \partial f\rangle &= \langle \nabla^*\nabla\partial f, \partial f\rangle - 2|\nabla\partial f|^2 + \langle \partial f, \nabla^*\nabla\partial f\rangle \\ &\leq 2|(\partial f)||\nabla^*\nabla(\partial f)| \\ &\leq C_1|(\partial f)|^2 + C_2|(\partial f)|^4 \\ &\leq C_1 e(f) + C_2 e^2(f). \qquad \square \end{aligned} \tag{2.22}$$

The only condition on our choice of metric μ on Σ is that its conformal class be the one determined by j. When we replace μ by a conformal metric $\lambda^{-2}\mu$ the pointwise energy rescales by $e_\lambda(f) = \lambda^2 e(f)$ (where $e_\lambda(f)$ is the energy function measured with respect to the metric $\lambda^{-2}\mu$) and the total energy $E(\Omega) = \int_\Omega e_\lambda(f)$ in any set Ω is invariant.

Theorem 2.3. (Main Energy Estimate) *There exist constants C and $\epsilon_0 > 0$, depending only on J and the metric on Σ, such that whenever $f : \Sigma \to N$ is a C^1 J-holomorphic map and $D(2r)$ is a geodesic disk of radius $2r$ with $E(2r) = \int_{D(2r)} e(f) \leq \epsilon_0$, then*

$$\sup_{D(r)} e(f) \leq \frac{CE(2r)}{r^2}. \tag{2.23}$$

Proof. Let ρ_0 be a maximum of the function

$$f(\rho) = \rho^2 \sup_{D(2r-2\rho)} e(f).$$

Set

$$e_0 = \sup_{D(2r-2\rho_0)} e(f)$$

and let x_0 be a point such that $e(x_0) = e_0$. It follows that $e(f) \leq e_0$ pointwise in the disk $D = D(x_0, \rho_0)$. Switching to the metric $\mu' = e_0\mu$, D has radius $R = \rho_0\sqrt{e_0}$, and $e'(f) = e_0^{-1}e(f) \leq 1$ pointwise on D. Hence by (2.27) e' satifies $(\Delta' - a)e' \leq 0$ where $a = C_1 + C_2$. The proof of Trudinger's Mean Value Theorem ([GT] Theorem 9.20) shows that there is a constant C such that

$$e'(x_0) \leq C(1 + aR^2)\frac{1}{R^2}\int_D e'.$$

Noting that $e'(x_0) = 1$ and $R^2 = \rho_0^2 e_0$ and returning to the original metric,

$$e_0 \leq \frac{C}{\rho_0^2}\int_D e + Cae_0E(2r).$$

When $E(2r) \leq \epsilon_0 = (2Ca)^{-1}$ we can rewrite this as

$$\rho_0^2 e_0 \leq 2C \int_{D(2r)} e(f)$$

and hence $f(\rho) \leq f(\rho_0) = \rho_0^2 e_0 \leq 2CE(2r)$. The result follows by taking $\rho = r/2$. $\square$

Versions of the above proof can be found in [S] and [W].

3. Isoperimetric inequalities and removable singularities

In this section we prove a removable singularity theorem for J-holomorphic curves. Our proof is based on the main energy estimate (2.30) and on an isoperimetric inequality for tamed J-holomorphic curves. The prototype removable singularity theorem is the result of Sacks–Uhlenbeck for harmonic maps [SU]. In our case the presence of the symplectic form leads to a strong isoperimetric inequality, which considerably simplifies the proof. We begin the section with a version of the isoperimetric inequality for J-holomorphic maps and one of its main corollaries, the monotonicity property of J-holomorphic maps. These results are used by Gromov and Pansu [Pa] to prove a C^0-removable singularity theorem (also see [McD2] and Lemma 3.7 below).

Lemma 3.1. (Isoperimetric Inequality) *Let h be a metric on N, hermitian for J. There exist constants ϵ_0 and $C > 0$ depending only on N, J, and h such that if $f : \tilde{\Omega} \to N$ is a J-holomorphic map with* $\operatorname{diam}(f(\tilde{\Omega})) \leq \epsilon_0$ *then any subdomain $\Omega \subset \tilde{\Omega}$ whose boundary is homeomorphic to a circle satisfies*

$$\text{Area}_h(f(\Omega)) \leq C \text{ length}_h^2(\partial f(\Omega)).$$

Proof. We first fix $x \in N$ and construct a local symplectic form ω that is taming for J. Let ω_0 be the constant 2-form on $T_x N$ defined by

$$\omega_0(v, J_x v) = h_x(v, v) \qquad v \in T_x N.$$

Then $\omega = (\exp^{-1})^* \omega_0$ is a symplectic form that is well defined and tames J on a neighborhood U of x.

Next define a hermitian metric g on U by

$$g(u, v) = \omega(u, Jv) \qquad u, v \in T_y N,\ y \in U. \tag{3.1}$$

Then g and h are both uniformly equivalent to a euclidean metric on a smaller set $V \subset U$. By compactness we can assume that V is a ball of radius ϵ_0 independent of x.

By our hypotheses and the compatibility condition (3.1), $f(\Omega)$ is a g-area minimizing surface for its boundary curve γ and lies in one of the neighborhoods V constructed above. After an arbitrarily small perturbation of Ω we can assume that γ is a smooth Jordan curve. Let Ω' be the solution to the Plateau problem for the euclidean metric in V with boundary γ (see [L]). This Ω' is a smooth minimal disk, so by the classical isoperimetric inequality

$$\mathrm{Area}_{\mathrm{euc}}(\Omega') \leq C \cdot \mathrm{length}^2_{\mathrm{euc}}(\gamma).$$

Since g is uniformly equivalent to the euclidean metric

$$\mathrm{Area}_g(\Omega) \leq \mathrm{Area}_g(\Omega') \leq C'\mathrm{Area}_{\mathrm{euc}}(\Omega') \leq C''\mathrm{length}^2_{\mathrm{euc}}(\gamma) \leq C'''\mathrm{length}^2_g(\gamma). \tag{3.2}$$

The result follows because g and h are uniformly equivalent. $\square$

The next two geometric results are immediate consequences of the isoperimetric inequality and the main energy estimate of the previous section. The first shows that the well-known monotonicity property of minimal surfaces also applies to J-holomorphic maps. The monotonicity formula says that, locally, the area of the image grows essentially like that of a two-dimensional plane. (Recall from Section 1 that energy and area are the same). Thus the images of holomorphic maps are not too sparse. The second result—applied to concentric disks $D(\epsilon) \subset D(2\epsilon)$—gives a bound for the diameter of the image in terms of the area. This shows, in particular, that the images of disks are not too elongated.

Corollary 3.2. (Monotonicity) *Suppose that $f : \Omega \to N$ is J-holomorphic on a domain Ω. There is a constant c such that for any sufficiently small ball $B(p,\delta)$ in N with center p in $f(\Omega)$ and with no boundary inside $B(p,\delta)$ we have*

$$\mathrm{Area}(f(\Omega) \cap B(p,\delta)) \geq c\delta^2. \tag{3.3}$$

Proof. By the isoperimetric inequality of Lemma 3.1, the function $A(r) = \mathrm{Area}(f(\Omega) \cap B(p,r))$ satisfies $\sqrt{A(r)} \leq c_1 L = c_1 A'(r)$ for sufficiently small r. Integrating form 0 to δ yields (3.3). $\square$

Lemma 3.3. *There is a constant $C = C(J,h)$ such that for any J-holomorphic map $f : \Omega \to N$ and any subdomain $D \subset \Omega$ with* $\mathrm{dist}(D, \partial\Omega) \geq \epsilon$ *we have*

$$\mathrm{diam}(f(D)) \leq \frac{C}{\epsilon}\,\mathrm{diam}(D)\sqrt{\mathrm{Area}(f(\Omega))}. \tag{3.4}$$

Here the diameter of D is measured along paths in D. In particular, when f is defined on a closed surface Σ

$$\mathrm{diam}(f(\Sigma)) \leq C'\sqrt{\mathrm{Area}(f(\Sigma))}.$$

Proof. Cover Ω by disks $D_\alpha(x_i, \epsilon_i)$ with $\epsilon_i \le \epsilon$ such that the energy of f on each disk is less than the constant ϵ_0 of Theorem 2.3. The main energy estimate (2.30) applies to each of these disks. For any two points $x, y \in D$ we have $\text{dist}(f(x), f(y)) \le \text{dist}(x, y)\ \sup_D |df|$ and hence

$$\frac{\text{diam}(f(D))}{\text{diam}(D)} \le \sup_D |df| \le \sup_\alpha \sup_{D_\alpha} \frac{C}{\epsilon}\sqrt{\text{Area}(f(D_\alpha))} \le \frac{C}{\epsilon}\sqrt{\text{Area}(f(\Omega))}. \qquad \square$$

The isoperimetric inequality above applies to J-holomorphic maps in coordinate neighborhoods, and this is not sufficient for some applications. However, we can use the symplectic form to prove a stronger version, valid for *tamed* J-holomorphic curves.

Proposition 3.4. (Strong Isoperimetric Inequality) *Let $\Omega \subset \Sigma$ be a domain with boundary components $\{\gamma_i\}$. There are constants ϵ_0, C such that any smooth tamed J-holomorphic map $f : \Omega \to N$ with* $\text{length}(f(\gamma_i)) \le \epsilon_0\ \forall i$ *has an associated homology class α and satisfies*

$$\text{Area}(f(\Omega)) \le C\left[A(\alpha) + \sum_i \text{length}^2(f(\gamma_i))\right] \tag{3.5}$$

where $A(\alpha) = \langle w, [\alpha]\rangle$ is the "symplectic area" of the homology class α.

Proof. Let ϵ_0 be the injectivity radius of the J-tamed metric on N. For each point $p \in N$ the ball $B(p, \epsilon_0)$ is contractible, so by the Poincaré Lemma $\omega = d\beta_p$ on $B(p, \epsilon_0)$. In fact, β_p can be determined by integrating outward from p, so satisfies a bound

$$|\beta_p(x)| \le C\text{dist}(p, x) \qquad \forall x \in B(p, \epsilon_0). \tag{3.6}$$

Here C depends on p, but by the compactness of N we can find a uniform constant C.

The hypothesis of the proposition ensures that the image of each boundary component γ_i lies in a ball B_i of radius $=$ $\text{length}(f(\gamma_i))$. Choose smooth disks $D_i \subset B_i$ with $\partial D_i = f(\gamma_i)$. The homology class of such disks D_i is well defined. Consequently, the closed surface

$$S = f(\Omega)\ \cup\ \bigcup_i D_i$$

defines a homology class α that is naturally associated to $f(\Omega)$. Its symplectic area is

$$A(\alpha) = \int_S \omega = \int_{f(\Omega)} \omega - \sum_i \int_{D_i} \omega,$$

so by the tamed condition we have

$$\text{Area}(f(\Omega)) \leq c \int_{f(\Omega)} \omega = c \left[A(\alpha) + \sum_i \int_{D_i} \omega \right].$$

On the other hand, $\omega = d\beta_i$ on each D_i, so by (3.6)

$$\int_{D_i} \omega = \int_{f(\gamma_i)} \beta_i \ \leq \ [\text{length}(f(\gamma_i))] \ \sup_{B_i} |\beta_i| \ \leq \ C \ \text{length}^2(f(\gamma_i)). \tag{3.7}$$

The result follows. $\square$

We now turn to the problem of removing a point singularity. Let $D = D(p, r_1)$ be a disk in a Riemann surface (Σ, j). We will use geodesic coordinates centered at p, writing $r = |x|$ for $x \in D$ and $D(r)$ for disks centered at $p = 0$. Suppose that f is a smooth finite-energy J-holomorphic map from a punctured disk $D - \{0\}$ into a symplectic manifold with tamed almost complex structure J. We will show that f extends to a smooth map on D. We will do this using the Regularity Theorem 2.1. That theorem has two hypotheses. The first is completely straightforward to verify:

Lemma 3.5. *The map f above is weakly jJ-holomorphic on D.*

Proof. Fix a smooth test function ξ compactly supported in D and a family $\{\beta_\epsilon\}$ of smooth cutoff functions with supp $\beta_\epsilon \subset D(\epsilon)$ and satisfying $0 \leq \beta_\epsilon \leq 1$ and $|d\beta_\epsilon| \leq C/\epsilon$. Then the integral (2.19) becomes (with the obvious notational shorthand)

$$\begin{aligned} \int \langle D\xi, \partial f\rangle + \langle \xi, B(\partial f)\rangle \quad &= \quad \int \langle D((1-\beta_\epsilon)\xi), \partial f\rangle + \langle (1-\beta_\epsilon)\xi, B(\partial f)\rangle \\ &\quad + \int \langle D(\beta_\epsilon \xi), \partial f\rangle + \langle (\beta_\epsilon \xi), B(\partial f)\rangle. \end{aligned} \tag{3.8}$$

The first integral on the right-hand side vanishes after integrating by parts since f is J-holomorphic on supp $(1-\beta_\epsilon) \subset D - \{0\}$. Noting that (again using shorthand) $D(\beta_\epsilon \xi) = \beta_\epsilon D\xi + d\beta_\epsilon \cdot \xi$ has support in $D(\epsilon)$ and applying Hölder's inequality, one sees that the last integral in (3.8) is dominated by

$$C\,(\epsilon \|D\xi\|_\infty + \|\xi\|_\infty) \left(\int_{D(\epsilon)} |df|^2 \right)^{\frac{1}{2}}.$$

This vanishes as $\epsilon \longrightarrow 0$. Thus f is weakly J-holomorphic. $\square$

We can now apply the Regularity Theorem 2.1 to remove the singularity provided we can show that f is Hölder continuous. By a well-known lemma of Morrey [M, Theorem 3.5.2] this is true if f satisfies an energy growth condition of the form

$$\int_{D(r)} |df|^2 \leq Cr^{\alpha}. \tag{3.9}$$

Of course, since we are assuming that f is smooth away from the origin it suffices to show this for disks centered at the origin. The standard technique for doing this—pioneered by Morrey—is to construct a comparison surface and apply an isoperimetric inequality. In our case, most of this work is done by the strong isoperimetric inequality (3.5).

Theorem 3.6. (Removable Singularities I) *Let (N, ω, J) be a smooth symplectic manifold with tamed almost complex structure. Then any smooth finite area J-holomorphic map from a punctured disk $D - \{0\}$ in Σ to N extends to a smooth J-holomorphic map on D.*

Proof. For each sufficiently small r we can apply the main energy estimate (2.30) on the disk $D(x, r/2)$. This shows that $|df|(x) \leq Cr^{-1}\sqrt{E(D(0,2r)}\ \forall x$ with $\mathrm{dist}(0, x) = r$, so

$$\text{length}(f(S_r)) \leq 2\pi r \sup_{S_r} |df| \leq C'\sqrt{E(D(2r))}. \tag{3.10}$$

where $S_r = \partial D(0, r)$. Thus for some $r_1 > 0$, we have

$$\text{length}(f(S_r)) \leq \epsilon_0 \qquad \forall r \leq r_1 \tag{3.11}$$

where ϵ_0 is the constant of Proposition 3.4 (the injectivity radius of N).

Now choose $\delta < \rho < r_1$ and apply Proposition 3.4 to the annulus $A(\delta, \rho)$. Using polar coordinates on the annulus, we can consider f as a smooth map $[\delta, \rho] \times S^1 \longrightarrow N$. But (3.11) implies that for each $r \leq r_1$ the image circle lies in a contractible set in N. It follows that the homology class α associated to $f(A(\delta, \rho))$ by Proposition 3.4 is the zero class. Thus

$$\text{Area}(f(A(\delta, \rho))) \leq C\left[\text{length}^2(f(S_\rho)) + \text{length}^2(f(S_\delta))\right].$$

We can then use (3.10) to take the limit as $\delta \longrightarrow 0$ and estimate using Hölder's inequality:

$$\begin{aligned}
\text{Area}(f(D(\rho))) \leq C\ \text{length}^2(f(S_\rho)) &= C\left(\int_{S(\rho)} |\partial_\theta f|\right)^2 \\
&\leq C\ \text{length}(S_\rho) \cdot \int_{S(\rho)} |\partial_\theta f|^2 \\
&\leq C'\rho \frac{d}{d\rho}\text{Area}(f(D_\rho)).
\end{aligned}$$

Hence

$$\frac{d}{d\rho}\ln \text{Area}(f(D_\rho)) \geq \frac{\alpha}{\rho}$$

where $\alpha = (C')^{-1}$. Integrating from r to r_1 gives

$$\text{Area}(f(D_r)) \leq Cr^\alpha$$

for all $r \leq r_1$ (where the constant C depends on r_1). This is exactly the needed energy growth rate (3.9), and the result follows, as described above, from Morrey's Lemma and the Regularity Theorem 2.1. □

Remark. When the almost complex structure J is ω-compatible, J-holomorphic curves are minimal surfaces and hence harmonic maps. Thus in the J-compatible case, the removable singularity theorem for J-holomorphic curves follows from the harmonic map removable singularity theorem of Sacks and Uhlenbeck. However, even in that case the above proof is easier and more geometric. Oh has used a version of this argument to show removability of singularities at certain boundary points [Oh]. □

The "tamed" condition in Theorem 3.6 is the appropriate one when studying symplectic manifolds. However, the removable singularity problem makes sense in a more general setting: for J-holomorphic maps between almost complex manifolds (without symplectic structure). It is easy to extend the above result to this general context using arguments similar to those of [Pa].

Theorem 3.7. (Removable Singularities II) *Let (N, J) be a smooth almost complex manifold. Then any smooth finite area J-holomorphic map from a punctured disk $D - \{0\}$ in Σ to N extends to a smooth J-holomorphic map on D.*

Proof. We first show that f above extends continuously across the origin. Given $\delta > 0$, choose r small enough that $E(D(4r)) < c\delta^2(1 + C')^{-2}$ where $c \leq 1$ is less than the constant in (3.3) and C' is greater than the constant in (3.10). Then (3.10) shows that $f(S_r) \subset B_1 = B(f(x), \delta)$ for some $x \in S_r$. If there is a $y \in D(r)$ with $f(y) \notin B(f(x), 4\delta)$ then (3.10) shows that $f(S_{|y|})$ lies in $B_2 = B(f(y), \delta)$ and by continuity we know there is a z with $|y| < |z| < r$ and with $B_3 = B(f(z), \delta)$ disjoint from B_1 and B_2. But then $f(D_r - D_{|y|})$ has no boundary in B_3 and

$$\text{Area}(f(D_r - D_{|y|}) \cap B_3) \leq \text{Area}(f(D_r)) = E(D_r) < c\delta^2,$$

contradicting (3.3). Therefore $f(D_r)$ lies in a ball of radius 4δ and f extends continuously to D.

We can now construct, as in the proof of Lemma 3.1, a symplectic form ω on a neighborhood U of $x = f(0) \in N$ that is taming for J. Since f is continuous we have $f(D_r)) \subset U$ for small r, and hence the singularity is removable by the previous theorem. □

4. Bubbling

In this section we describe the renormalization procedure that results in the "tower of bubbles." This procedure is quite general (see the remark at the end of this section). Our starting point is the following theorem that is based on the "bubbling argument" of Sacks and Uhlenbeck.

Theorem 4.1. *Let $\{J_n\}$ be a sequence of almost complex structures on N converging in C^1 to J and $\{f_n\}$ a sequence of J_n-holomorphic maps $\Sigma \to N$ with $E(f_n) < C$. Then there is a subsequence of the $\{f_n\}$, a finite set of points $\{x_1, \ldots, x_k\} \in \Sigma$, and a J-holomorphic map $f_0 : \Sigma \to N$ such that*

(a) $f_n \to f_0$ *in* C^1 *on* $\Sigma - \{x_1, \ldots, x_k\}$.

(b) The $e(f_n)$ *converge as measures to* $e(f_0)$ *plus a sum of point measures:*

$$e(f_n) \to e(f_0) + \sum_{i=1}^{k} m_i \ \delta(x_i) \tag{4.1}$$

where each $m_i \geq B_0$ (B_0 is the constant of Proposition 1.1b).

Remark. Here, and throughout these arguments, we use the convention of immediately renaming subsequences with the original subscripts, so a subsequence of $\{f_n\}$ is still denoted $\{f_n\}$. □

Proof. We will give the proof under the assumption that $J_n = J$ is a fixed almost complex structure; the general case follows because all estimates in the argument are uniform for a converging sequence of J_n.

Choose an $r_0 > 0$ and set $r_m = 2^{-m} r_0$, $m \in \mathbb{Z}_+$. For each m, choose a finite covering $\mathcal{C}_m = \{D(y_\alpha, r_m) \mid y_\alpha \in \Sigma\}$ of Σ such that each point of Σ is covered at most h times by the disks in $\mathcal{C}_m$ and such that $\{D(y_\alpha, r_m/2)\}$ is still a covering of Σ (here h depends on Σ only). Let Σ_0 be the constant determined by Theorem 2.3. For each j, C

$$\sum_\alpha \int_{D(y_\alpha, r_m)} e(f_j) \leq hC,$$

so for each j there are at most hC/ϵ_0 disks on which

$$\int_{D(y_\alpha, r_m)} e(f_j) \geq \epsilon_0. \tag{4.2}$$

The center points of these disks make at most hC/ϵ_0 sequences of points of Σ (by letting $j = 1, 2, \ldots$). Since $\mathcal{C}_m$ is a finite covering and Σ is compact we may assume these center points are fixed by passing to a subsequence of $\{f_j\}$. For each m, call these center points $\{x_{1,m}, \ldots, x_{l,m}\}$ where l is at most hC/ϵ. By the C^1 estimates and the Ascoli theorem we can successively choose a subsequence of $\{f_j\}$ that converges (in C^1) in every disk $\{D(y_\alpha, r_m/2)\}$ for each $D(y_\alpha, r_m) \in \mathcal{C}_m$ except for at most l disks of $\mathcal{C}_m$. Now let $m \to \infty$. We can then choose a subsequence of $\{m\}$ such that $\{x_{1,m}, \ldots, x_{l,m}\}$ converge to points $\{x_1, \ldots, x_l\}$. Choosing a diagonal subsequence of $\{f_j\}$ gives a sequence that converges in C^1 on $\Sigma - \{x_1, \ldots, x_l\}$. Denote the limit by f_0. By elliptic regularity f_0 is smooth and J-holomorphic on $\Sigma - \{x_1, \ldots, x_l\}$, and therefore extends to a smooth J-holomorphic map $f_0 : \Sigma \to N$ by the removable singularity theorem (Theorem 3.6). This completes the proof of (a).

Observe that if C^1 convergence fails at x_i then for any $\epsilon > 0$ the numbers

$$b_j^i = \sup \{|df_j| \ : \ x \in D(x_i, \epsilon)\}$$

are unbounded (otherwise the Ascoli theorem implies C^1 convergence on $D(x_i, \epsilon)$ for a subsequence). We henceforth assume that the $\{b_j^i\}$ are unbounded for each $i = 1, \ldots, l$.

Now fix an $\epsilon > 0$, less than half the injectivity radius of (Σ, μ), such that the disks $D(2\epsilon, x_i) \subset \Sigma$ are disjoint and set

$$m_i = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{D(x_i, \epsilon)} \left| \, |df_n|^2 - |df_0|^2 \, \right| dv_\mu. \tag{4.3}$$

It follows immediately that the energy density measures converge as in (4.1), so it remains only to show that $m_i \geq B_0$.

In the Sacks–Uhlenbeck argument [SU] this is done by renormalizing the maps f_n to construct a "bubble" at each x_i. For notational simplicity we will fix i and write x and b_n for x_i and b_n^i. Let $\overline{x}_n$ be the point in $D(x, \epsilon)$ at which $|df_n|$ achieves its supremum, so $|df_n(\overline{x}_n)| = b_n$. Then $\overline{x}_n \to x$ as $n \to \infty$ by part (a) above. Fix a coordinate system in each disk $D(x, 2\epsilon)$ and define renormalized maps by

$$\tilde{f}_n(y) = f_n(\overline{x}_n + y/b_n) \qquad \text{for } y \in D(0, \epsilon b_n) \tag{4.4}$$

Then $|d\tilde{f}_n(y)| \leq 1$ at each $y \in D(0, \epsilon b_n)$ and $|d\tilde{f}_n(0)| = 1$. The $\tilde{f}_n$ are then a sequence of J-holomorphic maps with bounded energy. By composing these with a fixed conformal identification (stereographic projection) of $S^2 - \{p\}$ with $\mathbb{R}^2$, we can regard these as maps from domains in $S^2 - \{p\}$ into N. They are still J-holomorphic with bounded energy, and the renormalization (4.4) ensures that the sequence $\{\tilde{f}_n\}$ is bounded in C^1. Repeating the above argument yields a subsequence $\{\tilde{f}_n\}$ that converges in C^1 on $S^2 - \{p\}$ to a J-holomorphic map $\tilde{f}$, and removing the singularity at p gives a J-holomorphic "bubble" map $\tilde{f}_{x_i} : S^2 \to N$ associated to x_i. This is not a map to a point since $|d\tilde{f}_{x_i}| = 1$ at at least one point. Using Lemma 1.1 we then have $m_i \geq E(\tilde{f}_{x_i}) \geq B_0$. $\square$

Theorem 4.1 provides one convergence result for J-holomorphic curves. It turns out, however, to have two related shortcomings. First, the inequality $m_i \geq E(\tilde{f}_{x_i})$ can be strict. This means that more energy has bubbled off at x_i than has been captured by the map $\tilde{f}_{x_i}$. Essentially the Sacks–Uhlenbeck procedure is sufficient to produce *some* J-holomorphic map from the bubble, but does not record *all* the J-holomorphic maps associated with the bubble. Second, the argument cannot be iterated. This is partly because it uses a variety of norms—C^1, sup, and energy—and one loses track of these in the renormalization (4.4). To rectify these deficiencies we modify the Sacks–Uhlenbeck renormalization procedure.

The idea is to replace the renormalization (4.4) by one that depends only on the energy. The new procedure involves four steps. In the first three, all quantities depend on the (arbitrary) choice of $\epsilon > 0$ made above; in the fourth step we let $\epsilon \longrightarrow 0$. The entire procedure depends on the choice of a "scaling constant" $C_0 > 0$; for now we only require that $C_0 < B_0/2$. We will continue to work with a fixed i and write x_i and m_i as simply x and m.

Step 1. (Pullback) We first identify a neighborhood of the bubble point with a domain in a sphere and pullback the maps.

Let S_x^2 denote the unit two-sphere in $\mathbb{R}^3$ with standard complex structure j, standard measure dv, and two distinguished antipodal points $p^+ = (0,0,1)$ and $p^- = (0,0,-1)$. Fix a stereographic projection $\sigma : S_x^2 \longrightarrow T_x\Sigma$, with $\sigma(p^-) = 0$ and $\sigma(p^+) = \infty$. Use σ and the exponential map to pullback f_n and f_0, obtaining maps (still denoted f_n and f_0) from a disk $D(\epsilon) \subset S_x^2$ to N. For each n let $q_n \in \mathbb{R}^3$ be the center of mass of the measure $\big|\, |df_n|^2 - |df_0|^2 \,\big|\, dv$, i.e., $q_n = (x_n^1, x_n^2, x_n^3)$ where

$$x_n^i = \int_{D(\epsilon)} x^i \left| \, |df_n|^2 - |df_0|^2 \, \right| dv, \qquad i = 1, 2, 3.$$

By (4.3) and the conformal invariance of the energy, the numbers

$$m(\epsilon, n) = \int_{D(\epsilon)} \left| \, |df_n|^2 - |df_0|^2 \, \right| dv$$

satisfy

$$\lim_{\epsilon \longrightarrow 0} \overline{\lim_{n \longrightarrow \infty}} \, m(\epsilon, n) = m \geq B_0.$$

Hence by making ϵ smaller and passing to a subsequence we can assume that $m(\epsilon, n) > B_0/2$ for all n.

Step 2. (Translation) Now compose with conformal transformations to move the center of mass onto the z-axis.

Each translation of $T_x\Sigma$ corresponds under σ to a conformal transformation of S_x^2 that fixes the north pole p^+. For each n there is a unique such transformation T_n such that the measure corresponding to the translated maps $\overline{f}_n = f_n \circ T_n$ and $\overline{f}_0 = f_0 \circ T_n$ has center of mass $\overline{q}_n$ on the z-axis. Since p^- corresponds under $\sigma \circ \exp$ to the bubble point x, it follows that $\overline{q}_n \longrightarrow p^-$. Hence each $\overline{f}_n$ and $\overline{f}_0$ are defined in some translated disk containing p^-.

Step 3. (Renormalization) We next compose with conformal transformations to make the energy of each f_n in the northern hemisphere $H^+ \subset S_x^2$ exactly equal to C_0.

Radial dilations of $T_x\Sigma$ correspond under σ to a one-parameter family ρ_t of conformal dilations of S_x^2. For each $t > 0$, the pullback maps $f_{n,t} = \overline{f}_n \circ \rho_t$ and $f_{0,t} = \overline{f}_0 \circ \rho_t$ are defined on a disk D_t in S_x^2. Since the energy is conformally invariant, the numbers

$$m(\epsilon, n) = \int_{D_t} \left| \, |df_{n,t}|^2 - |df_{0,t}|^2 \, \right| dv \tag{4.5a}$$

are independent of t and larger than $B_0/2$. Hence for each n there is a unique t_n such that

$$\int_{D_{t_n} - H^-} \left| \, |df_{n,t_n}|^2 - |df_{0,t_n}|^2 \, \right| dv = C_0, \tag{4.5b}$$

where H^- is the southern hemisphere. Since x is a bubble point and $C_0 < B_0/2$, it follows from Theorem 4.1 that $t_n \to \infty$ as $n \to \infty$.

Now the renormalization: for each n let R_n be the composition

$$R_n : S_x^2 \xrightarrow{\rho_{t_n}} S_x^2 \xrightarrow{T_n} S_x^2 \xrightarrow{\sigma} T_x\Sigma \xrightarrow{\exp} \Sigma. \tag{4.6}$$

and define the renormalized sequence $\{\tilde{f}_n\}$

$$\tilde{f}_n = f_{n,t_n} = R_n^* \left(f_n|_{B(x,\epsilon)} \right). \tag{4.7}$$

Then $\tilde{f}_n$ is defined on S_x^2 except on a disk around p^+ whose radius becomes arbitrarily small as $n \to \infty$. Again, because the energy is conformally invariant, the renormalized maps have energy $E(\tilde{f}_n) \le E(f_n) \le C$ and (4.5) implies

$$|E(\tilde{f}_n) - E(\tilde{f}_0)\,| > B_0/2 \quad \text{and} \quad \int_{H^+} \left| |d(\tilde{f}_n)|^2 - |d(\tilde{f}_0)|^2 \, \right| dv = C_0 \tag{4.8ab}$$

where H^+ is the northern hemisphere.

Now $\tilde{f}_n$ is holomorphic with respect to the induced complex structure $j_n = R_n^* j$, and the sequence $\{j_n\}$ converges to the standard complex structure j on compact sets $K \subset S_x^2 - \{p^+\}$. For each $\rho > 0$ apply the argument of Theorem 4.1 to the sequence $\{\tilde{f}_n\}$ on the compact set $S_x^2 - D(p^+, \rho)$, then let $\rho \to 0$ and take a diagonal subsequence. This yields a subsequence of the $\{\tilde{f}_n\}$ that converges in C^1 on $S_x^2 - \{y_1, \dots y_l, p^+\}$ to a smooth jJ-holomorphic map $\tilde{f}_x : S_x^2 \to N$, and

$$e(\tilde{f}_n) \to e(\tilde{f}_x) + \sum_{j=1}^{l} m_j \, \delta(y_j) + \tau_x \delta_{p^+}. \tag{4.9}$$

where $\tau_x = \overline{\lim}\, E(\tilde{f}_n) \; - \; E(\tilde{f}_x) - \sum_{j=1}^{l} m_j$.

Step 4. (The $\epsilon \to 0$ limit) Each of the quantities in (4.9)—the $\{m_j\}$, τ, the $\{y_j\}$, and the limit map $\{\tilde{f}_x\}$—depend on the choice of $\epsilon > 0$ made earlier. Now let $\epsilon \to 0$. Again we have a family of jJ-holomorphic maps $\{\tilde{f}_x(\epsilon)\}$ with bounded energy, and again Theorem 4.1 shows that there is a subsequence converging in C^1 off another set of points $\{z_j, p^+\}$ to a smooth jJ-holomorphic map $\tilde{f}_x$. As for the singular part of the measures on the left-hand side of (4.9), we know, using Proposition 1.1b, that each m_j lies between B_0 and C and $\tau_x < C_0$. Hence as $\epsilon \to 0$ the $2l + 1$-tuple $\{y_1, y_2, \ldots y_l, m_1, m_2 \ldots m_l, \tau_x\}$ ranges over the compact set

$$\bigcup_{l=0}^{[C/B_0]} \mathrm{Sym}^l(S^2) \times [B_0, C]^l \times [0, C_0],$$

and therefore has a convergent subsequence. Thus we obtain a limiting jJ-holomorphic map $\tilde{f}_x : S^2_x \to N$ and points $\{y_j\}$ (the limit points just found together with the z_j) such that a diagonal subsequence $\{\tilde{f}_n\}$ of the sequence $\{\tilde{f}_n(\epsilon)\}$ converges to $\tilde{f}_x$ in C^1 on $S^2_x - \{y_1, \ldots y_{l'}, p^+\}$, and

$$e(\tilde{f}_n) \to e(\tilde{f}_x) + \sum_{j=1}^{l'} m_j \; \delta(y_j) + \tau_x \delta_{p^+}. \tag{4.10}$$

For each $\epsilon > 0$ the center of mass of $|e(\tilde{f}_n(\epsilon)) - e(\tilde{f}_0(\epsilon))|$ lies on the z-axis. Moreover, as $\epsilon \to 0$ the $e(\tilde{f}_0(\epsilon))$ converge to 0 as measures. Hence the limit measure in (4.10) has center of mass on the z-axis and (4.8) becomes

$$E(\tilde{f}_n) \to m_x \geq B_0 \quad \text{and} \quad \int_{H^+} |d(\tilde{f}_n)|^2 \, dv \to C_0. \tag{4.11ab}$$

In summary, this renormalization procedure associates to each point x_i of (4.1) a sequence of J-holomorphic maps $S^2 \to N$ which converge to a jJ-holomorphic map $\tilde{f}_{x_i} : S^2 \to N$ in C^1 on $\Sigma - \{y_1, \ldots, y_{l'(x_i)}, p^+\}$. The map $\tilde{f}_{x_i}$ is called a *bubble* and the quantity τ is called the *energy loss* associated with the point x_i. We will examine the energy loss in detail in the next section. Note that we can repeat the renormalization with each y_j, $j = 1, \ldots, l'(x_i)$, obtaining bubbles on bubbles.

It is possible for $\tilde{f}_x$ to have $E(\tilde{f}_x) = 0$, so the bubble is a map to a single point. In this case we say that $\tilde{f}_x$ is a *ghost bubble*. The following lemma places an important constraint on this possibility.

Lemma 4.2. *If $E(\tilde{f}_x) = 0$ then either $l' \geq 2$, or $l' = 1$ and $\tau_x = C_0$.*

Proof. If $E(\tilde{f}_x) = 0$ then $e(\tilde{f}_x) \equiv 0$. If $l' = 0$ then (4.10) implies that $E(\tilde{f}_n) \to \tau_x \leq C_0 < B_0/2$, contradicting (4.11a). If $l' = 1$ then $e(\tilde{f}_n) \to m\delta_y + \tau_x \delta_{p^+}$ for some $y \in S^2_x$ and

$m = m_x \geq B_0$. The center of mass of the limit measure $m\delta_y + \tau_x \delta_{p^+}$ lies on the z-axis, so $y = p^+$ or p^-. The possibility $y = p^+$ contradicts (4.11b) because $m_x > C_0$. Thus $y = p^-$, and (4.11b) then shows that $\tau_x = C_0$. $\square$

Lemma 4.2 and Proposition 1.1b show that when the renormalization procedure is iterated each step reduces the energy by at least C_0. Hence the process terminates after at most C/C_0 iterations and the tower of bubbles is finite.

For some applications it is important to keep careful track of the choices and parameters involved in the bubbling process. One such parameter occurs in step 1, where we chose an element σ in the space $Ster$ of all orientation-preserving conformal maps $\sigma : S^2 \to \mathbb{R}^2$ with $\sigma(p^-) = 0$ and $\sigma(p^+) = \infty$. Note that the group $\mathbb{C}^*$ acts conformally on $\mathbb{R}^2 = \mathbb{C}$ by complex multiplication and hence acts on $Ster$ by composition.

The corresponding global picture is described in terms of bundles associated to the bundle $F\Sigma$ of oriented conformal—or equivalently complex—frames on Σ, which is a principal $\mathbb{C}^*$-bundle over Σ determined by the oriented conformal (complex) structure of Σ. The complex tangent bundle is $T\Sigma = F\Sigma \times_{\mathbb{C}^*} \mathbb{C}$ and its compactification is $S\Sigma = F\Sigma \times_{\mathbb{C}^*} S^2_x$. Sections of the bundle $F\Sigma \times_{\mathbb{C}^*} Ster$ are global (fiberwise) stereographic projections

$$\sigma : S\Sigma \to T\Sigma,$$

and the space $\mathcal{S} = \Gamma(F\Sigma \times_{\mathbb{C}^*} Ster)$ of such σ is acted on (by $\sigma \mapsto \lambda \circ \sigma$) by the "gauge group" $\mathcal{H}$ of nowhere vanishing complex-valued functions on Σ.

The other effective parameter in our bubbling process is a choice of a metric μ within the conformal class; this is the same as a section $\mu \in \Gamma(\Lambda)$ of the real line bundle $\Lambda = F\Sigma \times_{\mathbb{C}^*} \mathbb{C}^*/S^1$. The following proposition shows a conformal change of metric can be absorbed in a gauge transformation. To state it, we let the group $\mathcal{H}_{\mathbb{R}}$ of positive real functions act on the pair (μ, σ) by $(\mu, \sigma) \to (\lambda^{-1}\mu, \lambda\sigma)$.

Proposition 4.3. *The numbers $\{m_j\}$ and τ, the points $\{y_j\} \in S\Sigma$, and the limit map $\{\tilde{f}_x\} : S\Sigma \to N$ in (4.10) depend only on the subsequence $\{f_n\}$ and the orbit of the action of $\mathcal{H}_{\mathbb{R}}$ on (μ, σ).*

Proof. Replace μ with a conformal metric $\mu' = \phi^2\mu$. The exponential maps $\exp_x$ and $\exp'_x$ of these metrics then satisfy $\exp' = \lambda(Id. + O(\epsilon))\exp$ on $D(0,\epsilon) \subset T_x\Sigma$, where $\lambda = \phi(x_i) \in \mathbb{R}$. It follows that ${\sigma_n^\epsilon}' = \lambda\, \sigma_n^\epsilon(1 + O(\epsilon))$. Renormalizing by steps 2 and 3 above, one finds that the map $\tilde{f}_x^\epsilon$ in (4.9) is replaced by $\tilde{f}'_x = \tilde{f}_x \circ \lambda \circ \sigma(Id. + O(\epsilon))$. Letting $\epsilon \to 0$ and $n \to \infty$, we see that the renormalized map $\tilde{f}_x$ constructed from the map σ and the metric μ is the same as the one constructed from $\lambda^{-1}\sigma$ and $\lambda\mu$. $\square$

Now repeat the construction of $S\Sigma$. Compactifying the vertical tangent space of $S\Sigma \to \Sigma$ gives an S^2_x-bundle $S^2\Sigma$ over $S\Sigma$. Iterating yields a tower of S^2_x fibrations

$$\cdots \to S^k\Sigma \to S^{k-1}\Sigma \to \cdots \to S\Sigma \to \Sigma. \tag{4.13}$$

Definition. A *bubble domain* at level k is a fiber B of $S^k\Sigma \to S^{k-1}\Sigma$ (B is a copy of S^2_x). A *bubble domain tower* is a finite union T of bubble domains that form a tower, i.e., such that the projection of $T \cap S^k\Sigma$ lies in $T \cap S^{k-1}\Sigma$.

Given a sequence of J-holomorphic maps, the iterated renormalization procedure singles out a bubble domain tower $T = \cup B_i$. At each iteration, renormalization gives J-holomorphic maps $B_i \to N$—these are the bubbles referred to above. The union of the renormalization maps is a map from T to N that we will call a *bubble tower*. The image of such a bubble tower forms what Gromov calls a "cusp-curve" [G].

Remark. The renormalization procedure described above applies without change to sequences of harmonic maps and for sequences of α-harmonic maps (i.e., critical points of the Sacks–Uhlenbeck α-energy) with $\alpha \to 1$. The needed estimates and removable singularities theorem are in [SU]. $\square$

5. The bubble tree

The bubble tower constructed in Section 4 has the structure of a tree where the vertices are jJ-holomorphic maps and the edges are bubble points. The tree is constructed from the sequence $\{f_n\}$ as follows: The $\{f_n\}$ converge to $f_0 : \Sigma \to N$ on $\Sigma - \{x_1, \dots, x_k\}$. The base vertex of the tree is the map f_0, which we relabel f, and the edges emanating from the base vertex are the points $\{x_i\}$. Each edge has a mass m_i, which is the mass of the limiting measure at x_i. For each x_i the renormalization process gives a sequence $\{\tilde{f}_n\}$ converging to a J-holomorphic map $f_i : B_i \to N$ from the bubble domain $B_i \cong S^2_x$ at x_i, plus the sum of point measures $\sum_j m_{ij}\delta(x_{ij}) + \tau_{x_i}\delta_{p_i^+}$. We label each edge by the pair (x_i, τ_{x_i}), where τ_{x_i} is the energy loss associated with the bubble point x_i. The edge (x_i, τ_{x_i}) terminates in the vertex f_i which, in turn, is the source of new edges $\{x_{ij}\}$, and so on. An example of a bubble tree was given in the introduction.

Associated to each vertex $f_I = f_{i_1 \dots i_k}$ of the tree are two invariants: the homology class $[f_I]$ and the energy $E(f_I)$. If $E(f_I) = 0$ then f_I is a map to a point and we say f_I is a *ghost map* and its vertex is a *ghost vertex*; the above construction of the tree can include such vertices.

At each bubble point x we obtain one image point $f_0(x)$ by removing the singularity in the base map f_0, and another image point $\tilde{f}(p)$ by removing the singularity in the bubble map $\tilde{f} = \tilde{f}_x$ at the point p at infinity on the bubble domain. We next investigate the structure of the limiting map between these bubble points. When n is large there is an annulus around the point p^+ on S^2_x whose image under $\tilde{f}_n$ is a tube joining circles near $f_0(x)$ and $\tilde{f}(p^+)$. We call such tubes "connecting tubes." These connecting tubes may contain some energy which does not disappear as $n \to \infty$. If this occurs the renormalization scheme does not preserve energy—there is "energy loss." For sequences of J-holomorphic curves we will rule this out: the isoperimetric inequality and monotonicity imply that these tubes are impossible, and hence there is no energy loss and the points $f_0(x)$ and $\tilde{f}(p^+)$ coincide. We call this latter phenomenon "zero distance bubbling" (Lemma 5.3 below).

We now further restrict the size of C_0. For this we use the observation of Siu and Yau [SY] that there is a constant K_0 depending only on the geometry of N such that any C^1 map $f : S^2 \to N$ that is nontrivial in homotopy has

$$E(f) \geq K_0. \tag{5.1}$$

We henceforth take $C_0 = \frac{1}{2}\min(K_0, B_0)$.

Lemma 5.1. (Connecting tubes) *Suppose that $\{f_n\}$ is a sequence of harmonic or J-holomorphic maps that converge to $f : \Sigma \to N$ with a bubble point $x \in \Sigma$ and bubble map $\tilde{f} : S^2_x \to N$. Let p^+ be the point at infinity on the bubble domain. Then (after replacing $\{f_n\}$ by a subsequence) there are sequences $\{\delta_1(n)\}$ and $\{\delta_2(n)\}$ with $\delta_1 \leq \delta_2$ such that the annuli $A_n = D(p^+, \delta_2) - D(p^+, \delta_1)$ have the following properties: Given $\epsilon > 0$ there is an N such that when $n \geq N$,*

(a) *$\tilde{f}_n(\partial D(p^+, \delta_1))$ has length less than 2ϵ and lies in an ϵ-ball around $f_0(x)$,*

(b) *$\tilde{f}_n(\partial D(p^+, \delta_2))$ has length less than 2ϵ and lies in an ϵ-ball around $\tilde{f}(p^+)$,*

(c) *the homotopy class constructed from $\tilde{f}_n(A_n)$ by collapsing the boundary curves to $f_0(x)$ and $\tilde{f}(p^+)$ is zero.*

Proof. For each $k > 0$ set

$$\delta_1'(k) = \sup\left\{ \delta \;\middle|\; f_0(D(x, \delta)) \subset B\left(f_0(x), \frac{1}{k}\right)\right\} \tag{5.2}$$

and

$$\delta_2(k) = \sup\left\{ \delta \;\middle|\; \tilde{f}(D(p^+, \delta)) \subset B\left(\tilde{f}(p^+), \frac{1}{k}\right)\right\}. \tag{5.3}$$

Using the notation of (4.6) and (4.7) we see that $S^2_x - R_n^{-1}(D(x, \delta_1'(k)))$ is a disk $D(p^+, \delta_1)$ of some radius δ_1. Since $t_n \to \infty$ as $n \to \infty$ we have $\delta_1 \geq \delta_2$ for all large n. Then the annulus

$$A_{nk} = D(p^+, \delta_2(k)) \cap R_n^{-1} D(x, \delta_1'(k))$$

in S_x^2 corresponds to the annulus

$$R_n A_{nk} = D(x, \delta_1'(k)) \cap R_n D(p^+, \delta_2(k))$$

in Σ. Note that $\tilde{f}_n \longrightarrow \tilde{f}$ uniformly in C^1 outside disks around p^+, and similarly $f_n \longrightarrow f_0$ outside disks around x. Hence we can choose n_k such that on the outer boundary of A_{nk}

$$\tilde{f}_n(\partial D(p^+, \delta_2)) \subset B\left(\tilde{f}(p^+), \frac{2}{k}\right) \qquad \forall n \geq n_k \tag{5.4a}$$

and

$$\Big|\text{length}[\tilde{f}_n(\partial D(p^+, \delta_2))] - \text{length}[\tilde{f}_0(\partial D(p^+, \delta_2))]\Big| < 4/k, \tag{5.4b}$$

and on its inner boundary

$$\tilde{f}_n(\partial D(p^+, \delta_1(k))) = \tilde{f}_n(R_n^{-1}\partial D(x, \delta_1'(k))) = f_n(\partial D(x, \delta_1')) \subset B\left(f_0(x), \frac{2}{k}\right) \tag{5.5a}$$

$\forall n \geq n_k$ and

$$\Big|\text{length}[\tilde{f}_n(\partial D(p^+, \delta_1))] - \text{length}[f_0(\partial D(x, \delta_1'))]\Big| < 4/k. \tag{5.5b}$$

Then $\{n_k\}$ determines a subsequence of the $\{f_n\}$ and $\{\tilde{f}_n\}$, which has properties (a) and (b). It is clear from the construction of the A_n that

$$E(\tilde{f}_n(A_n)) \leq C_0 \leq K_0/2 \tag{5.6}$$

for all large n. Property (c) then follows by collapsing the ends of $\tilde{f}_n(A_n)$ to obtain a C^1 map $S^2 \longrightarrow N$ with energy less that K_0 and applying (5.1). $\square$

The bubbling process can be viewed as a surgery. For large n f_n is C^0 close to f_0 on $\Sigma - \cup D_i$ where the $D_i = D(x_i, \delta_1')$ are small disks around the bubble points and, by Lemma 5.1, $f_n(\partial D_i)$ lies in a geodesic ball around $f_0(x_i)$. Hence the homology class $[f_n]$ obtained by identifying the image of each ∂D_i to a point is $[f_n] = [f_0]$ for large n. Similarly, on each bubble domain $\tilde{f}_n(\partial D(p^+, \delta_2))$ lies in a geodesic ball around $\tilde{f}_0(p^+)$ for large n, so after identifying the image of $\partial D(p^+, \delta_2))$ to a point we have a homology class $[\tilde{f}_n]$. Since the tube of Lemma 5.1 does not carry any homology, we conclude that

Lemma 5.2. *If $\{f_n\}$ is a sequence of harmonic or J-holomorphic maps representing a homology class α, then for n sufficiently large,*

$$\alpha = [f_n] = [f_0] + \sum_i [\tilde{f}_{n,x_i}] \qquad \text{in } H_2(N, \mathbb{Z}).$$

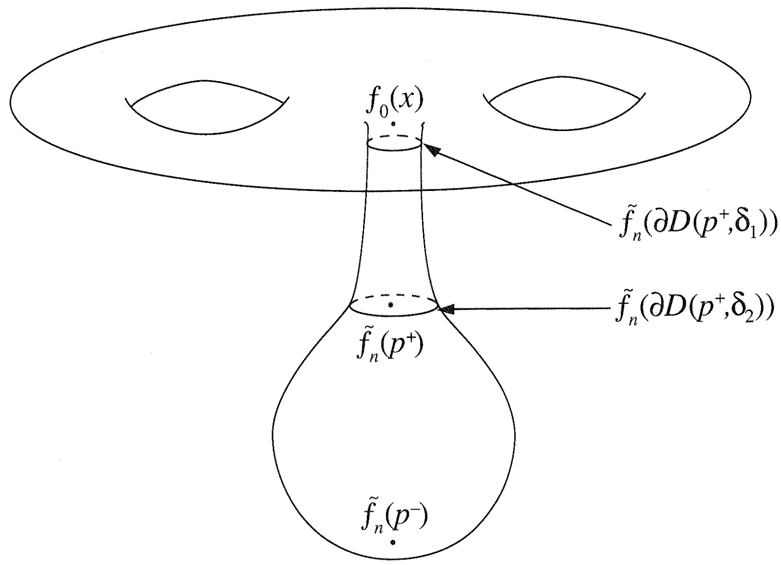

Figure 3. Bubbling with energy loss.

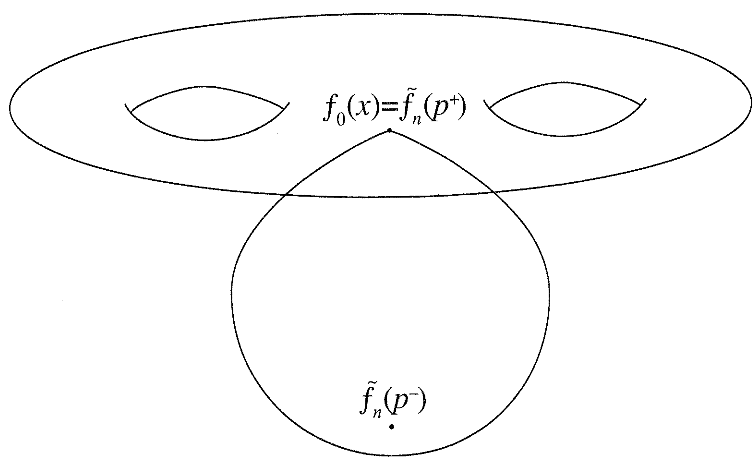

Figure 4. Bubbling without energy loss.

When $\{f_n\}$ is a sequence of tamed J-holomorphic curves, some special properties follow from the strong isoperimetric property and monotonicity.

Lemma 5.3. *Suppose $\{f_n\}$ is a sequence of tamed J-holomorphic maps that converge to $f_0 : \Sigma \to N$ with a bubble point x and bubble map $\tilde{f}_x : S^2_x \to N$ with point at infinity $p^+ \in S^2_x$. Then*

(i) $\tau_x = 0$, *i.e., there is no energy loss, and*

(ii) $\tilde{f}_x(p^+) = f_0(x)$ *(zero distance bubbling).*

Proof. Lemma 5.1 implies that the image of f_n contains a tube whose boundary curves have length going to zero. Since the tube carries no homology, the strong isoperimetric inequality implies that the area of the tube vanishes in the limit. Thus $\tau_x = 0$. If $\tilde{f}_x(p^+) \neq f_0(x)$, then since the area of the tube shrinks to zero, Proposition 3.2 (monotonicity) is eventually contradicted. $\square$

Lemma 5.3 implies that a configuration of two-spheres is the limit of a sequence of J-holomorphic two-spheres only if the spheres of the configuration intersect. This imposes strong restrictions on the structure of such limits, especially for J-holomorphic maps in dim $N = 4$. Such restrictions do not occur in the case of sequences on harmonic or α-harmonic maps.

There are many beautiful properties of the bubble trees determined by the renormalization process and Lemmas 4.2, 5.2, and 5.3. We list some of the properties below. These give constraints on the possible bubble trees and show, for example, that not every tree with vertices J-holomorphic two-spheres and edges labeled by points arises from a bubbling sequence of J-holomorphic maps.

Properties of the bubble tree

The following properties hold for the bubble trees associated with sequences $\{f_n\}$ of J-holomorphic, harmonic, or α-harmonic maps. We assume that these represent a fixed homology class $[f_n] = \alpha$ and that $E(f_n) < A$ for all n.

(1) Lemma 5.2 implies that

$$\alpha = [f_n] = \sum_{\text{vertices}} [f_{i_1 \dots i_k}]. \tag{5.8}$$

(2) While in general there may be energy loss, we do have

$$\overline{\lim}\, E(f_n) \geq \sum_{\text{vertices}} E(f_{i_1 \dots i_k}). \tag{5.9}$$

(3) By Lemma 4.2 no branch of the bubble tree can terminate in a ghost vertex.

When the maps are J-holomorphic, the associated the bubble trees have the following additional properties.

(4) Energy is preserved, so (5.9) is an equality. Equivalently,

$$m_{i_1 \dots i_k} = \text{Area}(f_{i_1 \dots i_k}) + \sum_j m_{i_1 \dots i_k j}. \tag{5.10}$$

(5) Since there is no energy loss, Lemma 4.2 implies that at least two edges must emanate from a ghost vertex (except when the ghost vertex is the base vertex, in which case only one edge may emanate).

(6) If two vertices are joined by an edge then the images of the corresponding bubble maps intersect.

(7) The Chern numbers $c(f) = c_1(N)[f(\Sigma)]$ are defined and by (5.8) satisfy

$$c(\alpha) = c(f_n) = \sum_{\text{vertices}} c(f_{i_1 \dots i_k}). \tag{5.11}$$

(8) From (1.7) we must have $\langle \omega, [f_{i_1 \dots i_k}] \rangle \geq 0$ with equality if and only if $f_{i_1 \dots i_k}$ is a ghost.

(9) Each vertex f_I is in a nonempty moduli space, so dim $\mathcal{M}_{f_I, J_\Sigma, J} \geq 0$ for generic J. From the index theorem (see, for instance, [G]) this means that the base vertex f satifies, for generic J,

$$c(f) \geq n(g-1) + \sigma_g, \qquad \text{where} \qquad \sigma_g = \begin{cases} 3 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 0 & \text{if } g > 1 \end{cases} \tag{5.12}$$

(where $2n = \dim N$ and $g = \text{genus}(\Sigma)$) and that all other vertices are either ghost vertices or satisfy

$$c(f_i) \geq 3 - n. \tag{5.13}$$

(10) When $\dim N = 4$ it follows from (6) above and the intersection properties of holomorphic curves that the homology classes of adjacent vertices must have positive intersection number.

6. The convergence theorem

Let $\{f_n\}$ be a sequence of jJ-holomorphic maps $\Sigma \to N$. The renormalization procedure of Section 4 associates to a subsequence of $\{f_n\}$ (still denoted $\{f_n\}$) a bubble domain tower T and a J-holomorphic map $f_\infty : T \to N$. In this section we formulate and prove a precise statement of the $C^0 \cap L^{1,2}$ convergence of the sequence $\{f_n\}$ to f_∞. Note that it is tricky to even state such a convergence result because the domains change: the maps f_n have domain

Σ whereas the map f_∞ has domain T. However, by modifying the "capping off" procedure of Lemma 5.2 we can construct a "prolongation" $\mathcal{P}_\epsilon(f_n)$ of each f_n to a map $T \to N$. The main point is to make this construction canonical in order to ensure that the prolongations of the f_n converge to f_∞ in suitable norms.

The construction requires a way of extending maps similar to the capping off procedure of Lemma 5.2, but which comes with an estimate on the energy of the extension. For this we use minimal surfaces. The basic idea of the following lemma goes back to Lebesgue and Courant; we follow Meeks and Yau [MY2].

Extension Lemma 6.1. *For each A there is an $\epsilon_1 > 0$ such that every continuous $L^{1,2}$ map defined off a closed disk of radius $\epsilon \le \epsilon_1$*

$$f : \Sigma - D(x, \epsilon) \to N$$

with energy $E(f) \le A$ extends to a continuous $L^{1,2}$ map $\overline{f} : \Sigma \to N$. There is a disk $D_f = D(x, r_f)$ with radius $\epsilon \le r_f \le \sqrt{\epsilon}$ such that this extension satisfies

$$\int_{D_f} |d\bar{f}|^2 \le C|\log \epsilon|^{-1} \tag{6.1}$$

and

$$\text{dist}(\overline{f}(y), f(z)) \le C|\log \epsilon|^{-1/2} \qquad \forall y \in D_f, \;\; z \in \partial D_f. \tag{6.2}$$

Proof. Cover N with convex geodesic balls and let $\lambda > 0$ be the Lebesgue number of the cover. Fix $\epsilon_1 \le \exp(-8\pi A/\lambda^2)$ small enough that the metric on each disk $D(x, \sqrt{\epsilon_1})$ in Σ is uniformly close to the euclidean metric. Then for $\epsilon \le \epsilon_1$ there is an r, $\epsilon \le r \le \sqrt{\epsilon}$, such that

$$\int_0^{2\pi} \left| \frac{\partial f}{\partial \theta} \right|^2 (r, \theta) \;, d\theta \le 4A|\log \epsilon|^{-1}. \tag{6.3}$$

Indeed, otherwise we would have

$$\int_\epsilon^{\sqrt{\epsilon}} \int_0^{2\pi} \frac{1}{\rho^2} \left| \frac{\partial f}{\partial \theta} \right|^2 (r, \theta) \rho \, d\theta \, d\rho > 4A|\log \epsilon|^{-1} \cdot \log \rho|_\epsilon^{\sqrt{\epsilon}} \ge 2A, \tag{6.4}$$

while

$$A \ge E(f) \ge \frac{3}{4} \int_\epsilon^{\sqrt{\epsilon}} \int_0^{2\pi} \left[\left| \frac{\partial f}{\partial \rho} \right|^2 + \frac{1}{\rho^2} \left| \frac{\partial f}{\partial \theta} \right|^2 \right] \rho \, d\theta \, d\rho. \tag{6.5}$$

Let r_f be the sup of the $r \in [\epsilon, \sqrt{\epsilon}]$ such that (6.3) holds, and set $D_f = D(x, r_f)$. For this choice of ϵ_1 and r_f the curve $\gamma = f(\partial D_f)$ has length

$$\operatorname{length}(f[\partial D_f]) \leq \sqrt{2\pi}\left(\int_0^{2\pi} |df|^2\right)^{\frac{1}{2}} \leq \sqrt{8\pi A}|\log \epsilon|^{-\frac{1}{2}} \leq \lambda \tag{6.6}$$

and hence lies in a convex geodesic ball B in N of radius $\sqrt{8\pi A}|\log \epsilon|^{-\frac{1}{2}}$. Assuming that γ is a Jordan curve, we can then solve the Plateau problem in N with this boundary curve (see [MY1] and [MY2]), obtaining a conformal harmonic map $\hat{f} : D_f \longrightarrow N$ which extends f and minimizes area amongst all such extensions. Define $\overline{f} = \hat{f}$ on D_f and $\overline{f} = f$ on its complement. Since the image of $\hat{f}$ lies in B (by the maximum principle) we obtain (6.2).

We can also solve the Plateau problem with respect to the euclidean metrics on $D(x, r_f)$ and B, obtaining a map $g : D_f \longrightarrow N$. Since the metrics on D_f and B are uniformly close to the euclidean metrics, minimality and the isoperimetric inequality for euclidean space give

$$\operatorname{Area}(\hat{f}) \leq \operatorname{Area}(g) \leq C\operatorname{Area}_{\text{eucl}}(g) \leq \frac{C}{4\pi}\operatorname{length}^2_{\text{eucl}}(\gamma) \leq C' \operatorname{length}^2(\gamma). \tag{6.7}$$

Since $\overline{f}$ is conformal its area and energy are equal, so (6.6) and (6.7) yield

$$\int_{D(x,\epsilon)} |d\overline{f}|^2 \leq \int_{D_f} |d\hat{f}|^2 \leq 8\pi AC'|\log \epsilon|^{-1}. \tag{6.8}$$

Finally, if γ is not a Jordan curve we can perturb it slightly to make it Jordan, maintaining the above estimate within a factor of two. $\square$

For $\epsilon \leq \epsilon_1$ (the constant of Lemma 6.1), the prolongation $\mathcal{P}_\epsilon(f_n)$ of f_n is defined as follows. First, on Σ choose disks $D_i = D(x_i, \epsilon)$ around the bubble points. Then the restriction of f_n to $\Sigma - \cup D_i$ extends (by the extension lemma) to a continuous map $\Sigma \longrightarrow N$. This defines $\mathcal{P}_\epsilon(f_n)$ on Σ.

The prolongation $\mathcal{P}_\epsilon(f_n)$ on each bubble is defined inductively as follows. Fix a bubble point x_i and consider the renormalized maps $\tilde{f}_n = \tilde{f}_{n,x_i}$ defined by (4.7). These are not defined on a neighborhood of the point at infinity $p^+ \in S^2_{x_i}$ but for sufficiently large n they are defined outside $D_{i0} = D(p^+, \epsilon)$. Of course, the sequence $\tilde{f}_n$ has its own bubble points $x_{ij} \in S^2_x$ which lie in disks $D_{ij} = D(x_{ij}, \epsilon)$. Define the prolongation $\mathcal{P}_\epsilon(f_n)$ on the bubble by restricting $\tilde{f}_n$ to $S^2_x - \cup_{j=0} D_{ij}$ and extending by Lemma 6.1.

Theorem 6.2. *Let $\{f_n\}$ be a sequence of jJ-holomorphic maps $\Sigma \longrightarrow N$. Then there is a bubble tower T and a sequence $\epsilon_n \searrow 0$ such that a subsequence of*

$$\mathcal{P}_{\epsilon_n}(f_n) : T \longrightarrow N$$

converges in $C^0 \cap L^{1,2}$ to a smooth jJ-holomorphic map $T \longrightarrow N$. The convergence is in $C^r(K)$

for each compact set $K \subset\subset T - \cup\{x_l\}$ where $\{x_l \mid 1 \leq l \leq L\}$ is the complete set of bubble points and points at infinity of T.

Proof. Convergence in C^r on compact sets $K \subset\subset T - \cup\{x_l\}$ follows immediately from Theorem 4.1 (applied on each bubble). Set $K_n = T - \cup D(x_l, \epsilon_n)$. For each n there is an $N(n)$ such that

$$\|\mathcal{P}_{\epsilon_n}(f_k) - f_\infty\|_{1,2,K_n} < 1/n \qquad \forall k \geq N(n). \tag{6.9}$$

Since $f_\infty \in C^1$,

$$\int_{D(x_l, \sqrt{\epsilon_n})} |df_\infty|^2 \leq \sup |df_\infty|^2 \text{ Vol } D(x_l, \sqrt{\epsilon_n}) \leq C'\epsilon_n. \tag{6.10}$$

Then since $\epsilon_n \leq r_{f_n}$ for each n, (6.9), (6.10), and (6.1) imply that

$$\|\mathcal{P}_{\epsilon_n}(f_k) - f_\infty\|_{1,2,T} \;<\; 1/n + LC|\log \epsilon_n|^{-1} + LC'\epsilon_n \qquad \forall k \geq N(n).$$

Hence the diagonal sequence $\{\overline{f}^{\epsilon_n}_{N(n)}\}$ converges in $L^{1,2}(T)$ to f_∞. The statement about continuity follows similarly using (6.2). □

We conclude with two corollaries, both valid for sequences of J-holomorphic maps. The first shows that energy is conserved. Thus the renormalization process of Section 4—unlike Sacks–Uhlenbeck renormalization—keeps track of all bubbles.

Corollary 6.3. *$E(f_n) \to E(f_\infty)$ as $n \to \infty$.*

Proof. Theorem 6.2 shows that $E(\mathcal{P}_{\epsilon_n}(f_n)) \to E(f_\infty)$. But from the definition of the prolongation the difference between $E(\mathcal{P}_{\epsilon_n}(f_n))$ and $E(f_n)$ is bounded by the sum of the energy in the tubes of Lemma 5.1 and the energy of the extensions. This vanishes in the limit by Lemma 5.3 and equation (6.1). □

Corollary 6.4. *If $\{f_n\}$ is a sequence of jJ-holomorphic maps $\Sigma \to N$ then there is a subsequence whose image in N converge pointwise to the image of $f_\infty : T \to N$.*

Proof. From Theorem 6.2 the image of the $\mathcal{P}_{\epsilon_n}(f_n)$ converge pointwise to the image of f_∞. But jJ-holomorphic maps have the "zero distance bubbling" property of Lemma 5.3, and so by construction the images of $\mathcal{P}_{\epsilon_n}(f_n)$ and $f_n : \Sigma \to N$ agree off the ϵ_n-balls around the image under f_∞ of each bubble point of T. □

This last result—the pointwise convergence of the image—is a crucial fact in applications. This is the convergence that is used, for example, at a key point in McDuff's construction of distinct symplectic manifolds with the same periods [McD1, Proposition 7.2].

Appendix

In this appendix we do a computation using the method of moving frames. The summation convention is used throughout.

Let (Σ, j) be a Riemann surface with metric μ compatible with j and let (N, J) be an almost complex manifold with hermitian metric h. Choose a hermitian connection ∇ on N. Let $\{\omega_\alpha\}$ be a unitary coframe on N and let $\{\omega_{\alpha\overline{\beta}}\}$ be the connection 1-forms with respect to this coframe. Then the $\{\omega_{\alpha\overline{\beta}}\}$ satisfy

$$\overline{\omega}_{\alpha\overline{\beta}} = -\,\omega_{\beta\overline{\alpha}} \tag{A.1}$$

and the first structure equation

$$d\omega_\alpha = \omega_{\alpha\overline{\beta}} \wedge \omega_\beta + \tau_\alpha \tag{A.2}$$

where τ_α is the torsion 2-form of the connection. The form τ_α involves the covariant derivative of the almost complex structure J (in fact, the integrability of J is implied by the vanishing of τ_α). Differentiating again yields the second structure equation

$$d\omega_{\alpha\overline{\beta}} - \omega_{\alpha\overline{\gamma}} \wedge \omega_{\gamma\overline{\beta}} = \Omega_{\alpha\overline{\beta}} \tag{A.3}$$

where $\Omega_{\alpha\overline{\beta}}$ is the curvature 2-form of the connection ∇. Similarly, let $\{\theta_1, \theta_2\}$ be an orthonormal coframe on Σ. Then $\phi = \theta_1 + i\theta_2$ is a $(1,0)$-form for j. The first structure equation is

$$d\phi = -i\rho \wedge \phi \tag{A.4}$$

where ρ is the connection 1-form of the Levi–Civita connection for the coframe $\{\theta_1, \theta_2\}$. The Gauss curvature K of μ is defined by the second structure equation

$$d\rho = -\frac{i}{2} K \phi \wedge \overline{\phi}. \tag{A.5}$$

Let $f : \Sigma \to N$ be a jJ-holomorphic map. Since df preserves type (cf. 2.3) we have

$$f^*\omega_\alpha = a_\alpha \phi \tag{A.6}$$

where the a_α are complex-valued functions. Taking the exterior derivative of (A.6) and using (A.2) and (A.4) gives (the pullback f^* will be suppressed in the following equations)

$$(da_\alpha - i\rho a_\alpha - \omega_{\alpha\overline{\beta}} a_\beta) \wedge \phi = \tau_\alpha. \tag{A.7}$$

The forms

$$Da_\alpha = da_\alpha - i\rho a_\alpha - \omega_{\alpha\overline{\beta}} a_\beta \tag{A.8}$$

are the components of the covariant derivative of $\partial f \in \Omega^{1,0}(f^*TN)$. Decomposing these into forms $D'a_\alpha$ and $D''a_\alpha$ of type (1,0) and (0,1) respectively, we can write (A.7) as

$$D''a_\alpha \wedge \phi = \tau_\alpha. \tag{A.9}$$

Differentiating (A.9) as in [W, Section 3] we obtain the explicit (local) formula

$$\Delta a_\alpha = \frac{1}{2}\partial(*\tau_\alpha) + \frac{K}{2} a_\alpha - a_\gamma \overline{a}_\beta a_\delta R_{\alpha\overline{\gamma}\beta\overline{\delta}} \tag{A.10}$$

where $\partial(*\tau_\alpha)$ is the function satisfying

$$\partial(*\tau_\alpha)\phi = i(D(*\tau_\alpha))^{1,0} = i\Big(d(*\tau_\alpha) - (*\tau_\beta)\omega_{\alpha\overline{\beta}}\Big)^{1,0}. \tag{A.11}$$

Equations (A.10) and (2.17) are equivalent.

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Received May 1, 1992

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Communicated by Dusa McDuff