EIGENVALUE GAP THEOREMS FOR A CLASS OF NONSYMMETRIC ELLIPTIC OPERATORS ON CONVEX DOMAINS

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0. Introduction

In the remarkable paper [AC] Andrews and Clutterbuck solve the "gap conjecture", that is, they show that the difference between the first and second eigenvalues of the laplacian with convex potential on a convex domain in euclidean space is at least $\frac{3\pi^2}{D^2}$. Here D is the diameter of the domain. Somewhat later, Lei Ni [N1] reformulated and expanded some of the techniques introduced in [AC]. Taken together, these papers suggest a general approach to estimating the eigenvalue gap of a large class of linear second-order elliptic operators on convex domains. In this paper we illustrate how this approach may work by estimating the eigenvalue gap of a class of nonsymmetric linear elliptic operators.

Let Ω be a strictly convex open domain in euclidean space with smooth boundary. The operators L we consider have the form:

$$Lu = \Delta u - B \cdot \nabla u - cu \tag{0.1}$$

where u is a C^2 function on $\bar{\Omega}$, Δ is the euclidean laplacian, B is a C^3 vector valued function on $\bar{\Omega}$ and c is a C^2 scalar function on $\bar{\Omega}$. Let β denote the one-form dual to B. We will require that, for all $x \in \Omega$, there is a constant K such that:

$$|d\beta(x)| \leq K \operatorname{dist}(x, \partial\Omega). \tag{0.2}$$

Note that this condition is satisfied if, for example, B has compact support in Ω or if $B = \nabla\phi$ for a C^4 function ϕ on Ω. We consider the eigenvalue problem:

$$Lu = -\lambda u,$$

where u satisfies Dirichlet boundary conditions. Since L is not symmetric the eigenvalues and eigenfunctions need not be real. However the principal eigenvalue λ_0 is real with eigenfunction u_0 that is positive on Ω, vanishes on the boundary and satisfies $|\nabla u_0| \neq 0$ on the boundary. It is also known that for any other eigenvalue λ, $\mathrm{Re}(\lambda) > \lambda_0$ [N2]. We will show that the gap $\mathrm{Re}(\lambda) - \lambda_0$ can be bounded below by a positive constant α that depends on the coefficients B and c and on u_0. The constant α is the eigenvalue gap of an associated regular Sturm-Liouville problem on the interval $[-\frac{D}{2}, \frac{D}{2}]$:

$$w''(s) + \sigma|s|w(s) = -\mu w(s)$$

with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$ where σ is a constant depending on B and c and on u_0. We note that the spectral gap of a regular Sturm-Liouville problem on an interval is relatively easy to estimate.

In [AC] the eigenvalue gap is bounded below by the eigenvalue gap of the regular Sturm-Liouville problem on the interval $[-\frac{D}{2}, \frac{D}{2}]$:

$$w'' + \lambda w = 0.$$

with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$. Much of the work in [AC] is the determination that this is the *associated Sturm-Liouville problem*, where *associated Sturm-Liouville problem* has a precise

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technical meaning described in the next section. In this paper the associated Sturm-Liouville problem is not obviously related to the operator L and is somewhat arbitrary. However the associated Sturm-Liouville problem must satisfy a number of conditions and it is these that are used to determine the particular problem.

The main theorem is:

Theorem 0.1. *Let Ω be a bounded connected strictly convex open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of diameter D. Suppose that the operator (0.1) has coefficients b^i and c that satisfy $c \geq 0$ and (0.2). Consider the eigenvalue problem for L on Ω:*

$$Lu = -\lambda u,$$

where u satisfies Dirichlet boundary conditions. Let λ_0 be the principal eigenvalue with eigenfunction u_0 and λ be any other eigenvalue. Then there is a constant $\sigma > 0$ depending on $||b^j||_{C^2(\bar{\Omega})}$, $||\Delta b^j||_{C^1(\bar{\Omega})}$, $||c||_{C^2(\bar{\Omega})}$, K and on u_0 such that:

$$Re\,(\lambda) - \lambda_0 > \frac{1}{4}(\mu_1 - \mu_0) > 0$$

where $\mu_0 < \mu_1$ are the first two eigenvalues of the Sturm-Liouville problem on $[-\frac{D}{2}, \frac{D}{2}]$:

$$w''(s) + \sigma|s|w(s) = -\mu w(s)$$

with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$.

The precise nature of the dependence of σ on the geometry of u_0 will be described in a later section.

An immediate consequence of our results are the following two applications to the Bakry-Emery Laplacian. Recall that for a C^2 function ϕ the Bakry-Emery Laplacian is given by

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla.$$

Consider the operator L:

$$Lu = \Delta_\phi u - cu.$$

The Bakry-Emery Laplacian is formally symmetric with respect to the weighed volume form $e^{-\phi}dv$. Therefore the eigenvalue problem:

$$Lu = -\lambda u,$$

with Dirichlet boundary conditions is has real eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ and real eigenfunctions [E]. The operator L is of the form (0.1) with $B = \nabla\phi$. Therefore $\beta = d\phi$ and thus (0.2) is trivially satisfied. Theorem 0.1 remains true but the constant σ now depends only on $||\phi||_{C^4(\bar{\Omega})}$ and $||c||_{C^2(\bar{\Omega})}$ and, in particular, is independent of the geometry of u_0.

We say a function c is *ϕ-convex* if the function $c - \frac{1}{2}\Delta\phi + \frac{1}{4}|\nabla\phi|^2 = c - \frac{1}{2}\Delta_{\frac{1}{2}\phi}\phi$ is convex in the usual sense. Note that if c is convex, in the usual sense, and $\Delta_{\frac{1}{2}\phi}\phi$ is concave, in the usual sense, then c is ϕ-convex. We show:

Theorem 0.2. *If ϕ is any C^4 function and c is ϕ-convex then the spectral gap for the operator $\Delta_\phi - c$ on a convex domain Ω satisfies:*

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}.$$

In the case that ϕ is a constant this is the result of [AC].

We are indebted to Lei Ni for introducing us both to his work [N1] and to [AC] and for interesting discussions. In particular, he pointed out that the gap problem for nonsymmetric elliptic operators.

1. The Method of Andrews-Clutterbuck

Suppose that Ω is a strictly convex domain in $\mathbb{R}^n$ and X is a vector field on Ω. A function $\omega(s) : \mathbb{R}_+ \to \mathbb{R}$ is called a *modulus of expansion* for X if for $x, y \in \Omega$, $x \neq y$

$$(X(y) - X(x)) \cdot \frac{y-x}{|y-x|} \geq 2\omega(\frac{|y-x|}{2}).$$

A function $\eta : \mathbb{R}_+ \to \mathbb{R}$ is called a *modulus of continuity* for a complex or real valued function f on Ω if for all $x, y \in \Omega$

$$|f(y) - f(x)| \leq 2\eta(\frac{|y-x|}{2}).$$

An important result of Andrews-Clutterbuck unifyng these two concepts in proved in [AC]:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a strictly convex domain with smooth boundary and with diameter D. Let $Y(x,t)$ be a real valued vector field. Let $z(x,t)$ be a smooth, possibly complex valued, solution of*

$$\frac{\partial}{\partial t} z = \Delta z - 2Y \cdot \nabla z. \tag{1.1}$$

with Neumann boundary condition. Suppose that:

(1) *$Y(\cdot, t)$ has modulus of expansion $\omega(\cdot, t)$ for each $t > 0$, where $\omega(s,t) : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ is smooth.*

(2) *$z(\cdot, 0)$ has modulus of continuity φ_0, where $\varphi_0(s) : [0, \frac{D}{2}] \to \mathbb{R}$ is smooth with $\varphi_0(0) = 0$ and $\varphi_0'(s) > 0$ on $[0, \frac{D}{2}]$.*

(3) *$\varphi(s,t) : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies:*
   (a) *$\varphi(s, 0) = \varphi_0(s)$ on $[0, \frac{D}{2}]$*
   (b) *$\frac{\partial \varphi}{\partial t} \geq \varphi'' - 2\omega\varphi'$ on $[0, \frac{D}{2}] \times \mathbb{R}_+$*
   (c) *$\varphi'(s,t) > 0$ on $[0, \frac{D}{2}] \times \mathbb{R}_+$*
   (d) *$\varphi(0,t) \geq 0$ for each $t \geq 0$.*

Then if $z(x,t)$ is real valued, $\varphi(s,t)$ is a modulus of continuity of $z(x,t)$. If $z(x,t)$ is complex valued, $2\varphi(s,t)$ is a modulus of continuity of $z(x,t)$.

Proof. The real case of this theorem is taken directly from [AC]. The case in which $z(x,t)$ is complex valued is needed in this paper and follows from the real case as follows: Since $Y(x,t)$ is real valued, both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ satisfy (1.1). Applying the real case to both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ implies that each has modulus of continuity $\varphi(s,t)$. The complex case follows. □

We next outline the method of Andrews-Clutterbuck. Let $\Omega \subset \mathbb{R}^n$ be a strictly convex domain with smooth boundary and with diameter D. Let L be the linear, elliptic, symmetric positive operator on scalar functions:

$$L(u) = \Delta u - cu, \tag{1.2}$$

on Ω where $\Delta u = \sum_{i=1}^n u_{x_i x_i}$ and $c \in C^\infty(\Omega)$ is a non-negative function. Let $0 < \lambda_0 < \lambda_1 \leq \ldots$ be the eigenvalues and $u_0, u_1, \ldots$ the corresponding eigenfunctions for Dirichlet boundary conditions. Using that $u_0 > 0$ on Ω it can be shown that there is a vector field Y so that the function

$$z(x,t) = \frac{e^{-\lambda_1 t} u_1}{e^{-\lambda_0 t} u_0}$$

satisfies the heat equation with drift (1.1) with Neumann boundary condition. The vector field Y is the drift velocity. Suppose that Y has modulus of expansion $\omega(\cdot,t)$ for each $t>0$, where $\omega(s,t):[0,\frac{D}{2}]\times\mathbb{R}_+\to\mathbb{R}$ is smooth.

On the interval $[-\frac{D}{2},\frac{D}{2}]$ consider the linear second order differential operator $\tilde{L}(w)=w''-\tilde{c}w$ where $\tilde{c}$ is a smooth function on $[-\frac{D}{2},\frac{D}{2}]$. Consider the Sturm-Liouville problem:

$$\begin{aligned}\tilde{L}(w)&=-\mu w \\ w(-\frac{D}{2})&=w(\frac{D}{2})=0\end{aligned}\tag{1.3}$$

with eigenvalues $0<\mu_0<\mu_1\leq\dots$ and corresponding eigenfunctions $w_0,w_1,\dots$. Set

$$\varphi(s,t)=\frac{e^{-\mu_1 t}w_1}{e^{-\mu_0 t}w_0}$$

for $s\in[0,\frac{D}{2}]$ and $t\geq 0$. Then

$$\frac{\partial\varphi}{\partial t}=\varphi''-2\tilde{\omega}\varphi' \text{ on } [0,\frac{D}{2}]\times\mathbb{R}_+\tag{1.4}$$

Here the function $\tilde{\omega}(s,t)$ is also called the drift velocity. It is not difficult to verify by direct computation that:

$$\tilde{\omega}=-(\log w_0)',\tag{1.5}$$

where w_0 is the first eigenfunction of (1.3). Suppose that the operators L and $\tilde{L}$ satisfy the condition that the potential function $\tilde{c}$ is even and that the potential function c is more convex then $\tilde{c}$ in the sense that for any $x\neq y$ in Ω:

$$(\nabla c(y)-\nabla c(x))\cdot\frac{(y-x)}{|y-x|}\geq 2\tilde{c}'(\frac{|y-x|}{2}).\tag{1.6}$$

Under this assumption [AC] prove that:

(1) $\tilde{\omega}=\omega$

(2) $\varphi'(s,t)>0$ on $[0,\frac{D}{2}]\times\mathbb{R}_+$

(3) $\varphi(0,t)\geq 0$ for each $t\geq 0$.

By Theorem 1.1, $\varphi(s,t)$ is a modulus of continuity of $z(x,t)$. Hence, for each $t\geq 0$

$$\text{osc } z(\cdot,t)\leq 2\sup\{\varphi(s,t):s\in[0,\frac{D}{2}]\}.$$

From this they derive that there is a constant C such that for each $t\geq 0$

$$e^{-(\lambda_1-\lambda_0)t}\text{ osc }\frac{u_1}{u_0}\leq 2Ce^{-(\mu_1-\mu_0)t}.$$

Hence,

$$\lambda_1-\lambda_0\geq\mu_1-\mu_0.\tag{1.7}$$

In this paper we will employ a variation of this argument. As above let Ω be a bounded connected strictly convex open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the uniformly elliptic operator:

$$Lu=\sum_{i=1}^n u_{x_ix_i}-\sum_{i=1}^n b^i u_{x_i}-cu.$$

with

$$|d\beta(x)|\leq K\,\mathrm{dist}(x,\partial\Omega),$$

where β is the one form dual to (b^j). For Dirichlet boundary conditions let λ_0 be the principal eigenvalue with corresponding eigenfunction u_0 and let $\lambda\in\mathbb{C}$ be any other eigenvalue with eigenfunction u. We will show that there is a vector field Y so that the function

$$z(x,t) = \frac{e^{-\lambda t}u}{e^{-\lambda_0 t}u_0}$$

satisfies the heat equation with drift (1.1) with Neumann boundary condition. Moreover Y has modulus of expansion $\omega(\cdot, t)$ for each $t > 0$.

We next consider the Sturm-Liouville problem on $[-\frac{D}{2}, \frac{D}{2}]$:

$$w''(s) + \sigma|s|w(s) = -\mu w(s) \tag{1.8}$$

with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$. Let $\mu_0 < \mu_1$ be the first two eigenvalues with corresponding eigenfunctions w_0 and w_1. We show that for suitable choice of σ and a scaling factor $\eta > \frac{1}{2}$ the functions $e^{-\mu_0\eta^2 t}w_0$ and $e^{-\mu_1\eta^2 t}w_1$ can be used to define a function φ on $[0, \frac{D}{2}] \times \mathbb{R}_+$ such that:

$$\frac{\partial\varphi}{\partial t} \geq \varphi'' - 2\omega\varphi'$$

on $[0, \frac{D}{2}] \times \mathbb{R}_+$, where ω is the modulus of expansion of Y. The other conditions needed for Theorem 1.1 can also be verified for this choice of φ. The argument proceeds as in [AC] to conclude that:

$$\text{Re }(\lambda) - \lambda_0 \geq \eta^2(\mu_1 - \mu_0) > \tfrac{1}{4}(\mu_1 - \mu_0) > 0 \tag{1.9}$$

We will call the Sturm-Liouville problem (1.8) an *associated Sturm-Liouville problem* for L. The difference $\mu_1 - \mu_0$ can be computed (or estimated) by applying ode techniques to the associated Sturm-Liouville problem, thus providing a lower bound on the eigenvalue gap of the operator L. In the case that the one-dimensional limit of the operator L coincides with an associated Sturm-Liouville problem the lower bound (1.9) is sharp.

Using our nomenclature the regular Sturm-Liouville problem on the interval $[-\frac{D}{2}, \frac{D}{2}]$:

$$w'' + \lambda w = 0.$$

with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$ is an associated Sturm-Liouville problem for (1.2) under the condition that c is convex. This is the technical meaning of "associated" referred to in the introduction.

2. The gap theorem for nonsymmetric elliptic operators

Let Ω be a bounded connected strictly convex open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume that $b^i \in C^3(\bar{\Omega})$ and that $c \in C^2(\bar{\Omega})$ and consider the uniformly elliptic operator:

$$Lu = \sum_{i=1}^{n} u_{x_i x_i} - \sum_{i=1}^{n} b^i u_{x_i} - cu. \tag{2.1}$$

Consider the one-form $\beta = \sum_{i=1}^n b^i dx_i$ dual to the vector field $\sum_{i=1}^n b^i \frac{\partial}{\partial x_i}$ and suppose that there is a constant $K > 0$ such that:

$$|d\beta(x)| \leq K \operatorname{dist}(x, \partial\Omega), \tag{2.2}$$

for all $x \in \Omega$. We will, throughout this paper, assume that the coefficients of L satisfy (2.2). Note that we can add a constant to the operator L without changing the spectral gap of L. Therefore we can, without loss of generality, assume that $c > 0$ on Ω. The operator L is not symmetric and the eigenvalues for Dirichlet boundary conditions may not all be real. However the following theorem quoted from [E] shows that, in part, the situation resembles the symmetric case.

Theorem 2.1. (1) *The principle eigenvalue λ_0 of L on Ω with zero boundary conditions is real and simple.*

(2) *If* $\lambda \in \mathbb{C}$ *is any other eigenvalue of* L *then*

$$Re\,\lambda \geq \lambda_0$$

(3) *The eigenfunction* u_0 *corresponding to* λ_0 *is positive in* Ω.

Recently Lei Ni [N2] has improved this result showing that if $\lambda \in \mathbb{C}$ is any other eigenvalue of L then the strict inequality $\mathrm{Re}\,\lambda > \lambda_0$ holds.

We will prove the gap theorem Theorem 0.1 for the eigenvalues of the operator (2.1). We begin with the following proposition.

Proposition 2.2. *Let* Ω *be a bounded connected strictly convex open domain in* $\mathbb{R}^n$ *with smooth boundary* $\partial\Omega$. *Suppose that* λ_0 *is the principal eigenvalue of the operator (2.1) with eigenfunction* u_0. *Then, there is a constant* κ *such that:*

$$\sup_{x\in\Omega} \frac{|\nabla u_0(x)|}{u_0(x)} |d\beta(x)| \leq \kappa. \tag{2.3}$$

Proof. The principal eigenfunction u_0 satisfies $u_0 > 0$ on Ω, $u_{0|\partial\Omega} = 0$, and $\frac{\partial u_0}{\partial \nu}_{|\partial\Omega} < 0$. There is a constant $A > 0$ such that:

$$|u_0|_{C^1(\bar{\Omega})} \leq A \tag{2.4}$$

Set $\Omega_\delta = \{x \in \Omega : u_0(x) \geq \delta\}$. Since $|\nabla u_0| > 0$ on $\partial\Omega$, there exist constants $\theta_0 > 0$ and $\delta_0 > 0$ such that on $\Omega \setminus \Omega_{\delta_0}$:

$$|\nabla u_0| \geq \theta_0. \tag{2.5}$$

It follows that all the critical points of u_0 occur in Ω_{δ_0}. Set $a = \min_{\Omega_{\delta_0}} u_0$. Then for $x \in \Omega_{\delta_0}$ we have:

$$\frac{|\nabla u_0(x)|}{u_0(x)} |d\beta(x)| \leq \frac{A}{a} KD,$$

where $D = \mathrm{diam}(\Omega)$. Choosing δ_0 smaller, if necessary, we can suppose that the line γ joining $x \in \Omega\backslash\Omega_{\delta_0}$ to its nearest boundary point y lies entirely in $\Omega\backslash\Omega_{\delta_0}$. Parameterizing γ by its arc length with $\gamma(0) = y$ and $\gamma(\ell) = x$ we can also suppose that $-\nabla u_0(\gamma(s))\cdot\gamma'(s) \geq \frac{\theta_0}{2}$ along γ. Thus,

$$u_0(x) = u_0(x) - u_0(y) = \int_0^\ell -\nabla u_0(\gamma(s)) \cdot \gamma'(s) ds \geq \frac{\theta_0 \ell}{2},$$

where ℓ is the length of γ. Hence for $x \in \Omega \setminus \Omega_{\delta_0}$ we have:

$$\frac{|\nabla u_0(x)|}{u_0(x)} |d\beta(x)| \leq \frac{2AK}{\theta_0 \ell} \mathrm{dist}(x, \partial\Omega) \leq \frac{2AK}{\theta_0}.$$

The result follows. $\square$

The dependence of the constant κ on the geometry of u_0 can be described as follows: Denote the set of critical points of u_0 in the interior of Ω by S. Set $\delta_0 = \frac{1}{2}\inf_{x\in S} u_0(x)$. Using the notation of the proof, we have that $\inf_{x\in\Omega\backslash\Omega_{\delta_0}} |\nabla u_0(x)| = \theta_0 > 0$. Then κ depends on $\delta_0, \theta_0, |\nabla u_0|_{C^1(\bar{\Omega})}$ and K.

The drift velocity of L. Let u_0 be the principal eigenfunction of L with eigenvalue λ_0 and set $u_0(x,t) = e^{-\lambda_0 t}u_0(x)$. Let u be any other eigenfunction with eigenvalue λ and set $u_1(x,t) = e^{-\lambda t}u(x)$. The following proposition is adapted from [AC].

Proposition 2.3. *Let Ω be a bounded strictly convex domain with smooth boundary in $\mathbb{R}^n$. Let u_0 and u_1 be two smooth solutions of the parabolic equation:*

$$\begin{aligned} \frac{\partial u}{\partial t} &= L(u) \quad on \quad \Omega \times \mathbb{R}_+ \\ u &= 0 \quad on \quad \partial\Omega \times \mathbb{R}_+ \end{aligned}$$

with u_0 is positive on the interior of Ω. Let $z(x,t) = \frac{u_1(x,t)}{u_0(x,t)}$ and

$$Y^j(x) = -\nabla_{x_j}(\log u_0)(x) + \frac{b^j}{2}(x).$$

Then z is smooth on $\Omega \times \mathbb{R}_+$ and satisfies the Neumann heat equation with drift:

$$\begin{aligned} \frac{\partial z}{\partial t} &= \Delta z - 2Y \cdot \nabla z \quad on \quad \Omega \times \mathbb{R}_+ &(2.6)\\ \nabla_\nu z &= 0 \quad on \quad \partial\Omega \times \mathbb{R}_+ &(2.7)\end{aligned}$$

Proof. The proof is essentially given in [AC] Proposition 3.1 (or [Y] Lemma 1.1, [SWYY] Appendix A). Both u_0 and u_1 are smooth on $\bar{\Omega} \times [0,\infty)$ and u_0 has negative derivative in the direction of the inward pointing unit normal. By the argument of [SWYY] z extends to $\bar{\Omega}$ as a smooth function and therefore $\frac{\partial z}{\partial t}, \Delta z$ and ∇z are smooth and bounded on $\bar{\Omega}$. By direct computation:

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{u_1}{u_0}\right) \\ &= \Delta z + (2\nabla \log u_0 - B) \cdot \nabla z &(2.8)\end{aligned}$$

On $\partial\Omega$, $\nabla u_0 = -k\nu$ with $k > 0$ and $u_0 = 0$. Therefore by (2.8), $\nabla_\nu z = 0$. □

Y is called the *drift velocity* of z.

Lemma 2.4. *Set $B = (b^j)$, $U^{ij} = \nabla_{x_j} b^i - \nabla_{x_i} b^j$ and*

$$V^j = V^j(c, B) = \left(\nabla_{x_j} c + \tfrac{1}{4}\nabla_{x_j}(|B|^2) - \tfrac{1}{2}\Delta b^j\right).$$

Then

$$\Delta Y = 2\nabla_Y Y - Y \cdot U - V \tag{2.9}$$

Proof. To begin we compute $\nabla_Y Y$.

$$\begin{aligned} \nabla_Y Y &= \sum_i (-\nabla_{x_i} \log u_0 + \frac{b^i}{2})\nabla_{x_i} Y^j \\ &= \sum_i (-\nabla_{x_i} \log u_0 + \frac{b^i}{2})(-\nabla_{x_j}\nabla_{x_i} \log u_0 + \frac{1}{2}\nabla_{x_i} b^j) \\ &= \sum_i \Big((\nabla_{x_i} \log u_0)(\nabla_{x_j}\nabla_{x_i} \log u_0) - \frac{b^i}{2}(\nabla_{x_j}\nabla_{x_i} \log u_0) \\ &- \frac{1}{2}(\nabla_{x_i} \log u_0)\nabla_{x_i} b^j + \frac{1}{4} b^i \nabla_{x_i} b^j\Big) &(2.10)\end{aligned}$$

Also we will need:

$$\begin{aligned} \sum_i \nabla_{x_i}\nabla_{x_i} \log u_0 &= \sum_i \Big(\frac{1}{u_0}(\nabla_{x_i}\nabla_{x_i} u_0) - (\nabla_{x_i} \log u_0)(\nabla_{x_i} \log u_0)\Big) \\ &= \sum_i \Big(b^i \nabla_{x_i} \log u_0 + c - \lambda_0 - (\nabla_{x_i} \log u_0)(\nabla_{x_i} \log u_0)\Big) \end{aligned}$$

Computing $\Delta(-\nabla_{x_j}\log u_0) = -\nabla_{x_j}(\Delta \log u_0)$ we have,

$$\begin{aligned}\sum_i -\nabla_{x_j}(\nabla_{x_i}\nabla_{x_i}\log u_0) &= \sum_i \big(-\nabla_{x_j} b^i(\nabla_{x_i}\log u_0) - b^i(\nabla_{x_j}\nabla_{x_i}\log u_0) \\ &- \nabla_{x_j} c + 2(\nabla_{x_j}\nabla_{x_i}\log u_0)(\nabla_{x_i}\log u_0)\big) \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) we have:

$$\begin{aligned}\sum_i -\nabla_{x_j}(\nabla_{x_i}\nabla_{x_i}\log u_0) &= 2\nabla_Y Y + \sum_i (\nabla_{x_i} b^j - \nabla_{x_j} b^i)(\nabla_{x_i}\log u_0) \\ &- \nabla_{x_j} c - \frac{1}{2}\sum_i b^i \nabla_{x_i} b^j \\ &= 2\nabla_Y Y + \sum_i (\nabla_{x_i} b^j - \nabla_{x_j} b^i)(\nabla_{x_i}\log u_0 - \frac{b^i}{2}) \\ &- (\nabla_{x_j} b^i)\frac{b^i}{2} - \nabla_{x_j} c \end{aligned}$$

Therefore

$$\begin{aligned}\Delta Y &= \Delta(-\nabla_{x_j}\log u_0 + \frac{b^j}{2}) \\ &= 2\nabla_Y Y - \sum_i (\nabla_{x_i} b^j - \nabla_{x_j} b^i) Y^i - \frac{1}{4}\nabla_{x_j}(\sum_i b^i b^i) - \nabla_{x_j} c + \Delta(\frac{b^j}{2}). \end{aligned}$$

□

Lemma 2.5. *On Ω there is a constant Λ depending on κ and on $\sup_j ||b^j||_{C^1(\bar{\Omega})}$ such that:*

$$\sup_{x\in\Omega} |Y(x)||U(x)| \leq \Lambda. \tag{2.12}$$

Proof. This follows easily from Proposition 2.2. □

We suppose that there is a function $\tau : \mathbb{R}_+ \to \mathbb{R}$ such that on Ω:

$$\big(V(y) - V(x)\big)\cdot \frac{y-x}{|y-x|} \geq 2\tau(\frac{|y-x|}{2}) \tag{2.13}$$

Let $\psi(s) : [0, \frac{D}{2}) \to \mathbb{R}$ be a C^2 function which satisfies for each $s \in [0, \frac{D}{2})$:

$$\psi(0) \geq 0, \quad \psi'(s) < 0 \tag{2.14}$$

$$2\Lambda - 2\tau(s) + 2\psi''(s) \leq -4\psi'(s)\psi(s). \tag{2.15}$$

The following theorem is motivated by a similar result in [AC] and [N1]. It is the first step in deriving a modulus of expansion for Y.

Theorem 2.6. *Suppose that Y satisfies (2.9) on Ω. Let ψ be the function defined above. Then*

$$\mathcal{C}(x,y) = \big(Y(y) - Y(x)\big)\cdot \frac{y-x}{|y-x|} + 2\psi\left(\frac{|y-x|}{2}\right)$$

can not attain a negative minimum in the interior of Ω.

Proof. We argue by contradiction and assume that at (x_0, y_0), $\mathcal{C}(x, y)$ attains a negative minimum. Clearly $x_0 \neq y_0$ since $\mathcal{C}(x, x) \geq 0$. Following [AC] and [N1] we choose a local orthonormal frame at x_0, denoted $\{e_1, \ldots, e_n\}$, with $e_n = \frac{y_0 - x_0}{|y_0 - x_0|}$ and parallel translate this frame along the line interval joining x_0 to y_0. Then at (x_0, y_0) we derive that:

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}\mathcal{C}(x + se_i, y)_{|s=0} \quad \text{for } 1 \leq i \leq n-1, \\ &= -\nabla_{e_i} Y(x) \cdot \frac{y - x}{|y - x|} - \frac{Y(y) - Y(x)}{|y - x|} \cdot e_i. \end{aligned} \tag{2.16}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}\mathcal{C}(x, y + se_i)_{|s=0} \quad \text{for } 1 \leq i \leq n-1, \\ &= \nabla_{e_i} Y(y) \cdot \frac{y - x}{|y - x|} + \frac{Y(y) - Y(x)}{|y - x|} \cdot e_i. \end{aligned} \tag{2.17}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}\mathcal{C}(x + se_n, y)_{|s=0}, \\ &= -\nabla_{e_n} Y(x) \cdot \frac{y - x}{|y - x|} - \psi'\left(\frac{|y - x|}{2}\right). \end{aligned} \tag{2.18}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}\mathcal{C}(x, y + se_n)_{|s=0}, \\ &= \nabla_{e_n} Y(y) \cdot \frac{y - x}{|y - x|} + \psi'\left(\frac{|y - x|}{2}\right). \end{aligned} \tag{2.19}$$

Let $E_i = e_i \oplus e_i \in T_{(x_0, y_0)}\mathbb{R}^n \times \mathbb{R}^n$ for $1 \leq i \leq n-1$ and $E_n = e_n \oplus (-e_n)$. Since $\mathcal{C}(x, y)$ attains its minimum at (x_0, y_0) we have:

$$\nabla^2_{E_i E_i}\mathcal{C}_{|(x_0, y_0)} \geq 0, \text{ for } 1 \leq i \leq n.$$

Along the path $(x + se_i, y + se_i)$ for $1 \leq i \leq n-1$ we note that $y - x$ is constant. Thus, computing as in [AC], we see that at (x_0, y_0):

$$0 \leq \nabla^2_{E_i E_i}\mathcal{C} = \left(\nabla^2_{e_i e_i} Y(y) - \nabla^2_{e_i e_i} Y(x)\right) \cdot \frac{y - x}{|y - x|} \text{ for } 1 \leq i \leq n-1. \tag{2.20}$$

Along the path $(x + se_n, y - se_n)$, $\frac{y - x}{|y - x|}$ is constant, $\frac{d}{ds}|y - x| = -2$ and $\frac{d^2}{ds^2}|y - x| = 0$. Again, computing as in [AC], we derive that at (x_0, y_0):

$$0 \leq \nabla^2_{E_n E_n}\mathcal{C} = \left(\nabla^2_{e_n e_n} Y(y) - \nabla^2_{e_n e_n} Y(x)\right) \cdot \frac{y - x}{|y - x|} + 2\psi'' \tag{2.21}$$

Using Lemma 2.4 we have that at (x_0, y_0):

$$\begin{aligned} &\sum_{i=1}^{n} \nabla^2_{E_i E_i}\mathcal{C} = \left(\Delta Y(y) - \Delta Y(x)\right) \cdot \frac{y - x}{|y - x|} + 2\psi'' \\ = \;& 2\left(\nabla_{Y(y)} Y(y) - \nabla_{Y(x)} Y(x)\right) \cdot \frac{y - x}{|y - x|} + \left(-Y(y) \cdot U(y) + Y(x) \cdot U(x)\right) \cdot \frac{y - x}{|y - x|} \\ - \;& \left(V(y) - V(x)\right) \cdot \frac{y - x}{|y - x|} + 2\psi'' \end{aligned}$$

As in [AC] and [N1] note that at (x_0, y_0):

$$\begin{aligned}\nabla_{Y(y)}Y(y)\cdot\frac{y-x}{|y-x|} &= \langle\nabla_{Y(y)}Y(y), e_n\rangle\\ &= \sum_{i=1}^{n}\langle Y(y), e_i\rangle\langle\nabla_{e_i}Y(y), e_n\rangle\\ &= \frac{-1}{|y-x|}\sum_{i=1}^{n-1}\langle Y(y), e_i\rangle\langle Y(y)-Y(x), e_i\rangle - \psi' Y(y)\cdot\frac{y-x}{|y-x|}\end{aligned}$$

By Lemma 2.5:

$$\Big|\big(-Y(y)\cdot U(y)+Y(x)\cdot U(x)\big)\cdot\frac{y-x}{|y-x|}\Big|\leq 2\Lambda \tag{2.22}$$

Putting these inequalities together we have at (x_0, y_0):

$$\begin{aligned}\sum_{j=1}^{n}\nabla^2_{E_jE_j}\mathcal{C} &\leq \frac{-2}{|y-x|}\sum_{j=1}^{n-1}\langle Y(y)-Y(x), e_i\rangle^2\\ &- 2\psi'(s)(Y(y)-Y(x))\cdot\frac{y-x}{|y-x|}+2\Lambda-2\tau(s)+2\psi''(s)\\ &\leq -2\psi'(s)(Y(y)-Y(x))\cdot\frac{y-x}{|y-x|}-4\psi'(s)\psi(s)\\ &\leq -2\psi'(s)\mathcal{C}(x,y)\\ &< 0\end{aligned}$$

Here $s=\frac{|y-x|}{2}$. The second inequality uses (2.15). The conclusion contradicts the assumption that (x_0, y_0) is a minimum point of $\mathcal{C}(x, y)$. $\square$

Boundary asympotics of $\mathcal{C}(x, y)$. The next step in the derivation of a modulus of expansion of Y is to study the boundary behavior of the function $\mathcal{C}(x, y)$. This analysis is similar to that done in [N1] and [AC], though we must modify the argument to our situation. In particular, unlike the situation in [N1] and [AC] we do not have available the log convexity of the first eigenfunction u_0. We continue under the assumption that there is a C^2 function ψ on $[0, \frac{D}{2})$ that satisfies (2.14) and (2.15). Note first that $\psi(s)$ is not defined at $s=\frac{D}{2}$. To rectify this, for fixed $D'>D$ consider ψ to be a solution of (2.15) on $[0, \frac{D'}{2})$. Then ψ is uniformly continuous on $[0, \frac{D}{2}]$.

Set $X=-\nabla\log u_0$ so that $Y=X+\frac{1}{2}B$. Then:

$$\mathcal{C}(x,y)=\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}+\frac{1}{2}\big(B(y)-B(x)\big)\cdot\frac{y-x}{|y-x|}+2\psi\left(\frac{|y-x|}{2}\right)$$

Given $\varepsilon>0$ we want to show that $\mathcal{C}(x, y)\geq-\varepsilon$ on $\Omega\times\Omega$. To begin we note that $\frac{1}{2}\big(B(y)-B(x)\big)\cdot\frac{y-x}{|y-x|}+2\psi(\frac{|y-x|}{2})\geq 0$ on the diagonal $\Delta=\{(x, x): x\in\Omega\}$. Use the uniform continuity of ψ on $[0, \frac{D}{2}]$ to find a neighborhood of the diagonal $\Delta_\eta=\{(x, y)\in\Omega\times\Omega: |x-y|<\eta\}$ on which:

$$\frac{1}{2}\big(B(y)-B(x)\big)\cdot\frac{y-x}{|y-x|}+2\psi\left(\frac{|y-x|}{2}\right)\geq-\frac{\varepsilon}{2} \tag{2.23}$$

The main work of this subsection involves the term $\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}$. Here u_0 is the first eigenfunction of the operator L on Ω with Dirichlet boundary conditions. It satisfies

$u_0 > 0$ on Ω, $u_{0|\partial\Omega} = 0$, and $\frac{\partial u_0}{\partial \nu}_{|\partial\Omega} < 0$. There is a constant $A > 0$ such that:

$$|u_0|_{C^2(\bar{\Omega})} \leq \frac{A}{2} \tag{2.24}$$

A neighborhood of the diagonal must be treated separately. This is because on the diagonal $\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|} = 0$ and therefore its behavior as $x \to \partial\Omega$ differs from its behavior away from the diagonal. We require two results. Set $\Omega_\delta = \{x \in \Omega : u_0(x) \geq \delta\}$. The first result states that for δ and η sufficiently small, if $x, y \in (\Omega \setminus \Omega_\delta) \cap \Delta_\eta$ then

$$\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|} \geq 0.$$

Thus there is an $\eta_0 \leq \eta$ so that on Δ_{η_0}, $\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|} \geq -\frac{\varepsilon}{2}$. The second result shows that for δ sufficiently small, if $x, y \in (\Omega \setminus \Omega_\delta)$ and $|x - y| > \eta_0$ then $\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|}$ is large. In fact, on this set $\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|} \to \infty$ as $\delta \to 0$.

To prove both results we study $\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|}$ on $\Omega \times \Omega$. Since $|\nabla u_0| > 0$ on $\partial\Omega$, there exist constants $\theta_0 > 0$ and $\delta' > 0$ such that on $\Omega \setminus \Omega_{\delta'}$:

$$|\nabla u_0| \geq \theta_0 \tag{2.25}$$

By the implicit function theorem this implies that for each $\delta \leq \delta'$ the set $\partial\Omega_\delta$ is a smooth hypersurface. By the convexity of Ω it follows that for δ' sufficiently small there is a constant $\theta_1 > 0$ such that for each $\delta \leq \delta'$ the second fundamental form $II(\cdot,\cdot)$ of the hypersurface $\partial\Omega_\delta$ satisfies the inequality:

$$II(\cdot,\cdot) \geq \theta_1 I(\cdot,\cdot), \tag{2.26}$$

where $I(\cdot,\cdot)$ is the metric on $\partial\Omega_\delta$. On $\partial\Omega_\delta$ the second fundamental form is given by:

$$II(\cdot,\cdot) = \frac{\nabla^2 u_0(\cdot,\cdot)}{|\nabla u_0|}, \tag{2.27}$$

since $\partial\Omega_\delta$ is a level set of u_0. For use below we set $C_1 = \frac{1}{2}\Big(\frac{A^2}{\theta_0\theta_1} + A\Big)$ and $\delta'' = \min\{\delta', \frac{\theta_0^2}{4C_1}\}$.

Lemma 2.7. *There is a $\bar{\delta} < \frac{1}{2}\delta''$ and an $\eta_0 < \eta$ such that for $x, y \in (\Omega \setminus \Omega_{\bar{\delta}}) \cap \Delta_{\eta_0}$, $x \neq y$:*

$$\big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|} \geq 0$$

Proof. Choose $\bar{\delta} < \frac{1}{2}\delta''$ and $\eta_0 < \eta$ such that for any two points $x, y \in (\Omega \setminus \Omega_{\bar{\delta}}) \cap \Delta_{\eta_0}$, $x \neq y$, the line segment joining x to y lies in $\Omega \setminus \Omega_{\delta''}$. Let $\gamma(s)$ denote this line segment parameterized by arc length. Without loss of generality we can suppose that $u_0(x) \leq u_0(y) \leq \bar{\delta}$. Set $\delta = u_0(x)$. Then,

$$\begin{aligned}
(X(y) - X(x)) \cdot \frac{y-x}{|y-x|} &= \langle X(\gamma(s)), \gamma'(s)\rangle|_0^{|y-x|} \\
&= \int_0^{|y-x|} \frac{d}{ds}(\langle X(\gamma(s)), \gamma'(s)\rangle) ds \\
&= \int_0^{|y-x|} \nabla^2(-\log u_0)(\gamma'(s), \gamma'(s)) ds
\end{aligned} \tag{2.28}$$

Set $\gamma'(s) = W$. At each point $\gamma(s)$, decompose W into a component, $W^\top$, tangent to $T_{\gamma(s)}\Omega_{u_0(\gamma(s))}$ and a normal component $W^\perp$ with respect to the inward pointing normal

$-\nu_{\gamma(s)}$. We have:

$$\begin{aligned}
\nabla^2 u_0(W,W) &= \nabla^2 u_0(W^\top,W^\top)+2\nabla^2 u_0(W^\top,W^\perp)+\nabla^2 u_0(W^\perp,W^\perp)\\
&\le -|\nabla u_0| II(W^\top,W^\top)+A|W^\top||W^\perp|+\frac{A}{2}|W^\perp|^2\\
&\le -\theta_0\theta_1|W^\top|^2+A|W^\top||W^\perp|+\frac{A}{2}|W^\perp|^2\\
&\le -\frac{\theta_0\theta_1}{2}|W^\top|^2+C_1|W^\perp|^2
\end{aligned}$$

In the last inequality we have used:

$$A|W^\top||W^\perp| = (\theta_0\theta_1)^{\frac{1}{2}}|W^\top|\frac{A}{(\theta_0\theta_1)^{\frac{1}{2}}}|W^\perp| \le \tfrac{1}{2}\Big(\theta_0\theta_1|W^\top|^2+\tfrac{A^2}{\theta_0\theta_1}|W^\perp|^2\Big)$$

Since $u_0(x)=\delta\le\bar\delta$, we let k be the integer such that

$$2^k\delta\le\bar\delta<2^{k+1}\delta<\delta''.$$

For $j=1,\dots,k+1$ set $\delta_j=2^j\delta$. Then for $\delta_{j-1}\le u_0\le\delta_j$ we have:

$$\begin{aligned}
\nabla^2\log u_0(W,W) &= \frac{\nabla^2 u_0(W,W)}{u_0}-\frac{|\nabla u_0|^2}{u_0^2}|W^\perp|^2\\
&\le -\frac{\theta_0\theta_1}{2u_0}|W^\top|^2+\frac{C_1}{u_0}|W^\perp|^2-\frac{\theta_0^2}{u_0^2}|W^\perp|^2\\
&\le -\frac{\theta_0\theta_1}{2\delta_j}|W^\top|^2+\frac{2C_1}{\delta_j}|W^\perp|^2-\frac{\theta_0^2}{{\delta_j}^2}|W^\perp|^2\\
&\le -\frac{\theta_0\theta_1}{2\delta_j}|W^\top|^2-\frac{\theta_0^2}{2{\delta_j}^2}|W^\perp|^2
\end{aligned}$$

where the final inequality uses the definition of δ'' and that $\delta_j<\delta''$. From this inequality it follows immediately that:

$$\int_\gamma \nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds\ge 0 \tag{2.29}$$

The result follows from (2.28). □

It follows from the lemma that there is an $\eta_1\le\eta_0$ so that on Δ_{η_1}, $\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}\ge-\frac{\varepsilon}{2}$. We next study $\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}$ on $\Omega\times\Omega\setminus\Delta_{\eta_1}$. We continue to use $C_1=\frac{1}{2}\big(\frac{A^2}{\theta_0\theta_1}+A\big)$ and $\delta''=\min\{\delta',\frac{\theta_0^2}{4C_1}\}$ but with the additional assumption that $\delta''<<\eta_1$.

Lemma 2.8. *For $\delta<\delta''$ sufficiently small and $x,y\in\Omega\setminus\Omega_\delta$ with $|x-y|>\eta_1$ there are constants C_2,C_3 independent of δ such that:*

$$(X(y)-X(x))\cdot\frac{y-x}{|y-x|}\ge\frac{C_2}{\delta}-C_3$$

Proof. Using the strict convexity of $\partial\Omega$, choose $\delta''<<\eta_1$ so that if $x,y\in\Omega\setminus\Omega_{\delta''}$ with $|y-x|>\eta_1$ then the line segment $\gamma(s)$ joining x to y intersects $\Omega_{\delta''}$. Suppose $\delta<\delta''$, $x\in\partial\Omega_{\frac{\delta}{2}}$ and $y\in\Omega$ with $|y-x|>\eta_1$ where we assume that $u_0(x)\le u_0(y)$. Let $\gamma(s)$ be the line segment joining x to y, parameterized by arc length. Divide $\gamma(s)$ into two disjoint curves: γ_1 lying in $\Omega \setminus \Omega_{\delta''}$ and γ_2 lying in $\Omega_{\delta''}$. Then:

$$\begin{aligned}(X(y)-X(x))\cdot\frac{y-x}{|y-x|} &= \langle X(\gamma(s)),\gamma'(s)\rangle|_0^{|y-x|}\\ &= \int_0^{|y-x|}\frac{d}{ds}(\langle X(\gamma(s)),\gamma'(s)\rangle)ds\\ &= \int_0^{|y-x|}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds\\ &= \int_{\gamma_1}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds+\int_{\gamma_2}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds\end{aligned} \tag{2.30}$$

From

$$|\nabla^2\log u_0|\le\frac{|\nabla^2 u_0|}{|u_0|}+\frac{|\nabla u_0|^2}{{u_0}^2}$$

and (2.24) we get the estimate:

$$\int_{\gamma_2}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds > -\big(\frac{A}{2\delta''}+\frac{A^2}{4{\delta''}^2}\big)D \tag{2.31}$$

To estimate the other integral, set $\gamma'(s)=W$. At each point $\gamma(s)$, decompose W into a component, $W^\top$, tangent to $T_{\gamma(s)}\Omega_{u_0(\gamma(s))}$ and a normal component $W^\perp$ with respect to the inward pointing normal $-\nu_{\gamma(s)}$. For the curve γ_1 lying in $\Omega\setminus\Omega_{\delta''}$ we have, as in the proof of Lemma 2.7:

$$\begin{aligned}\nabla^2u_0(W,W) &= \nabla^2u_0(W^\top,W^\top)+2\nabla^2u_0(W^\top,W^\perp)+\nabla^2u_0(W^\perp,W^\perp)\\ &\le -\frac{\theta_0\theta_1}{2}|W^\top|^2+C_1|W^\perp|^2\end{aligned}$$

For $\delta<\delta''$ let k be the integer such that

$$2^k\delta\le\delta''<2^{k+1}\delta.$$

Set $\delta_j=2^j\delta$. For $\delta_{j-1}\le u_0\le\delta_j$ with $j=0,\dots,k$, we have, as in the proof of Lemma 2.7:

$$\begin{aligned}\nabla^2\log u_0(W,W) &= \frac{\nabla^2u_0(W,W)}{u_0}-\frac{|\nabla u_0|^2}{u_0^2}|W^\perp|^2\\ &\le -\frac{\theta_0\theta_1}{2\delta_j}|W^\top|^2-\frac{\theta_0^2}{2{\delta_j}^2}|W^\perp|^2\end{aligned} \tag{2.32}$$

We subdivide the curve γ_1 via the level sets of u_0. Let s_j be the first s satisfying $u_0(\gamma(s))=\delta_j$, for $j=-1,0,\dots,k$. Let s' be the first s satisfying $u_0(\gamma(s))=\delta''$. Then $s'>s_k$ and we can write:

$$\begin{aligned}\int_{\gamma_1}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds &\ge \sum_{j=-1}^{k-1}\int_{s_j}^{s_{j+1}}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds\\ &+\int_{s_k}^{s'}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds\\ &\ge \sum_{j=-1}^{k-1}\int_{s_j}^{s_{j+1}}\nabla^2(-\log u_0)(\gamma'(s),\gamma'(s))ds,\end{aligned}$$

where the last inequality follows from (2.29). Since $|\nabla u_0| \le \frac{A}{2}$ it follows that $s_{j+1} - s_j \ge \frac{\delta_{j+1}}{A}$. Thus $s_{j+1} \ge \frac{\delta_{j+1}}{A}$. Similarly $s' \ge \frac{\delta''}{A}$. Hence $|y-x| \ge \frac{\delta''}{A}$. Since Ω is strictly convex, for $\delta \le \delta''$, if $x \in \partial\Omega_{\frac{\delta}{2}}$ and $y \in \Omega$ satisfy $|y-x| \ge \frac{\delta''}{A} > 0$ then there exists a constant $\theta_2 > 0$, depending only on $\frac{\delta''}{A}$, δ' and the convexity of Ω such that:

$$\langle -\nu_x, \frac{y-x}{|y-x|} \rangle \ge \theta_2. \tag{2.33}$$

This estimate can also be written $|W^{\perp}| \ge \theta_2$. Therefore using (2.32):

$$\begin{aligned} \sum_{j=-1}^{k-1} \int_{s_j}^{s_{j+1}} \nabla^2(-\log u_0)(\gamma'(s), \gamma'(s)) ds &\ge \sum_{j=-1}^{k-1} (s_{j+1} - s_j) \frac{\theta_0^2}{2{\delta_j}^2} |W^{\perp}|^2 \\ &\ge \sum_{j=-1}^{k-1} \frac{\theta_0^2 \theta_2^2}{A\delta_j} \\ &\ge \frac{\theta_0^2\theta_2^2}{A\delta} \sum_{j=-1}^{k-1} \frac{1}{2^j} \end{aligned} \tag{2.34}$$

Set $C_2 = \frac{2\theta_0^2\theta_2^2}{A}$ and $C_3 = \left(\frac{A}{2\delta''} + \frac{A^2}{4\delta''^2}\right)D$ then the estimates (2.31) and (2.34) imply:

$$(X(y) - X(x)) \cdot \frac{y-x}{|y-x|} \ge \frac{C_2}{\delta} - C_3 \tag{2.35}$$

□

Theorem 2.9. *On $\Omega \times \Omega$:*

$$\mathcal{C}(x,y) = \big(Y(y) - Y(x)\big) \cdot \frac{y-x}{|y-x|} + 2\psi\left(\frac{|y-x|}{2}\right) \ge 0$$

Proof. Given $\varepsilon > 0$ by Lemma 2.7 there is an $\eta_1 > 0$ such that $\mathcal{C}(x,y) \ge -\varepsilon$ for $x, y \in \Delta_{\eta_1}$. By Lemma 2.8 there is a $\delta > 0$ such that for $x \in (\Omega \setminus \Omega_\delta) \setminus \Delta_{\eta_1}$ and for any $y \in \Omega \setminus \Delta_{\eta_1}$, $\mathcal{C}(x,y) \ge 0$. By Theorem 2.6 this implies that $\mathcal{C}(x,y) \ge -\varepsilon$ for $x, y \in \Omega$. Since ε is arbitrary this implies the result for ψ defined on $[0, \frac{D'}{2})$. Let $D' \to D$ to conclude the result for ψ satisfying (2.14) and (2.15) on $[0, \frac{D}{2})$. □

Under the assumption that there is a C^2 function ψ on $[0, \frac{D}{2})$ that satisfies (2.14) and (2.15), this result implies that $-\psi = \omega$ is a modulus of expansion of Y.

Differential inequalities and a Sturm-Liouville problem. To apply Theorem 2.9 we must find a solution to the differential inequalities (2.14) and (2.15). The inequality (2.15) becomes:

$$2\psi''(s) + 4\psi'(s)\psi(s) + 2\Lambda - 2\tau(s) \le 0 \tag{2.36}$$

To proceed we consider two cases:

(I) $\tau(s) \ge 0$. This is a "convexity" condition on the vector field V. In this case (2.36) follows from:

$$2\psi''(s) + 4\psi'(s)\psi(s) + 2\Lambda \le 0 \tag{2.37}$$

(II) In general, let

$$\Lambda' = \max\big(\sup_{s\in[0,\frac{D}{2}]} -\tau(s), 0\big)$$

and set $\tilde{\Lambda} = \Lambda + \Lambda'$. Then (2.36) follows from:

$$2\psi''(s) + 4\psi'(s)\psi(s) + 2\tilde{\Lambda} \leq 0 \tag{2.38}$$

In both cases the inequality has the same form. We will set $\Lambda = \tilde{\Lambda}$ and use the inequality:

$$2\psi''(s) + 4\psi'(s)\psi(s) + 2\Lambda \leq 0, \text{ on } [0, \tfrac{D}{2}). \tag{2.39}$$

If $g(s)$ is a continuous piecewise differentiable function on $[0, \frac{D}{2}]$ such that $g'(s) \geq \Lambda$ for $s \in [0, \frac{D}{2}]$ then this inequality follows from:

$$\psi'(s) + \psi(s)^2 + g(s) = -\nu, \text{ on } [0, \tfrac{D}{2}). \tag{2.40}$$

by differentiation, where ν is an arbitrary constant.

Make the substitution $\omega = -\psi$. Then (2.40) becomes:

$$\omega'(s) - \omega(s)^2 - g(s) = \nu, \text{ on } [0, \tfrac{D}{2}). \tag{2.41}$$

This is a Riccati equation. We will show that this equation can be solved for suitable ν with $\omega'(s) > 0$ for $s \in [0, \frac{D}{2})$ and $\omega(0) \leq 0$.

Let $F_\sigma(s)$ be the continuous piecewise differentiable function:

$$F_\sigma(s) = \begin{cases} \sigma s & \text{if } 0 \leq s \leq \frac{D}{2}, \\ -\sigma s & \text{if } -\frac{D}{2} \leq s \leq 0. \end{cases} \tag{2.42}$$

Consider the Sturm-Liouville eigenvalue problem on $[-\frac{D}{2}, \frac{D}{2}]$:

$$w'' + F_\sigma w = -\mu w \tag{2.43}$$

with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$. This is a regular Sturm-Liouville eigenvalue problem in normal form. It has an infinite sequence of real eigenvalues $\mu_0 < \mu_1 < \mu_2 < \dots$ with $\lim_{n\to\infty} \mu_n = \infty$. The eigenfunction $w_n(s)$ belonging to the eigenvalue μ_n has exactly n zeros in the interval $(-\frac{D}{2}, \frac{D}{2})$ and is uniquely determined up to a constant factor [BR].

There is a scalar $\sigma_0 > 0$ such that the linear operator on smooth functions on $[-\frac{D}{2}, \frac{D}{2}]$:

$$\mathcal{L}(w) = -w'' - F_\sigma w$$

is positive definite for $\sigma < \sigma_0$ and is not positive definite for $\sigma > \sigma_0$. Therefore the first eigenvalue $\mu_0(\sigma)$ is positive if $\sigma < \sigma_0$, zero if $\sigma = \sigma_0$ and negative otherwise. In all cases the corresponding eigenfunction w_0 is positive on $(-\frac{D}{2}, \frac{D}{2})$, vanishes at the endpoints and satisfies:

$$w_0'' = -(\mu_0 + F_\sigma(s))w_0 \tag{2.44}$$

Using the variational characterization of the first eigenvalue it follows that:

(i) $-\mu_0 = -\mu_0(\sigma)$ is an increasing function of σ,

(ii) $-\mu_0 < \sigma\frac{D}{2}$.

Since F_σ is an even function, so is w_0 and therefore $w_0'(0) = 0$. Thus w_0 satisfies the boundary value problem on $[0, \frac{D}{2}]$:

$$\begin{aligned} &w''(s) + F_\sigma(s)w(s) = -\mu_0 w(s) \\ &w'(0) = 0, w(\tfrac{D}{2}) = 0. \end{aligned} \tag{2.45}$$

Proposition 2.10. *As $\sigma \to \infty$, $\frac{-\mu_0}{\sigma} \to \frac{D}{2}$.*

Proof. Set $y(s) = w_0(s\sigma^{-\frac{1}{3}})$. Then,

$$\begin{aligned} y''(s) &= \sigma^{-\frac{2}{3}} w_0''(s\sigma^{-\frac{1}{3}}) \\ &= \sigma^{-\frac{2}{3}}\big(-\sigma s\sigma^{-\frac{1}{3}} w_0(s\sigma^{-\frac{1}{3}}) - \mu_0 w_0(s\sigma^{-\frac{1}{3}})\big) \\ &= -sy(s) - \mu_0\sigma^{-\frac{2}{3}} y(s). \end{aligned}$$

Hence y is the first eigenfunction with eigenvalue $-\mu_0\sigma^{-\frac{2}{3}}$ of the boundary value problem on $[0, \frac{D}{2}\sigma^{\frac{1}{3}}]$:

$$\begin{aligned} &y''(s) + sy(s) = -\mu_0\sigma^{-\frac{2}{3}}y(s) \\ &y'(0) = 0, \;\; y(\tfrac{D}{2}\sigma^{\frac{1}{3}}) = 0. \end{aligned} \tag{2.46}$$

Scaling we can suppose that y satisfies: $\int_0^{\frac{D}{2}\sigma^{\frac{1}{3}}} y(s)^2 ds = 1$. Set $\beta = \mu_0\sigma^{-\frac{2}{3}}$. Then β is characterized as the infimum

$$\beta = \inf\Big(\int_0^{\frac{D}{2}\sigma^{\frac{1}{3}}} \big((y'(s))^2 - s(y(s))^2\big) ds\Big)$$

over functions y satisfying $\int_0^{\frac{D}{2}\sigma^{\frac{1}{3}}} y(s)^2 ds = 1$. Clearly $\beta > -\sigma^{\frac{1}{3}}\frac{D}{2}$. On the other hand define the test function:

$$z(s) = \begin{cases} \sqrt{2}\sin\pi(s - \frac{D}{2}\sigma^{\frac{1}{3}}) & \text{if } \frac{D}{2}\sigma^{\frac{1}{3}} - 1 \le s \le \frac{D}{2}\sigma^{\frac{1}{3}}, \\ 0 & \text{if } 0 \le s \le \frac{D}{2}\sigma^{\frac{1}{3}} - 1. \end{cases} \tag{2.47}$$

Then,

$$\int_0^{\frac{D}{2}\sigma^{\frac{1}{3}}} z(s)^2 ds = 1$$

and

$$\int_0^{\frac{D}{2}\sigma^{\frac{1}{3}}} \big((z'(s))^2 - s(z(s))^2\big) ds = -\tfrac{D}{2}\sigma^{\frac{1}{3}} + \pi^2 + \tfrac{1}{2}.$$

Hence,

$$-\tfrac{D}{2}\sigma^{\frac{1}{3}} < \beta \le -\tfrac{D}{2}\sigma^{\frac{1}{3}} + \pi^2 + \tfrac{1}{2}.$$

Since $\beta = \mu_0\sigma^{-\frac{2}{3}}$ this implies,

$$\tfrac{D}{2} - (\pi^2 + \tfrac{1}{2})\sigma^{-\frac{1}{3}} \le \tfrac{-\mu_0}{\sigma} < \tfrac{D}{2}.$$

The result follows. □

Introduce the function

$$v_\sigma(s) = \frac{w_0'(s)}{w_0(s)} \quad 0 \le s \le \frac{D}{2}.$$

On $[0, \frac{D}{2}]$, $v_\sigma(s)$ satisfies the initial value problem:

$$\begin{aligned} &v'(s) + v(s)^2 = -\sigma s - \mu_0 \\ &v(0) = 0 \end{aligned} \tag{2.48}$$

Proposition 2.11. *For each $\sigma > \sigma_0$, there is a unique point $s_0 = s_0(\sigma) \in (0, \frac{D}{2})$ such that $v_\sigma'(s_0) = 0$, $v_\sigma'(s) > 0$ on $(0, s_0)$ and $v_\sigma'(s) < 0$ on $(s_0, \frac{D}{2})$. Moreover, as $\sigma \to \infty$, $s_0(\sigma) \to 0$.*

Proof. Differentiating (2.48) we have:

$$v''(s) = -\sigma - 2v(s)v'(s)$$

Therefore every critical point of v in $(0, \frac{D}{2})$ is a local maximum. Note that provided $\mu_0 < 0$, $v'(0) > 0$ so that v is initially increasing and positive. On the other hand, since $0 < -\mu_0 < \sigma$, by (2.44) there is an $s_1 \in (0, \frac{D}{2})$ such that $w''(s) < 0$ for $s \in (s_1, \frac{D}{2})$. Thus on $(s_1, \frac{D}{2})$, $v'(s) = \frac{w''(s)}{w(s)} - (\frac{w'(s)}{w(s)})^2 < 0$. It follows that v has at least one local maximum point in $(0, \frac{D}{2})$. Thus there is a unique critical point $s_0 \in (0, \frac{D}{2})$ and it is a local maximum. Therefore $v'(s) > 0$ on $[0, s_0)$ and $v'(s) < 0$ on $(s_0, \frac{D}{2})$. From (2.48) it follows at that the maximum point s_0:

$$v(s_0)^2 = -\sigma s_0 - \mu_0$$

Set $\lambda = -\mu_0$. Thus

$$\max_{[0,\frac{D}{2}]} v(s) = v(s_0) = \sqrt{\lambda - \sigma s_0} < \sqrt{\lambda}$$

Let $0 < a < \frac{D}{2}$. We will show that for σ sufficiently large (depending on $\frac{1}{a^2}$) the point $s_0 \in (0, a)$. Suppose not. Then on $[0, a)$, $v'(s) > 0$ and v is strictly increasing. We deduce a contradiction by showing that then $v(a) > \max_{[0,\frac{D}{2}]} v(s)$. Let $0 = t_0 < t_1 < \cdots < t_n = a$ be a subdivision of $[0, a]$ with $t_{i+1} - t_i = \Delta t$. Denote $v(t_i) = v_i$. We construct an iterative scheme that successively estimates v_i above and below. We denote the lower estimate $\bar{v}_i$, the upper estimate $\bar{\bar{v}}_i$ so that $\bar{v}_i \leq v_i \leq \bar{\bar{v}}_i$. At $t_0 = 0$ set $\bar{v}_0 = v_0 = \bar{\bar{v}}_0 = 0$. On the interval $[t_0, t_1]$ we have:

$$-\sigma t_1 + \lambda - v_1^2 \leq v' \leq \lambda.$$

Hence,

$$v_1 \leq \lambda \Delta t,$$

so that,

$$-\sigma t_1 + \lambda - (\lambda \Delta t)^2 \leq v' \leq \lambda$$

Therefore,

$$\begin{aligned} \bar{\bar{v}}_1 &= \lambda \Delta t, \\ \bar{v}_1 &= (-\sigma t_1 + \lambda - (\lambda \Delta t)^2) \Delta t. \end{aligned}$$

Suppose that $\bar{v}_i$ and $\bar{\bar{v}}_i$ are defined. Then on $[t_i, t_{i+1}]$ we have:

$$-\sigma t_{i+1} + \lambda - \bar{\bar{v}}_{i+1}^2 \leq v' \leq -\sigma t_i + \lambda - \bar{v}_i^2.$$

Hence we define:

$$\bar{\bar{v}}_{i+1} = (-\sigma t_i + \lambda - \bar{v}_i^2)\Delta t + \bar{\bar{v}}_i,$$

so that, $v_{i+1} \leq \bar{\bar{v}}_i$. Then on $[t_i, t_{i+1}]$:

$$-\sigma t_{i+1} + \lambda - \Big((-\sigma t_i + \lambda - \bar{v}_i^2)\Delta t + \bar{\bar{v}}_i \Big)^2 \leq v'.$$

Define,

$$\bar{v}_{i+1} = \left(-\sigma t_{i+1} + \lambda - \Big((-\sigma t_i + \lambda - \bar{v}_i^2)\Delta t + \bar{\bar{v}}_i \Big)^2 \right) \Delta t + \bar{v}_i.$$

Thus since,

$$\bar{\bar{v}}_i = (-\sigma t_{i-1} + \lambda - \bar{v}_{i-1}^2)\Delta t + \bar{\bar{v}}_{i-1}$$

we derive,

$$\begin{aligned} \bar{v}_{i+1} &\equiv (-\sigma t_{i+1} + \lambda - \bar{\bar{v}}_i^2)\Delta t + \bar{v}_i \mod (\Delta t)^2, \\ &\equiv (-\sigma t_{i+1} + \lambda - \bar{\bar{v}}_{i-1}^2)\Delta t + \bar{v}_i \mod (\Delta t)^2, \\ &\equiv (-\sigma t_{i+1} + \lambda - \bar{\bar{v}}_{i-2}^2)\Delta t + \bar{v}_i \mod (\Delta t)^2, \\ &\cdots \\ &\equiv (-\sigma t_{i+1} + \lambda)\Delta t + \bar{v}_i \mod (\Delta t)^2. \end{aligned}$$

Hence,

$$v(a) \geq \bar{v}_n \equiv (-\sigma t_n + \lambda)\Delta t + (-\sigma t_{n-1} + \lambda)\Delta t + \cdots + \lambda \Delta t \mod (\Delta t)^2.$$

Letting $n \to \infty$ we get,

$$v(a) \geq -\sigma \int_0^a t dt + \lambda a = -\sigma \frac{a^2}{2} + \lambda a.$$

Recall that,

$$v(a) \leq \max_{[0,\frac{D}{2}]} v \leq \sqrt{\lambda}.$$

Hence,

$$-\sigma\frac{a^2}{2} + \lambda a < \sqrt{\lambda}.$$

Using Proposition 2.10 there is a scalar σ_1 such that if $\sigma \geq \sigma_1$ then $\sigma\frac{D}{3} < \lambda < \sigma\frac{D}{2}$. Hence:

$$a(-\sigma\frac{a}{2} + \sigma\frac{D}{3}) < -\sigma\frac{a^2}{2} + \lambda a < \sqrt{\lambda} < \sqrt{\sigma\frac{D}{2}}$$

Since $a < \frac{D}{2}$ we get,

$$\sigma < \frac{72}{D}\frac{1}{a^2}.$$

Therefore, if,

$$\sigma \geq \max(\sigma_1, \frac{72}{D}\frac{1}{a^2})$$

we have a contradiction, proving that, in this case, the maximum of v occurs in $(0, a)$. $\square$

In particular, there is a $\sigma_2 > \sigma_0$ such that for all $\sigma > \sigma_2$ we have $s_0(\sigma) \in (0, \frac{D}{4})$.
Set:

$$\eta_\sigma = (\tfrac{D}{2})^{-1}(\tfrac{D}{2} - s_0(\sigma))$$

and note that for $\sigma > \sigma_2$, $\eta_\sigma > \frac{1}{2}$, independent of the choice of σ. Let η satisfy:

$$\eta_\sigma \geq \eta > \tfrac{1}{2} \tag{2.49}$$

We next explain how to use this Sturm-Liouville problem to solve (2.41).

For $s \in [0, \frac{D}{2})$ and η satisfying (2.49) set:

$$\omega(s) = -\frac{\eta w_0'(\eta s + s_0)}{w_0(\eta s + s_0)}.$$

Set

$$\tilde{F}_\sigma(s) = \eta^2 F_\sigma(\eta s + s_0)$$

Then

$$\begin{aligned}
\omega'(s) - \omega^2(s) - \tilde{F}_\sigma(s) &= -\eta^2\left(\frac{w_0''(\eta s + s_0)}{w_0(\eta s + s_0)} + F_\sigma(\eta s + s_0)\right) \\
&= -\eta^2\left(\frac{1}{w_0(\eta s + s_0)}\big(w_0''(\eta s + s_0) + F_\sigma(\eta s + s_0)w_0(\eta s + s_0)\big)\right) \\
&= \eta^2\mu_0 \qquad (2.50)
\end{aligned}$$

Hence if we chose $\nu = \eta^2\mu_0$ then $\omega(s)$ satisfies (2.41) with $g(s) = \tilde{F}_\sigma(s)$. Since,

$$\frac{d}{ds}\tilde{F}_\sigma(s) = \eta^3 F_\sigma'(\eta s + s_0) = \eta^3\sigma$$

Choose

$$\sigma = \max(\sigma_2, 8\Lambda), \tag{2.51}$$

then $g'(s) \geq \Lambda$ and (2.41) follows. Note that for $s \in [0, \frac{D}{2})$:

$$\omega(s) = -\eta v_\sigma(\eta s + s_0)$$

Therefore for $s \in [0, \frac{D}{2})$:

$$\omega'(s) = -\eta^2 v_\sigma'(\eta s + s_0) > 0$$

Clearly, $\omega(0) = -\eta v_\sigma(s_0) < 0$. It follows that $\psi(s) = -\omega(s)$ on $[0, \frac{D}{2})$ satisfies the requirements of Theorem 2.6.

Notice that if we choose the scaling factor η satisfying $\eta_\sigma > \eta > \frac{1}{2}$ then $\omega(s)$ is defined on some interval $[0, \frac{D'}{2})$ with $D' > D$. In particular $\omega(s)$ is defined on $[0, \frac{D}{2}]$. This choice is used in the next subsection.

Associated Sturm-Liouville problem. In this subsection we show that the Sturm-Liouville problem:

$$\mathcal{L}[w] = w'' + F_\sigma(s)w = -\mu w \tag{2.52}$$

with $w(\pm\frac{D}{2}) = 0$, where F_σ is defined above, is an associated Sturm-Liouville problem and thus yields a spectral gap result.

The second eigenfunction w_1 has a unique zero and because F_σ is even this zero occurs at the origin. After scaling we can assume that $w_0(s) > 0$ for $s \in (-\frac{D}{2}, \frac{D}{2})$ and $w_1(s) > 0$ for $s \in (0, \frac{D}{2})$.

Given any two smooth functions u, v on $[-\frac{D}{2}, \frac{D}{2}]$ the Lagrange identity is easily derived:

$$u\mathcal{L}[v] - v\mathcal{L}[u] = \frac{d}{ds}\big(u(s)v'(s) - v(s)u'(s)\big).$$

Set $u = w_1$ and $v = w_0$, then the Lagrange identity yields:

$$w_1\mathcal{L}[w_0] - w_0\mathcal{L}[w_1] = \frac{d}{ds}\big(w_1 w_0' - w_0 w_1'\big). \tag{2.53}$$

By our scaling assumptions $\int_0^{\frac{D}{2}} w_1 w_0 > 0$. Since $\int_{-\frac{D}{2}}^{\frac{D}{2}} w_1 w_0 = 0$ and $w_0 w_1 < 0$ on $(-\frac{D}{2}, 0)$, it follows that $\int_s^{\frac{D}{2}} w_1 w_0 > 0$ for every $s > -\frac{D}{2}$. Integrating (2.53) from s to $\frac{D}{2}$ we have:

$$0 < -(\mu_0 - \mu_1)\int_s^{\frac{D}{2}} w_0 w_1 = -\big(w_1(s)w_0'(s) - w_0(s)w_1'(s)\big),$$

where we have used that $w_0(\frac{D}{2}) = w_1(\frac{D}{2}) = 0$. Hence, for any $s \in (-\frac{D}{2}, \frac{D}{2})$:

$$\big(w_0(s)w_1'(s) - w_1(s)w_0'(s)\big) > 0. \tag{2.54}$$

In particular, this implies that for $s \in (-\frac{D}{2}, \frac{D}{2})$ we have:

$$\frac{d}{ds}\big(\frac{w_1}{w_0}\big)(s) = \frac{1}{w_0^2}(s)\big(w_0(s)w_1'(s) - w_1(s)w_0'(s)\big) > 0 \tag{2.55}$$

Set $\rho(s,t) = \frac{e^{-\mu_1 t} w_1(s)}{e^{-\mu_0 t} w_0(s)}$. Denote $\frac{\partial}{\partial s}\rho(s,t) = \rho'(s,t)$. Then

Lemma 2.12. *For* $s \in (-\frac{D}{2}, \frac{D}{2})$,

$$\frac{\partial}{\partial t}\rho(s,t) = -(\mu_1 - \mu_0)\rho(s,t) = \rho''(s,t) + 2\frac{w_0'(s)}{w_0(s)}\rho'(s,t).$$

Proof. This is a direct computation. □

To exploit the lemma, choose the scaling factor η to satisfy $\eta_\sigma > \eta > \frac{1}{2}$ and the point $s_0(\sigma)$ as defined above. Set:

$$\varphi(s,t) = Ce^{-(\mu_1-\mu_0)\eta^2 t}\frac{w_1(\eta s + s_0)}{w_0(\eta s + s_0)}, \tag{2.56}$$

where C is a constant to be determined. Note that since η satisfies $\eta_\sigma > \eta > \frac{1}{2}$, $\varphi(s,t)$ is defined on some interval $[0, \frac{D'}{2})$ with $D' > D$. Hence $\varphi(s,t)$ is defined on $[0, \frac{D}{2}]$. Recall that: $\omega(s) = -\frac{\eta w_0'(\eta s + s_0)}{w_0(\eta s + s_0)}$. By the same computation as in the lemma we get:

$$\frac{\partial \varphi}{\partial t} = \varphi'' - 2\omega\varphi' \quad \text{on} \quad [0, \frac{D}{2}] \times \mathbb{R}_+,$$

Set

$$\varphi_0(s) = \varphi(s,0) = C\frac{w_1(\eta s + s_0)}{w_0(\eta s + s_0)}.$$

Since $w_0(s_0) > 0$ and $w_1(s_0) > 0$ it follows that both $\varphi_0(0) > 0$ and $\varphi(0,t) > 0$ for all $t > 0$. It is also true that $\frac{\partial}{\partial s}\varphi(s,t) = Ce^{-(\mu_1-\mu_0)\eta^2 t}\frac{d}{ds}(\frac{w_1}{w_0})(\eta s + s_0) > 0$ for all $t \geq 0$ and $0 \leq s \leq \frac{D}{2}$.

We prove:

Theorem 2.13. *Let Ω be a strictly convex bounded domain with smooth boundary in $\mathbb{R}^n$. Suppose the diameter of Ω is D. Then the gap between the first eigenvalue, λ_0, and the real part of any other eigenvalue, λ, of the linear elliptic operator L given by (2.1) satisfies:*

$$\mathrm{Re}(\lambda) - \lambda_0 > \frac{1}{4}(\mu_1 - \mu_0) = \alpha > 0,$$

where α is a constant depending on Λ and hence on b^i, c, κ.

Proof. We derive the theorem from Theorem 1.1. Suppose that the eigenfunction corresponding to λ is denoted u. Let $z(x,t) = e^{-(\lambda-\lambda_0)t}\frac{u(x)}{u_0(x)}$. We wish to conclude that for suitable constant C, $\varphi(s,t)$ is a modulus of continuity of $z(x,t)$.

The drift velocity $Y(\cdot,t)$ has modulus of expansion

$$\omega(\cdot,t) = -\psi(\cdot,t) = -(\log w_0(\eta s + s_0))' = -(\frac{\eta w_0'}{w_0})(\eta s + s_0),$$

by Theorem 2.9. As shown above the function $\varphi(s,t)$ satisfies the equation:

$$\frac{\partial \varphi}{\partial t} \geq \varphi'' - 2\omega\varphi' \quad \text{on } [0, \frac{D}{2}] \times \mathbb{R}_+,$$

Using (2.55) $\frac{d}{ds}\big(\frac{w_1}{w_0}(\eta s + s_0)\big) > 0$ on $[0, \frac{D}{2}]$. Therefore there exists a constant C such that $\frac{d}{ds}\big(C\frac{w_1}{w_0}(\eta s + s_0)\big)$ is a modulus of continuity of $z(x,0)$. Set

$$\varphi(s,t) = Ce^{-(\mu_1-\mu_0)\eta^2 t}(\frac{w_1}{w_0})(\eta s + s_0),$$

with this constant C. The hypotheses of Theorem 1.1 are satisfied and we conclude that $2\varphi(s,t)$ is a modulus of continuity of $z(x,t)$. Thus,

$$|e^{-(\lambda-\lambda_0)t}|\left|\frac{u(y)}{u_0(y)} - \frac{u(x)}{u_0(x)}\right| \leq C'e^{-(\mu_1-\mu_0)\eta^2 t}\big(\frac{w_1}{w_0}\big)(\eta\tfrac{|y-x|}{2} + s_0),$$

for some constant C' and any $t \geq 0$. Therefore $\mathrm{Re}(\lambda) - \lambda_0 \geq \eta^2(\mu_1 - \mu_0) > \frac{1}{4}(\mu_1 - \mu_0)$. $\square$

3. Special case: The ϕ-Laplacian

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth strictly convex boundary and let ϕ and c be smooth functions of $\bar{\Omega}$. Consider the eigenvalue problem with Dirichlet boundary conditions on Ω:

$$\Delta u - \nabla\phi \cdot \nabla u - cu = -\lambda u. \tag{3.1}$$

Introduce the ϕ-Laplacian $\Delta_\phi = \Delta - \nabla\phi \cdot \nabla$, also called the Bakry-Emery Laplacian, to rewrite (3.1) as:

$$\Delta_\phi u - cu = -\lambda u. \tag{3.2}$$

with $u = 0$ on $\partial\Omega$. The operator $\Delta_\phi - c$ is not L^2-symmetric with respect to the euclidean volume form dv however introducing the weighted volume form $e^{-\phi}dv$ it is easy to show that this operator is symmetric with respect to the L^2 inner product with Dirichlet boundary conditions. Hence from L^2 elliptic theory [E] the eigenvalues of (3.1) are real.

Comparing with the eigenvalue problem $L(u) = -\lambda u$ with L as in (2.1) we have:

$$b^i = \nabla_{x_i}\phi.$$

Hence,

$$U^{ij} = \nabla_{x_j} b^i - \nabla_{x_i} b^j = 0.$$

It follows that Proposition 2.2 is not needed and in Theorem 0.1 the parameter σ depends only on $||\phi||_{C^4(\bar\Omega)}$ and $||c||_{C^2(\bar\Omega)}$ and does not depend on K or on the geometry of u_0. Moreover,

$$V^j = \nabla_{x_j} c + \tfrac{1}{4}\nabla_{x_j}(|B|^2) - \tfrac{1}{2}\Delta b^j = \nabla_{x_j}\left(c + \tfrac{1}{4}|\nabla\phi|^2 - \tfrac{1}{2}\Delta\phi\right)$$

Definition 3.1. A function c is called ϕ-convex if the function $c - \frac{1}{2}\Delta\phi + \frac{1}{4}|\nabla\phi|^2 = c - \frac{1}{2}\Delta_{\frac{1}{2}\phi}\phi$ is convex in the usual sense. In particular, if

$$(V(y) - V(x)) \cdot \frac{y-x}{|y-x|} \geq 0$$

In the notation of the previous section this is equivalent to $\tau(s) = 0$.

Theorem 3.1. *If ϕ is any C^4 function and g is ϕ-convex then an associated Sturm-Liouville problem to the eigenvalue problem (3.1) is:*

$$w'' + \mu w = 0$$

on $[-\frac{D}{2}, \frac{D}{2}]$ with $w(-\frac{D}{2}) = w(\frac{D}{2}) = 0$. Hence the spectral gap for the operator $\Delta_\phi - c$ on a convex domain Ω satisfies:

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}.$$

In the case that ϕ is a constant this is the result of [AC].

4. **General remarks**

It is well known that, for example, the Schrödingier operator with a double well potential on $\mathbb{R}^n$ does not satisfy a uniform non zero gap between the first and second eigenvalues. Harrell [H] gives a family of such operators with a separation R between the pairs of wells. As $R \to \infty$ the eigenvalue gap goes to zero. Of course, theses examples do not apply to a bounded domain. However, we note that the Sturm-Liouville problem used above (2.42) and (2.43) has the property that as $\sigma \to \infty$ the eigenvalue gap of the the Sturm-Liouville problem goes to zero. This show that the method used here does not yield useful results without suitable bounds. It suggests, though does not prove, that as the constant $\Lambda \to \infty$ the eigenvalue gap of the Dirichlet problem for the operator (2.1) also goes to zero.

The Sturm-Liouville problem (2.42) and (2.43), while natural for the problem, is somewhat arbitrary. It is not difficult to find other potential functions whose eigenfunctions yield solutions to the differential inequalities (2.14) and (2.15). The choice of F_σ in (2.42) was made because, since F_σ is even, the unique zero of the second eigenfunction w_2 is at the origin. Control of the location of this zero is necessary to complete the proof. It is likely that other choices of potential functions can also determine associated Sturm-Liouville problems. Perhaps some of these problems give better spectral gap results.

References

[AC] Andrews, B., and Clutterbuck, J., Proof of the fundamental gap conjecture, Journal AMS **24** 2011(899-916).

[BR] Birkhoff, G., and Rota, G-C., Ordinary Differential Equations, 3rd edition, John Wiley and Sons, New York, (1978).

[E] Evans, Lawrence, Partial Differential Equations, Graduate Studies in Math. Vol. 19, AMS (1998).

[H] Harrel, Evans M., Double Wells, Commun. Math. Phys., **75**, 1980 (239-261).

[N1] Ni, Lei., Estimates on the modulus of expansion for vector fields solving nonlinear equations, to appear in Jour. Math. Pure Appl.

[N2] Ni, Lei., A Perron type theorem on the principal eigenvalue for nonsymmetric elliptic operators, preprint.

[SWYY] Singer, I. M., Wong, Bun, Yau, Shing-Tung and Yau, Stephen S.-T. , An estimate of the gap of the first two eigenvalues in the Schrödinger operator, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 2, 319333.

[Y] Yau, Shing-Tung, An estimate of the gap of the first two eigenvalues in the Schrödinger operator, Lectures on partial differential equations, New Stud. Adv. Math., vol. 2, Int. Press, Somerville, MA, 2003, 223235

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