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Calculus of Variations

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A new approach to some nonlinear geometric equations in dimension two

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Abstract We give new arguments for several Liouville type results related to the equation $-\Delta u = Ke^{2u}$. The new approach is based on the holomorphic function associated with any solution, which plays a similar role as the Hopf differential for harmonic maps from a surface.

Keywords Liouville type results · Constant curvature surfaces · Constant geodesic curvature

1 Introduction

The elliptic equation $-\Delta u = Ke^{2u}$ on a domain in $\mathbb{R}^2$ and its various ramifications have been studied intensively from both analytic and geometric viewpoints. In the important special case when K is constant (with certain special boundary condition if there is a boundary), the equation has some canonical solutions which are derived by elementary geometric considerations. It is often a fundamental question whether these are all the solutions satisfying certain analytic conditions. Many results of this type have been established, mostly by the method of moving planes. In this approach the main step is to prove that a solution in question must be rotational symmetric with a proper choice of the origin. This usually involves some delicate analysis. With this done the problem is then reduced to an ODE.

In this note we present a new approach to this type of results. It is based on a simple observation that a solution gives rise to a holomorphic function which is zero if and only if the solution is canonical. It is reminiscent of the Hopf differential for harmonic maps from a surface. The required analysis to show that this

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holomorphic function is indeed zero under some analytic assumption seems to be simpler and more transparent. It is our hope that this new approach may also be of use in some other situations.

2 Classification of solutions of $-\Delta u = e^{2u}$

Let S be the south pole of S^2 and $\pi_{\mathrm{S}} : S^2 \backslash \{\mathrm{S}\} \to \mathbb{R}^2$ be the stereographic projection, then

$$\left(\pi_{\mathrm{S}}^{-1}\right)^* g_{S^2} = \frac{4}{(1+|x|^2)^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2).$$

For $\lambda > 0$ and $\xi \in \mathbb{R}^2$, let $d_\lambda x = \lambda x$ be the dilation and $\tau_{-\xi} x = x - \xi$ be the translation, then

$$\tau_{-\xi}^* d_\lambda^* \left(\pi_{\mathrm{S}}^{-1}\right)^* g_{S^2} = \frac{4\lambda^2}{(1+\lambda^2|x-\xi|^2)^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) = e^{2u_{\lambda,\xi}} g_{\mathbb{R}^2},$$

here

$$u_{\lambda,\xi}(x) = \log \frac{2\lambda}{1+\lambda^2 |x-\xi|^2} \quad \text{for } x \in \mathbb{R}^2.$$

Since the metric $e^{2u_{\lambda,\xi}} g_{\mathbb{R}^2}$ has curvature 1 and total area 4π, we see

$$-\Delta u_{\lambda,\xi} = e^{2u_{\lambda,\xi}} \text{ on } \mathbb{R}^2 \quad \text{and} \quad \int_{\mathbb{R}^2} e^{2u_{\lambda,\xi}} dx = 4\pi.$$

On the other hand, the following interesting statement was discovered in [3] through the method of moving planes.

Theorem 2.1 **([3])** *Let u be a smooth function on $\mathbb{R}^2$ such that*

$$\begin{cases} -\Delta u = e^{2u} \text{ on } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{2u} dx < \infty. \end{cases} \tag{2.1}$$

Then for some $\lambda > 0$ and $\xi \in \mathbb{R}^2$,

$$u(x) = \log \frac{2\lambda}{1+\lambda^2 |x-\xi|^2}.$$

Remark 2.1 It was known that the condition $\int_{\mathbb{R}^2} e^{2u} dx < \infty$ can not be discarded because if we take any nonpolynomial entire function f such that f' never vanishes (e.g. $f(z) = e^z$), then the metric $f^*(\pi_{\mathrm{S}}^{-1})^* g_{S^2} = \frac{4|f'|^2}{(1+|f|^2)^2} g_{\mathbb{R}^2}$ has constant curvature 1 and infinite total area. If we let $u = \log \frac{2|f'|}{1+|f|^2}$, then $-\Delta u = e^{2u}$ and $\int_{\mathbb{R}^2} e^{2u} dx = \infty$. We note that there are other proofs of the above theorem in [4] by complex analysis and in [2] by the isoperimetric inequality and Pohozaev identities.

We will present a new approach to this result and other related problems. The new method will be local in nature. The starting point is the following simple observation. For convenience, we identify $\mathbb{R}^2 = \mathbb{C}$ by $z = x_1 + ix_2$.

Lemma 2.1 *Let u be smooth solution of the equation $-\Delta u = e^{2u}$ on a domain in $\mathbb{R}^2$. Then $u_{zz} - u_z^2$ is a holomorphic function.*

Proof First we observe that u satisfies $-4u_{z\bar{z}} = e^{2u}$, hence $-4u_{zz\bar{z}} = 2e^{2u}u_z = -8u_z u_{z\bar{z}}$. This implies $(u_{zz} - u_z^2)_{\bar{z}} = 0$ i.e. $u_{zz} - u_z^2$ is a holomorphic function. □

Remark 2.2 It is worth pointing out that the function $u_{zz} - u_z^2$ is crucial because as we will see soon that the solution u is of the canonical form if and only if $u_{zz} - u_z^2$ vanishes. In this sense the holomorphic function plays a similar role as the Hopf differentials for harmonic maps from surfaces: a harmonic map from a surface gives rise to a holomorphic quadratic differential called the Hopf differential and the harmonic map is conformal if and only if its Hopf differential vanishes (see [8, p.6]).

Next we recall the following standard fact which follows from scaling and elliptic estimate. For reader's convenience, we present the proof here.

Lemma 2.2 *Assume φ is a nonnegative smooth function on $\bar{B}_r \subset \mathbb{R}^2$ such that $-\Delta\varphi \le \varphi^2$. Then there exists a universal constant $\eta_0 > 0$ such that $\int_{B_r} \varphi dx \le \eta_0$ implies $\varphi(x) \le \frac{c}{r^2} \int_{B_r} \varphi$ for $x \in B_{r/2}$. Here c is an absolute constant.*

Proof By scaling we may assume $r = 1$. Put $K = \max_{|x|\le 1}(1 - |x|)^2\varphi(x)$. We claim $K \le 1$. Otherwise $K > 1$, choose $\xi \in B_1$ such that $(1 - |\xi|)^2\varphi(\xi) = K$. Set $\sigma = 1 - |\xi|$, then for $x \in B_{\frac{\sigma}{2}}(\xi)$, $\varphi(x) \le \frac{4K}{\sigma^2}$. Hence $\psi(x) = \frac{\sigma^2}{4K}\varphi(\xi + \frac{\sigma}{2\sqrt{K}}x)$ is well defined on B_1. It satisfies

$$-\Delta\psi \le \psi^2, \quad \psi \le 1 \text{ on } B_1, \quad \psi(0) = \frac{1}{4}, \quad \int_{B_1} \psi = \int_{B_{\frac{\sigma}{2\sqrt{K}}}(\xi)} \varphi(x)dx \le \eta_0.$$

Hence $-\Delta\psi \le \psi$. From elliptic estimate ([5, p.67]), we know $\psi(0) \le c\int_{B_1}\psi \le c\eta_0$. Here c is an absolute constant. Choose η_0 small enough such that $c\eta_0 < \frac{1}{4}$, we get a contradiction. Hence $K \le 1$. On $B_{\frac{3}{4}}$ we have $\varphi(x) \le 16$. $-\Delta\varphi \le 16\varphi$. So again by the elliptic estimate we get $\varphi(x) \le c\int_{B_{\frac{3}{4}}}\varphi \le c\int_{B_1}\varphi \le c\eta_0$ for $x \in B_{1/2}$. □

We now present the proof of Theorem 2.1. Let u be a smooth solution of (2.1). Then by Lemma 2.1 we have an entire function $u_{zz} - u_z^2$. Our goal is to prove it is identically zero. First observe

$$-\Delta e^{2u} = -4e^{2u}|\nabla u|^2 + 2e^{4u} \le 2e^{4u}, \quad \int_{\mathbb{R}^2} e^{2u}dx < \infty.$$

There exists $R > 0$, such that $\int_{\mathbb{R}^n \setminus B_R} e^{2u}dx < \eta_0$, the number given by Lemma 2.2. Then for every $x \in \mathbb{R}^2 \setminus B_{2R}$ we have $\int_{B_{|x|/2}(x)} e^{2u(y)}dy < \eta_0$. It follows from Lemma 2.2 that $e^{2u(x)} \le c|x|^{-2}$. In particular $e^{u(x)} \to 0$ as $|x| \to \infty$, and u is bounded from above (this statement also follows from [1, theorem 2]). Define

$$v(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} (\log|x-y| - \log|y|)\, e^{2u(y)} dy.$$

Then $-\Delta v = e^{2u}$ and $|v(x)| \le c\log(|x|+2)$. Indeed, for $|x| = r \ge 2$,

$$\begin{aligned}|v(x)| &\le c\int_{B_1\cup B_1(x)} (|\log|x-y|| + |\log|y||)\, e^{2u(y)} dy \\ &\quad + c\int_{B_{2r}\backslash(B_1\cup B_1(x))} \left|\log\frac{|x-y|}{|y|}\right| e^{2u(y)} dy \\ &\quad + c\int_{\mathbb{R}^2\backslash B_{2r}} \left|\log\frac{|x-y|}{|y|}\right| e^{2u(y)} dy \\ &\le c\log(|x|+2).\end{aligned}$$

Hence $u - v$ is harmonic and $u - v \le c\log(|x|+2)$. This implies $u - v \equiv$ const by the following standard Liouville type statement.

Lemma 2.3 *Let h be a harmonic function on $\mathbb{R}^2$ such that for some $m > 0$, $h(x) \le c(|x|+1)^m$. Then h must be a polynomial.*

Proof We will show that for some $c > 0$, $|h(x)| \le c(|x|+1)^m$, then it follows from the gradient estimate for harmonic functions that h is a polynomial. Let $r = |x| > 0$, by the mean value inequality, we have

$$\begin{aligned}h(0) &= \frac{1}{4\pi r^2}\int_{B_{2r}} h(y)dy = \frac{1}{4\pi r^2}\int_{B_{2r}} h^+(y)dy - \frac{1}{4\pi r^2}\int_{B_{2r}} h^-(y)dy \\ &\le c(r+1)^m - \frac{1}{4\pi r^2}\int_{B_{2r}} h^-(y)dy.\end{aligned}$$

Hence

$$\int_{B_{2r}} h^-(y)dy \le cr^2(r+1)^m - 4\pi r^2 h(0) \le cr^2(r+1)^m.$$

It follows that

$$\begin{aligned}|h(x)| &\le \frac{1}{\pi r^2}\int_{B_r(x)} h^+(y)dy + \frac{1}{\pi r^2}\int_{B_r(x)} h^-(y)dy \\ &\le c(r+1)^m + \frac{1}{\pi r^2}\int_{B_{2r}} h^-(y)\,dy \\ &\le c(r+1)^m.\end{aligned}$$

□

Since

$$u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} (\log|x-y| - \log|y|)\, e^{2u(y)} dy + c,$$

we see

$$\partial_j u(x) = u_j(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_j - y_j}{|x-y|^2} e^{2u(y)} dy.$$

In particular, by Lebesgue dominated convergence theorem

$$\begin{aligned}|\nabla u(x)| &\le \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{2u(y)}}{|x-y|}dy\\ &= \frac{1}{2\pi}\int_{B_1(x)}\frac{e^{2u(y)}}{|x-y|}dy+\frac{1}{2\pi}\int_{\mathbb{R}^2\backslash B_1(x)}\frac{e^{2u(y)}}{|x-y|}dy\\ &\le c\sup_{y\in B_1(x)}e^{2u(y)}+\frac{1}{2\pi}\int_{\mathbb{R}^2}\chi_{\mathbb{R}^2\backslash B_1(x)}(y)\frac{e^{2u(y)}}{|x-y|}dy\to 0\end{aligned}$$

as $|x|\to\infty$. Differentiate Eq. (2.1), we see $-\Delta u_j = 2e^{2u}u_j$ for $j=1,2$. Hence $|\nabla u_j(x)|\le c|u_j|_{L^3(B_1(x))}\to 0$ as $|x|\to\infty$. Therefore the holomorphic function $u_{zz}-u_z^2$ approaches zero at infinity. It follows from Liouville theorem that

$$u_{zz}-u_z^2=0. \tag{2.2}$$

This together with Eq. (2.1) give us

$$D^2u = du\otimes du-\frac{1}{2}(|du|^2+e^{2u})g_{\mathbb{R}^2}.$$

Let $v=e^{-u}$, then the above equation becomes

$$D^2v=\frac{|dv|^2+1}{2v}g_{\mathbb{R}^2}. \tag{2.3}$$

In particular, we see $v_{12}=0$ and $v_{11}=v_{22}$. This implies $v_{111}=v_{221}=v_{122}=0$. Similarly $v_{222}=0$. It follows that $D^3v=0$ and thus v is a quadratic polynomial. Since $v>0$, it it easy to see from (2.3) that $v(x)=\frac{\lambda}{2}|x-\xi|^2+\frac{1}{2\lambda}$ for some $\lambda>0$ and $\xi\in\mathbb{R}^2$. This shows $u(x)=\log\frac{2\lambda}{1+\lambda^2|x-\xi|^2}$.

Remark 2.3 It is interesting to note that the gradient estimate for the equation $-\Delta u = e^{2u}$ can not be localized. Indeed, for $\varepsilon>0$ small, we let $f_\varepsilon(z)=\varepsilon^2 e^{\frac{z-1}{\varepsilon}}$, then the metric

$$f_\varepsilon^*(\pi_S^{-1})^*g_{S^2}=e^{2u_\varepsilon}g_{\mathbb{R}^2}$$

has constant curvature 1 i.e. $-\Delta u_\varepsilon = e^{2u_\varepsilon}$. Here

$$u_\varepsilon(x)=\log 2\varepsilon+\frac{x_1-1}{\varepsilon}-\log\left(1+\varepsilon^4e^{\frac{2(x_1-1)}{\varepsilon}}\right).$$

In particular,

$$\int_{B_1}e^{2u_\varepsilon(x)}dx\le 4\pi\varepsilon^2\to 0,$$

but

$$|\nabla u_\varepsilon(0)|=\frac{1}{\varepsilon}-\frac{2\varepsilon^3e^{-2/\varepsilon}}{1+\varepsilon^4e^{-2/\varepsilon}}\to\infty \text{ as } \varepsilon\to 0^+.$$

3 Constant positive curvature surface with a boundary of constant geodesic curvature

Our method also works well on a domain with boundary on which the constant geodesic curvature condition is imposed. We first illustrate this by working in the compact case where geometry is clear and analysis is simple.

For $\zeta \in \mathbb{C}$ with $|\zeta| < 1$, we let $\phi_\zeta(z) = \frac{z-\zeta}{1-\bar{\zeta}z}$. Then ϕ_ζ is a biholomorphic map from $\bar{B_1}$ to itself. For $a > 0$, let $d_a z = az$. Then the metric

$$\begin{aligned} g_{\zeta,a} &= \phi_\zeta^* d_a^* \left(\pi_{\mathrm{S}}^{-1}\right)^* g_{S^2} \\ &= \frac{\frac{4(a^2+|\zeta|^2)^2}{a^2(1-|\zeta|^2)^2}}{\left(1+\frac{(a^2+|\zeta|^2)^2}{a^2(1-|\zeta|^2)^2}\left|x-\frac{a^2+1}{a^2+|\zeta|^2}\zeta\right|^2\right)^2}(dx_1\otimes dx_1+dx_2\otimes dx_2) \end{aligned}$$

is isometric to $(\overline{B_{2\arctan a}(\mathrm{N})}, g_{S^2}) = (\overline{B_{\operatorname{arccot}\kappa}(\mathrm{N})}, g_{S^2})$ with $\kappa = \frac{1-a^2}{2a}$. Here the $\overline{B_{2\arctan a}(\mathrm{N})}$ means the geodesic ball on standard S^2.

Theorem 3.1 *Let g be a smooth metric on S_+^2 such that the Gaussian curvature $K = 1$ and the geodesic curvature of S^1 (with respect to inner normal direction) is equal to a constant κ, then (S_+^2, g) is isometric to $(\overline{B_{\operatorname{arccot}\kappa}(\mathrm{N})}, g_{S^2})$. Here $\overline{B_{\operatorname{arccot}\kappa}(\mathrm{N})}$ means the geodesic ball around* N.

Proof By Riemann mapping theorem, we may assume g is conformal to g_{S^2}. Denote $(\pi_{\mathrm{S}}^{-1})^* g = e^{2u} g_{\mathbb{R}^2}$. Then

$$\begin{cases} -\Delta u = e^{2u} \text{ in } B_1, \\ \partial_\nu u = \kappa e^u - 1 \text{ on } \partial B_1. \end{cases}$$

Here ν is the outer normal direction. By Lemma 2.1 we know $u_{zz} - u_z^2$ is a holomorphic function. On the other hand, the boundary condition may be written as

$$z u_z + \bar{z} u_{\bar{z}} = \kappa e^u - 1 \text{ on } \partial B_1.$$

Differentiating along the tangential direction, which may be written as $i(z\partial_z - \bar{z}\partial_{\bar{z}})$, we get

$$z^2 u_{zz} - \bar{z}^2 u_{\bar{z}\bar{z}} + z u_z - \bar{z} u_{\bar{z}} = \kappa e^u \left(z u_z - \bar{z} u_{\bar{z}}\right).$$

Using the boundary condition again, we get

$$z^2\left(u_{zz} - u_z^2\right) = \bar{z}^2\left(u_{\bar{z}\bar{z}} - u_{\bar{z}}^2\right).$$

This means the holomorphic function $z^2(u_{zz} - u_z^2)$ is real on ∂B_1. Hence it must be identically zero. This implies $u_{zz} = u_z^2$. Same argument as in the previous section shows for some $\lambda > 0$ and $\xi \in \mathbb{R}^2$, $u(x) = \log \frac{2\lambda}{1+\lambda^2|x-\xi|^2}$. Using the boundary condition, we get $\kappa = \frac{1-\lambda^2+\lambda^2|\xi|^2}{2\lambda}$. Let a be the positive number such that

$$a^2 + \frac{1-\lambda^2+\lambda^2|\xi|^2}{\lambda} a - 1 = 0,$$

and $\zeta = \frac{\lambda a}{1+\lambda a}\xi$ (note that $\frac{\lambda a|\xi|}{\lambda a+1} = \frac{\lambda-a}{\lambda|\xi|}$, we see $|\zeta| < 1$), then $(\pi_{\mathrm{S}}^{-1})^* g = g_{\zeta,a}$. In particular, (S^2_+, g) is isometric to $(\overline{B_{\operatorname{arccot}\kappa}(\mathrm{N})}, g_{S^2})$. □

4 Half plane case

In this section, we shall present a new proof in the spirit of arguments in Section 2 for a result more general than Theorem 3.1 in the analytical sense. This result under further finiteness condition was first proved in [6] and then strengthened to the present version in [9], all by the method of moving planes.

Let $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 \ge 0\}$. For $a > 0$, let $d_a x = ax$ be the dilation, then

$$d_a^*\left(\pi_{\mathrm{S}}^{-1}\right)^* g_{S^2} = \frac{4a^2}{(1+a^2|x|^2)^2}\left(dx_1 \otimes dx_1 + dx_2 \otimes dx_2\right).$$

For this metric on $\overline{B_1}$, the geodesic curvature of S^1 with respect to inner normal direction is equal to $\kappa = \frac{1-a^2}{2a}$. Let $\phi(z) = \frac{1}{z-i} - \frac{i}{2}$, then ϕ maps $\overline{B_1}\backslash\{i\}$ onto $\mathbb{R}^2_+$. Moreover, for $b > 0$, $c \in \mathbb{R}$, $\tau_c x = (x_1 + c, x_2)$,

$$\begin{aligned}
&\tau_c^* d_b^* (\phi^{-1})^* d_a^*\left(\pi_{\mathrm{S}}^{-1}\right)^* g_{S^2}\\
&= \frac{4\frac{b^2\left(1+a^2\right)^2}{a^2}}{\left(1+\frac{b^2(1+a^2)^2}{a^2}\left|x+\left(c, \frac{1-a^2}{2b(1+a^2)}\right)\right|^2\right)^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2)\\
&= \frac{4\lambda^2}{\left(1+\lambda^2\left|x+\left(c,\frac{\kappa}{\lambda}\right)\right|^2\right)^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) = e^{2u^\lambda_{a,c}} g_{\mathbb{R}^2}.
\end{aligned}$$

Here $\lambda = \frac{b(1+a^2)}{a}$,

$$u^\lambda_{a,c}(x) = \log \frac{2\lambda}{1+\lambda^2\left|x+\left(c,\frac{\kappa}{\lambda}\right)\right|^2}.$$

It is clear that

$$\begin{aligned}
-\Delta u^\lambda_{a,c} &= e^{2u^\lambda_{a,c}} \text{ on } \mathbb{R}^2_+\\
\partial_2 u^\lambda_{a,c}(x_1,0) &= -\kappa e^{u^\lambda_{a,c}(x_1,0)} \text{ for } x_1 \in \mathbb{R},\\
\int_{\mathbb{R}^2_+} e^{2u^\lambda_{a,c}(x)}dx &= \frac{4\pi a^2}{1+a^2},\\
\int_{\mathbb{R}} e^{u^\lambda_{a,c}(t,0)}dt &= \frac{4\pi a}{1+a^2}.
\end{aligned}$$

On the other hand, the following boundary version of Theorem 2.1 was proved in [6, 9].

Theorem 4.1 ([6, 9]) *Assume* $u \in C^{\infty}(\mathbb{R}^2_+)$ *such that*

$$\begin{cases} -\Delta u = e^{2u} \text{ on } \mathbb{R}^2_+ \\ \partial_2 u\,(x_1, 0) = -\kappa e^{u(x_1,0)} \quad \text{for } x_1 \in \mathbb{R}, \\ \int_{\mathbb{R}^2_+} e^{2u(x)} dx < \infty, \end{cases}$$

here κ *is a constant, then for some* $\lambda > 0$ *and* $\xi \in \mathbb{R}^2$,

$$u(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - \xi|^2},$$

moreover,

$$\lambda \xi_2 = -\kappa.$$

We will present an argument using the idea from Section 2. Denote $B_r^+ = \{x \in B_r : x_2 > 0\}$. For $x \in \mathbb{R}^2$, let $\overline{x} = (x_1, -x_2)$. To get some control on u, we need the following boundary version of Lemma 2.1.

Lemma 4.1 *Let* φ *be a nonnegative smooth function on* $\overline{B_r^+} \subset \mathbb{R}^2$ *such that*

$$\begin{cases} -\Delta \varphi \le \varphi^2 \text{ in } B_r^+, \\ |\partial_2 \varphi\,(t, 0)| \le \varphi\,(t, 0)^{3/2} \quad \text{for } |t| < r, \end{cases}$$

then there exists a universal constant $\eta_0 > 0$ *such that* $\int_{B_r^+} \varphi dx \le \eta_0$ *implies*

$$\varphi(x) \le \frac{c}{r^2} \int_{B_r^+} \varphi \quad \text{for } x \in B_{r/2}^+.$$

Here c *is an absolute constant.*

Proof To reach the above estimate, we need the following claim.

Claim 4.1 For any $\varepsilon > 0$, there exists a $\eta = \eta(\varepsilon) > 0$ such that if φ is a smooth nonnegative function on $\overline{B_1^+}$ with

$$\begin{cases} -\Delta \varphi \le \varphi^2 \text{ in } B_1^+, \\ |\partial_2 \varphi\,(t, 0)| \le \varphi\,(t, 0)^{3/2} \text{ for } |t| < 1, \end{cases}$$

and $\int_{B_1^+} \varphi dx \le \eta$, then $\max_{x \in \overline{B_1^+}} (1 - |x|)^2 \varphi(x) \le \varepsilon$.

Proof of the claim Let $K = \max_{x \in \overline{B_1^+}} (1 - |x|)^2 \varphi(x)$. If $K > \varepsilon$, then choose $\xi \in \overline{B_1^+}$ such that $(1 - |\xi|)^2 \varphi(\xi) = K$. Denote $\sigma = 1 - |\xi|$, then for $x \in B_{\sigma/2}(\xi) \cap B_1^+$, $\varphi(x) \le \frac{4K}{\sigma^2}$. $\psi(x) = \frac{\sigma^2 \varepsilon}{4K} \varphi\left(\xi + \frac{\sigma\sqrt{\varepsilon}}{K} x\right)$ is well defined on $\{x \in B_1 : x_2 \ge -a\}$ for some $a \in [0, 1]$. Moreover,

$$-\Delta \psi \le \psi^2, \quad \psi \le \varepsilon \text{ on } \{x \in B_1 : x_2 > -a\}, \quad \psi\,(0) = \varepsilon/4,$$

and

$$\int_{\{x\in B_1:x_2>-a\}}\psi(x)dx\le\eta,\quad |\partial_2\psi(x)|\le\psi(x)^{3/2}\quad \text{for } x\in B_1, x_2=-a.$$

In particular, $-\Delta\psi\le\psi$. From this we conclude $a<1/4$. Because otherwise, it follows from elliptic estimate ([5, p.67]) that $\psi(0)=\varepsilon/4\le c\int_{B_{1/4}}\psi\le c\eta$ and we get a contradiction by choosing η small enough. Let $\psi_1(x)=\psi(x_1,x_2-a)$ for $x\in B_{1/2}^+$. Define

$$\psi_2(x)=\frac{1}{2\pi}\int_{-1/2}^{1/2}\log((x_1-t)^2+(x_2)^2)\partial_2\psi_1(t,0)dt\quad \text{for } x\in B_{1/2}^+$$

and $\psi_3=\psi_1-\psi_2$. We have

$$\begin{aligned}&|\psi_2(x)|\le c\varepsilon^{3/2},\quad \psi_3\,(0,a)=\frac{\varepsilon}{4}-\psi_2\,(0,a)\,,\\ &-\Delta\psi_3\le\psi_1^2 \text{ in } B_{1/2}^+,\quad \partial_2\psi_3\,(t,0)=0\quad \text{for } |t|\le 1/2.\end{aligned}$$

We may set $\psi_3(x)=\psi_3(\overline{x})$, $\psi_1(x)=\psi_1(\overline{x})$ for $x\in B_{1/2}^-$, then $\psi_3\in C^2(B_{1/2})$ and $-\Delta\psi_3\le\psi_1^2$ on $B_{1/2}$. It follows from elliptic estimate ([5, p.67]) that

$$\psi_3\,(0,a)\le c\int_{B_{1/2}}\psi_3^+(x)dx+c\left(\int_{B_{1/2}}\psi_1^4(x)dx\right)^{1/2}.$$

Hence $\frac{\varepsilon}{4}\le c(\sqrt{\eta}+\varepsilon^{3/2})$. Since ε is small, it follows that $\varepsilon\le c\eta$. We get a contradiction when η is small enough. □

Now we go back to the proof of the lemma. By scaling, we may assume $r=1$. Let $\varepsilon>0$ be a tiny positive number to be determined later. Then we see when $\int_{B_1^+}\varphi(x)dx$ is small enough, $\max_{x\in\overline{B_1^+}}(1-|x|)^2\varphi(x)\le\varepsilon$. In particular, for $x\in B_{3/4}$, $\varphi(x)\le 16\varepsilon$. Let

$$\varphi_1(x)=\frac{1}{2\pi}\int_{-3/4}^{3/4}\log((x_1-t)^2+(x_2)^2)\partial_2\varphi(t,0)dt\quad \text{for } x\in B_{3/4}^+$$

and $\varphi_2=\varphi-\varphi_1$. Then we have

$$|\varphi_1(x)|\le c\varepsilon^{1/2}|\varphi|_{L^\infty(B_1^+)},\quad -\Delta\varphi_2\le\varphi^2,\quad \partial_2\varphi_2\,(t,0)=0\quad \text{for } |t|\le 3/4.$$

For $x\in B_{3/4}^-$, let $\varphi_2(x)=\varphi_2(\overline{x})$, $\varphi(x)=\varphi(\overline{x})$, then $\varphi_2\in C^2(B_{3/4})$ and $-\Delta\varphi_2\le\varphi^2$ on $B_{3/4}$. It follows from elliptic estimate ([5, p. 67]) that

$$|\varphi_2^+|_{L^\infty(B_{1/2})}\le c\int_{B_{3/4}}\varphi_2^+(x)dx+c\left(\int_{B_{3/4}}\varphi^4(x)dx\right)^{1/2}.$$

Hence

$$|\varphi|_{L^\infty(B_{1/2})} \le c\varepsilon^{1/2} |\varphi|_{L^\infty(B_1)} + c\int_{B_1} \varphi(x)dx \le \frac{1}{2} |\varphi|_{L^\infty(B_1)} + c\int_{B_1} \varphi(x)dx,$$

if ε is small enough. In particular, it follows from scaling argument that under the assumption of the lemma, we have for any $1/2 \le r < s \le 1$,

$$|\varphi|_{L^\infty(B_r^+)} \le \frac{1}{2} |\varphi|_{L^\infty(B_s^+)} + \frac{c}{(s-r)^2}\int_{B_1^+} \varphi(x)dx.$$

It follows from usual iteration procedure ([5, Lemma 4.3 on p. 75]) that

$$|\varphi|_{L^\infty(B_{1/2}^+)} \le c\int_{B_1^+} \varphi(x)dx.\square$$

We will also need the following Liouville type result.

Lemma 4.2 *Let h be a harmonic function on $\mathbb{R}^2$ such that for some positive number m and some positive constant c_1 we have*

$$|h(x)| \begin{cases} \le \frac{c_1}{|x_2|^m} \text{ when } |x_2| \le 1, \\ \le c_1 \text{ when } |x_2| \ge 1. \end{cases}$$

Then h must be a constant function.

Proof Let $p = \frac{1}{m+1}$. It follows from elliptic estimate ([5, p.67]) that for any $\xi \in \mathbb{R}^2$,

$$|h(\xi)| \le c(m)\left(\int_{B_1(\xi)} |h(x)|^p\, dx\right)^{1/p} \le c(m, c_1).$$

Here we have used the fact $pm < 1$. Then it follows from standard Liouville theorem that h is identically constant. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1 Let $\varphi = e^{2u}$, then we know

$$-\Delta\varphi \le 2\varphi^2 \text{ on } \mathbb{R}^2_+, \quad \partial_2\varphi = -2\kappa\varphi^{3/2} \text{ on } \mathbb{R}.$$

Using Lemma 2.1 and Lemma 4.1, we see $e^{2u(x)} = o(|x|^{-2})$ as $|x| \to \infty$. Let

$$\begin{aligned} v(x) &= -\frac{1}{2\pi}\int_{\mathbb{R}^2_+} (\log|x-y| + \log|\overline{x}-y| - 2\log|y|)\, e^{2u(y)}dy, \\ w(x) &= -\frac{\kappa}{\pi}\int_{\mathbb{R}} (\log\sqrt{(t-x_1)^2 + (x_2)^2} - \log|t|)e^{u(t,0)}dt, \\ h &= u - v - w \text{ on } \mathbb{R}^2_+. \end{aligned}$$

Note that w is well defined since $e^{u(t,0)} = o(t^{-1})$ as $|t| \to \infty$. Similar arguments as before give us $|v(x)| \le c \log(|x|+2)$ for $x \in \mathbb{R}^2$. On the other hand, we have $|w(x)| \le c \log^2(|x|+2)$. Indeed, let $|x| = r \ge 2$, then

$$|w(x)| \le c \int_{\mathbb{R}} |\log \sqrt{(t-x_1)^2 + (x_2)^2} - \log |t|| \frac{dt}{|t|+1}.$$

Observing that

$$\int_{|t|\ge 2r} |\log \sqrt{(t-x_1)^2 + (x_2)^2} - \log |t|| \frac{dt}{|t|+1} \le c \int_{|t|\ge 2r} \frac{r\,dt}{t^2} = c,$$

$$\int_{|t|<1} |\log \sqrt{(t-x_1)^2 + (x_2)^2} - \log |t|| \frac{dt}{|t|+1} \le c \log(|x|+2),$$

$$\int_{\sqrt{(t-x_1)^2+(x_2)^2}<1} |\log \sqrt{(t-x_1)^2 + (x_2)^2} - \log |t|| \frac{dt}{|t|+1} \le c$$

and

$$\int_{\{t:1\le|t|<2r,\sqrt{(t-x_1)^2+(x_2)^2}\ge 1\}} |\log \sqrt{(t-x_1)^2 + (x_2)^2} - \log |t|| \frac{dt}{|t|+1} \\ \le c \log^2(|x|+2),$$

we see

$$h(x) \le c \log^2(|x|+2).$$

We may show that h is harmonic on $\mathbb{R}^2_+$ and $\partial_2 h(t,0) = 0$. For $x \in \mathbb{R}^2_-$, let $h(\overline{x}) = h(x)$. Then h becomes a harmonic function on the whole plane with estimate $h(x) \le c \log^2(|x|+2)$. By Lemma 2.2 we see $h(x) = \text{const}$, this implies $u = v + w + c$. Since

$$v_j(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2_+} \left(\frac{x_j - y_j}{|x-y|^2} + \frac{x_j - \overline{y}_j}{|x-\overline{y}|^2} \right) e^{2u(y)} dy \quad \text{for } j = 1, 2,$$

similar arguments as before show that

$$|\nabla v(x)| \le c, \quad |\nabla v(x)| \to 0 \text{ as } |x| \to \infty.$$

On the other hand,

$$w_1(x) = -\frac{\kappa}{\pi} \int_{\mathbb{R}} \frac{x_1 - t}{(x_1-t)^2 + (x_2)^2} e^{u(t,0)} dt,$$

$$w_2(x) = -\frac{\kappa}{\pi} \int_{\mathbb{R}} \frac{x_2}{(x_1-t)^2 + (x_2)^2} e^{u(t,0)} dt,$$

hence

$$
\begin{aligned}
|\nabla w(x)| &\le c\int_{\mathbb{R}} \frac{|x_1-t|+x_2}{(x_1-t)^2+(x_2)^2}\cdot\frac{1}{|t|+1}dt\\
&\le c\int_{\mathbb{R}} \frac{|t|+x_2}{t^2+(x_2)^2}\cdot\frac{1}{|t+x_1|+1}dt\\
&= c\int_{\mathbb{R}} \frac{|t|+1}{t^2+1}\cdot\frac{1}{x_2|t+x_1(x_2)^{-1}|+1}dt\\
&\le c\int_{\mathbb{R}} \frac{1}{|t|+1}\cdot\frac{1}{x_2|t+x_1(x_2)^{-1}|+1}dt.
\end{aligned}
$$

Next we observe the following inequality follows from elementary calculation: for any $b \in \mathbb{R}$,

$$
\int_{\mathbb{R}} \frac{1}{|t|+1}\cdot\frac{1}{a\,|t-b|+1}dt \begin{cases} \le \dfrac{c\log a}{a}, & \text{when } a \ge 2;\\ \le \dfrac{c}{a}, & \text{when } 0 < a \le 2. \end{cases}
$$

Here c is an absolute constant. It follows that

$$
|\nabla w(x)| \begin{cases} \le \dfrac{c\log x_2}{x_2}, & \text{when } x_2 \ge 2;\\ \le \dfrac{c}{x_2}, & \text{when } 0 < x_2 \le 2. \end{cases}
$$

Hence we know $|\nabla u(x)| \le \frac{c}{x_2}$ for $0 < x_2 < 1$, $|\nabla u(x)| \le c$ for $x_2 \ge 1$ and $|\nabla u(x)| \to 0$ as $x_2 \to \infty$. Since for any $j = 1, 2$, $-\Delta u_j = 2e^{2u}u_j$, it follows from elliptic estimate that $|D^2u(x)| \le \frac{c}{(x_2)^2}$ for $0 < x_2 < 1$, $|D^2u(x)| \le c$ for $x_2 \ge 1$ and $|D^2u(x)| \to 0$ as $x_2 \to \infty$.

Let $f(z) = u_{zz} - u_z^2$, then f is holomorphic and $\operatorname{Im} f = \frac{1}{2}(u_1u_2 - u_{12})$. On the other hand, since on $\mathbb{R}$, $u_2 = -\kappa e^u$, we see $u_{12} = -\kappa e^u u_1 = u_1u_2$. This implies f is real on $\mathbb{R}$, we may extend f to a holomorphic function on $\mathbb{C}$ by $f(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{R}^2_-$. Moreover we know $|f(z)| \le c|Imz|^{-2}$ when $|Imz| \le 1$, $|f(z)| \le c$ when $|\operatorname{Im} z| \ge 1$, and $f(z) \to 0$ as $|\operatorname{Im} z| \to \infty$, it follows from Lemma 4.2 that f is identically zero. Theorem 4.1 follows. ☐

5 Nonpositive curvature case

In this section, we shall apply ideas from Section 2 to those results in nonpositive curvature cases derived previously in [7, 9] by complex analysis and moving plane methods.

5.1 Flat surface having boundary of constant geodesic curvature

For $a > 0$, $d_a^* g_{\mathbb{R}^2} = a^2 g_{\mathbb{R}^2}$ is a flat metric on $\overline{B_1}$ and the geodesic curvature of S^1 with respect to inner normal direction is equal to $\kappa = \frac{1}{a}$. Let $\phi(z) = \frac{1}{z-i} - \frac{i}{2}$, then for $b > 0$ and $c \in \mathbb{R}$,

$$\begin{aligned}\tau_c^* d_b^* (\phi^{-1})^* d_a^* g_{\mathbb{R}^2} &= \frac{1}{\left(\frac{b}{a}\right)^2 \left|x + \left(c, \frac{1}{2b}\right)\right|^4} (dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \\ &= \frac{1}{\lambda^2 \left|x + \left(c, \frac{\kappa}{2\lambda}\right)\right|^4} (dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \\ &= e^{2u_{a,c}^\lambda} g_{\mathbb{R}^2}.\end{aligned}$$

Here $\lambda = \frac{b}{a}$,

$$u_{a,c}^\lambda(x) = \log \frac{1}{\lambda \left|x + \left(c, \frac{\kappa}{2\lambda}\right)\right|^2}.$$

It follows that

$$-\Delta u_{a,c}^\lambda = 0 \text{ on } \mathbb{R}_+^2, \quad \partial_2 u\,(x_1, 0) = -\kappa e^{u(x_1,0)} \quad \text{for } x_1 \in \mathbb{R},$$

and

$$\int_{\mathbb{R}_+^2} e^{2u_{a,c}^\lambda(x)} dx = \pi a^2, \quad \int_{\mathbb{R}} e^{2u_{a,c}^\lambda(t,0)} dt = 2\pi a.$$

Theorem 5.2 **([7, 9])** *Assume $u \in C^\infty(\mathbb{R}_+^2)$ such that*

$$\begin{cases} -\Delta u = 0 \text{ on } \mathbb{R}_+^2 \\ \partial_2 u\,(x_1, 0) = -\kappa e^{u(x_1,0)} \text{ for } x_1 \in \mathbb{R}, \\ \int_{\mathbb{R}_+^2} e^{2u(x)} dx < \infty, \end{cases}$$

here κ is a constant, then for some $\lambda > 0$ and $\xi \in \mathbb{R}^2$ with $\xi_2 < 0$ such that

$$u(x) = \log \frac{1}{\lambda |x - \xi|^2},$$

moreover,

$$\kappa = -2\lambda\xi_2 > 0.$$

Indeed, if we let $\varphi = e^{2u}$, then

$$-\Delta\varphi = -4e^{2u}|\nabla u|^2 \leq 0 \text{ on } \mathbb{R}_+^2, \quad \int_{\mathbb{R}_+^2} \varphi\,(x)\,dx < \infty.$$
$$\partial_2\varphi\,(t, 0) = -2\kappa\varphi^{3/2} \text{ for } t \in \mathbb{R},$$

It follows from Lemma 4.1 that $\varphi(x) = e^{2u(x)} = o(|x|^{-2})$ as $|x| \to \infty$. In particular, $e^{u(t,0)} = o(t^{-1})$ as $|t| \to \infty$. Hence we may set

$$w(x) = -\frac{\kappa}{\pi} \int_{\mathbb{R}} (\log \sqrt{(t - x_1)^2 + (x_2)^2} - \log |t|) e^{u(t,0)} dt$$

for $x \in \mathbb{R}_+^2$ and $h = u - w$. Since $|w(x)| \leq c \log^2(|x| + 2)$, we see $h(x) \leq c \log^2(|x|+2)$. Moreover, h is harmonic in $\mathbb{R}_+^2$ and $\partial_2 h(t, 0) = 0$. Let $h(x) = h(\overline{x})$

for $x \in \mathbb{R}^2_-$, then h becomes a harmonic function on $\mathbb{R}^2$ with estimate $h(x) \leq c\log^2(|x|+2)$. It follows from Lemma 2.2 that h must be a constant function. Hence $u = w + c$. Using the representation formula of w, we see

$$|\nabla u(x)| \begin{cases} \leq \dfrac{c\log x_2}{x_2}, & \text{when } x_2 \geq 2; \\ \leq \dfrac{c}{x_2}, & \text{when } 0 < x_2 \leq 2. \end{cases}$$

It follows from estimates for harmonic function that $|D^2u(x)| \leq \frac{c}{(x_2)^2}$ for $0 < x_2 < 1$, $|D^2u(x)| \leq c$ for $x_2 \geq 1$ and $|D^2u(x)| \to 0$ as $x_2 \to \infty$. We may rewrite $-\Delta u = 0$ as $u_{z\bar{z}} = 0$, then u_z is a holomorphic function, this implies $f(z) = u_{zz} - u_z^2$ is also a holomorphic function. From the boundary condition of u, we see f is real on $\mathbb{R}$. Hence it may be extended to a holomorphic function on the whole plane by $f(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{R}^2_-$. It follows from the estimate of f and Lemma 4.2 that $f = 0$. Hence $u_{zz} = u_z^2$. This together with $u_{z\bar{z}} = 0$ imply

$$D^2u = du \otimes du - \frac{|du|^2}{2} g_{\mathbb{R}^2}.$$

If we let $v = e^{-u}$, then the equation becomes

$$D^2v = \frac{|dv|^2}{2v} g_{\mathbb{R}^2}.$$

This clearly implies $D^3v = 0$. Moreover v is either equal to constant function or for some $\lambda > 0$ and $\xi \in \mathbb{R}^2$, $v(x) = \lambda|x-\xi|^2$. In view of the fact $\int_{\mathbb{R}^2_+} e^{2u(x)}dx < \infty$ and $v > 0$ on $\mathbb{R}^2_+$, we see v can not be constant and $\xi_2 < 0$. The theorem follows.

5.2 Constant negative curvature surface with boundary of constant geodesic curvature

The standard model of hyperbolic space (B_1, g_H) is given by

$$g_H = \frac{4}{(1-|x|^2)^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2).$$

For $0 < a < 1$, we know the geodesic curvature of ∂B_a with respect to inner normal direction is equal to $\kappa = \frac{1+a^2}{2a}$. Let $\phi_a(z) = \frac{a}{z-ai} - \frac{i}{2}$, then

$$\left(\phi_a^{-1}\right)^* g_H = \frac{4\frac{(1-a^2)^2}{a^2}}{\left(\frac{(1-a^2)^2}{a^2}\left|x+\left(0,\frac{1+a^2}{2(1-a^2)}\right)\right|^2-1\right)^2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2).$$

For $b > 0$ and $c \in \mathbb{R}$, we have

$$\begin{aligned}\tau_c^* d_b^* (\phi_a^{-1})^* g_H &= \frac{4\frac{b^2(1-a^2)^2}{a^2}}{\left(\frac{b^2(1-a^2)^2}{a^2}\left|x + \left(0, \frac{1+a^2}{2b(1-a^2)}\right)\right|^2 - 1\right)^2} \\ &\quad \times (dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \\ &= \frac{4\lambda^2}{\left(\lambda^2\left|x + \left(c, \frac{\kappa}{\lambda}\right)\right|^2 - 1\right)^2} (dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \\ &= e^{2u_{a,c}^\lambda} g_{\mathbb{R}^2},\end{aligned}$$

here $\lambda = \frac{b(1-a^2)}{a}$,

$$u_{a,c}^\lambda(x) = \log \frac{2\lambda}{\lambda^2\left|x + \left(c, \frac{\kappa}{\lambda}\right)\right|^2 - 1}.$$

Clearly we have

$$-\Delta u_{a,c}^\lambda = -e^{2u_{a,c}^\lambda} \text{ in } \mathbb{R}_+^2, \quad \partial_2 u_{a,c}^\lambda (x_1, 0) = -\kappa e^{u_{a,c}^\lambda (x_1,0)} \quad \text{for } x_1 \in \mathbb{R},$$

and

$$\int_{\mathbb{R}_+^2} e^{2u_{a,c}^\lambda(x)} dx = \frac{4\pi a^2}{1-a^2}, \quad \int_{\mathbb{R}} e^{u(t,0)} dt = \frac{4\pi a}{1-a^2}.$$

Theorem 5.3 **([9])** *Assume $u \in C^\infty(\mathbb{R}_+^2)$ such that*

$$\begin{cases} -\Delta u = -e^{2u} \text{ in } \mathbb{R}_+^2, \\ \partial_2 u (x_1, 0) = -\kappa e^{u(x_1,0)} \quad \text{for } x_1 \in \mathbb{R}, \\ \int_{\mathbb{R}_+^2} e^{2u(x)} dx < \infty, \end{cases}$$

here κ is a constant, then for some $\lambda > 0$, $\xi \in \mathbb{R}^2$,

$$u(x) = \log \frac{2\lambda}{\lambda^2 |x - \xi|^2 - 1}.$$

Moreover,

$$\xi_2 < 0, \quad \lambda\xi_2 < -1, \quad \kappa = -\lambda\xi_2 > 1.$$

Proof We may rewrite $\Delta u = e^{2u}$ as $4u_{z\bar{z}} = e^{2u}$. Differentiate the equation we get $4u_{zz\bar{z}} = 2e^{2u}u_z = 8u_z u_{z\bar{z}} = 4(u_z^2)_{\bar{z}}$, hence $u_{zz} - u_z^2$ is a holomorphic function. We want to show this holomorphic function is identically zero.

To get some control on u, we observe that $-\Delta e^{2u} = -4e^{2u}|\nabla u|^2 - 2e^{4u}$ and $\partial_2 e^{2u}(t, 0) = -2\kappa e^{2u}(t, 0)$ for $t \in \mathbb{R}$, it follows from Lemma 4.1 and the fact $\int_{\mathbb{R}_+^2} e^{2u(x)} dx < \infty$ that $e^{2u(x)} = o(|x|^{-2})$ as $|x| \to \infty$. Denote

$$v(x) = \frac{1}{2\pi} \int_{\mathbb{R}_+^2} (\log |x-y| + \log |\bar{x} - y| - 2\log |y|)\, e^{2u(y)} dy,$$

$$w(x) = -\frac{\kappa}{\pi} \int_{\mathbb{R}} (\log \sqrt{(t-x_1)^2 + (x_2)^2} - \log |t|) e^{u(t,0)} dt.$$

Then we may show

$$|v(x)| \le c\log(|x|+2), \quad |\nabla v(x)| \le c, \quad |\nabla v(x)| \to 0 \text{ as } |x| \to \infty,$$
$$|w(x)| \le c\log^2(|x|+2), \quad |\nabla w(x)| \begin{cases} \le c\dfrac{\log x_2}{x_2}, & \text{when } x_2 \ge 2, \\ \le \dfrac{c}{x_2}, & \text{when } 0 < x_2 \le 2. \end{cases}$$

It follows from Lemma 2.2 that $u = v + w + c$. Hence

$$|\nabla u(x)| \begin{cases} \le c, & \text{when } x_2 \ge 2, \\ \le \dfrac{c}{x_2}, & \text{when } 0 < x_2 \le 2, \end{cases}$$

and $|\nabla u(x)| \to 0$ as $x_2 \to \infty$. Since for $j = 1, 2$, $\Delta u_j = 2e^{2u} u_j$, it follows from elliptic estimate that $|D^2 u(x)| \le \frac{c}{(x_2)^2}$ for $0 < x_2 \le 1$, $|D^2 u(x)| \le c$ for $x_2 \ge 1$ and $|D^2 u(x)| \to 0$ as $x_2 \to \infty$. Let $f(z) = u_{zz} - u_z^2$, then f is holomorphic in $\mathbb{R}^2_+$ and real on $\mathbb{R}$(in view of the boundary condition). It follows from Lemma 4.2 and estimate on u that f must be equal to 0. Combine $4u_{z\bar{z}} = e^{2u}$ with $u_{zz} = u_z^2$, we get

$$D^2 u = du \otimes du + \frac{e^{2u} - |du|^2}{2} g_{\mathbb{R}^2}.$$

Let $v = e^{-u}$, then it becomes

$$D^2 v = \frac{|dv|^2 - 1}{2v} g_{\mathbb{R}^2}.$$

This implies $D^3 v = 0$. Hence v is a quadratic polynomial. Since $v_{11} = v_{22} = \frac{|dv|^2 - 1}{2v}$, $v_{12} = 0$ and $v(x) \to \infty$ as $|x| \to \infty$, we see for some $\lambda > 0$ and $\xi \in \mathbb{R}^2$,

$$v(x) = \frac{\lambda^2 |x - \xi|^2 - 1}{2\lambda}.$$

In view of the fact $v > 0$ on $\mathbb{R}^2_+$, we see $\xi_2 < 0$ and $\lambda\xi_2 < -1$. □

Remark 5.1 It was observed in [9] that the area finiteness condition $\int_{\mathbb{R}^2_+} e^{2u(x)} dx < \infty$ may be replaced by the length finiteness condition $\int_{\mathbb{R}} e^{u(t,0)} dt < \infty$. Indeed, under the length finiteness condition, it follows from maximum principle that $e^{2u(x)} = O(|x|^{-2})$ as $|x| \to \infty$ (see [9, Lemma 2.5]). Hence we have well defined

$$v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} (\log|x - y| + \log|\overline{x} - y| - 2\log|y|)\, e^{2u(y)} dy,$$
$$w(x) = -\frac{\kappa}{\pi} \int_{\mathbb{R}} (\log\sqrt{(t - x_1)^2 + (x_2)^2} - \log|t|) e^{u(t,0)} dt.$$

Elementary calculation shows

$$|v(x)| \le c\log^2(|x|+2), \quad |\nabla v(x)| \to 0 \text{ as } |x| \to \infty,$$

$$|w(x)| \le c\log(|x|+2), \quad |\nabla w(x)| \begin{cases} \le \dfrac{c}{x_2}, & \text{when } x_2 \ge 1/2, \\ \le c\log\dfrac{1}{x_2}, & \text{when } 0 < x_2 \le 1/2. \end{cases}$$

Based on these estimates, one may proceed as in the argument above.

Remark 5.2 As in Remark 2.3, the gradient estimate for the equation $-\Delta u = -e^{2u}$ can not be localized. Indeed, for $\varepsilon > 0$ small, we let $f_\varepsilon(z) = \varepsilon^2 e^{\frac{z-1}{\varepsilon}}$, then the metric

$$f_\varepsilon^* g_H = e^{2u_\varepsilon} g_{\mathbb{R}^2}$$

has constant curvature -1 i.e. $-\Delta u_\varepsilon = -e^{2u_\varepsilon}$. Here

$$u_\varepsilon(x) = \log 2\varepsilon + \frac{x_1-1}{\varepsilon} - \log\left(1-\varepsilon^4 e^{\frac{2(x_1-1)}{\varepsilon}}\right).$$

In particular,

$$\int_{B_1} e^{2u_\varepsilon(x)}dx \le c\varepsilon^2 \to 0,$$

but

$$|\nabla u_\varepsilon(0)| = \frac{1}{\varepsilon} + \frac{2\varepsilon^3 e^{-2/\varepsilon}}{1-\varepsilon^4 e^{-2/\varepsilon}} \to \infty \text{ as } \varepsilon \to 0^+.$$

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