SHARP INTEGRAL INEQUALITIES FOR HARMONIC FUNCTIONS

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Abstract. Motivated by Carleman's proof of isoperimetric inequality in the plane, we study some sharp integral inequalities for harmonic functions on the upper halfspace. We also derive the regularity for nonnegative solutions of the associated integral system and some Liouville type theorems.

Contents

1. Introduction

The classical isoperimetric inequality in the plane states that for any bounded domain with area A and boundary length L we have

$$4\pi A \leq L^2 \tag{1.1}$$

and equality holds if and only if the domain is a disk. Inequality (1.1) remains true for bounded domains in a simply connected surface with nonpositive curvature. Among the proofs of this fact the one due to Carleman [C] is particularly interesting. Indeed, let $\left(M^2, g\right)$ be any simply connected compact surface with boundary and nonpositive curvature, it follows from Riemann mapping theorem that $\left(M^2, g\right)$ is isometric to $\left(\overline{B_1^2}, e^{2w} g_{\mathbb{R}^2}\right)$, here B_1^2 is the two dimensional open unit disk and $g_{\mathbb{R}^2}$ is the Euclidean metric on $\mathbb{R}^2$. It follows from the nonpositivity of curvature that w is a subharmonic function. Let u be the harmonic function on B_1 with the same boundary value as w, then $w \leq u$. In [C] it was proved that for any smooth harmonic function on $\overline{B_1^2}$ we have

$$\int_{B_1} e^{2u} dx \leq \frac{1}{4\pi} \left(\int_{S^1} e^u d\theta \right)^2 \tag{1.2}$$

and equality holds if and only if $u(x) = c$ or $-2\log|x - x_0| + c$ for some $x_0 \in \mathbb{R}^2 \backslash \overline{B}_1$ and constant c. We may ask for natural generalizations to higher dimensions. Without an analog of the Riemann mapping theorem, we may start with a metric $g = \rho^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ on $\overline{B_1^n}$ with nonpositive scalar curvature, here $n \geq 3$, B_1^n is the open unit ball in $\mathbb{R}^n$ and $g_{\mathbb{R}^n}$ is the Euclidean metric on $\mathbb{R}^n$. It follows that ρ is a subharmonic function. Under the metric g the volume of $\overline{B_1}$ is equal to $\int_{B_1} \rho^{\frac{2n}{n-2}} dx$ and the area of ∂B_1 is equal to $\int_{\partial B_1} \rho^{\frac{2(n-1)}{n-2}} dS$. We would like to know whether the inequality

$$\int_{B_1} \rho^{\frac{2n}{n-2}} dx \leq n^{-\frac{n}{n-1}} \omega_n^{-\frac{1}{n-1}} \left(\int_{\partial B_1} \rho^{\frac{2(n-1)}{n-2}} dS \right)^{\frac{n}{n-1}}$$

is still true. Here ω_n is the Euclidean volume of the unit ball in $\mathbb{R}^n$. Since ρ is bounded above by the harmonic function with same boundary value, we only need to know whether the inequality

$$|u|_{L^{\frac{2n}{n-2}}(B_1)} \leq n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} |u|_{L^{\frac{2(n-1)}{n-2}}(\partial B_1)} \tag{1.3}$$

is true for any smooth harmonic function u on $\overline{B_1^n}$. The answer to this problem is affirmative and the inequality may be proved by subcritical approximation (see [HWY]). However, for future purpose it seems helpful to transfer this problem to upper halfspace and derive some Liouville type results. Indeed, assume u is a positive harmonic function on $\overline{B_1}$, let $e_n = (0, \cdots, 0, 1)$ and ϕ be the Mobius transformation given by

$$\phi(x) = \frac{x + \frac{e_n}{2}}{\left|x + \frac{e_n}{2}\right|^2} - e_n.$$

Then $\phi(\mathbb{R}^n_+) = B_1$ and

$$\phi^* g_{\mathbb{R}^n} = \frac{1}{\left|x + \frac{e_n}{2}\right|^4} \sum_{i=1}^n dx_i \otimes dx_i.$$

Here $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. Let

$$v(x) = \frac{1}{\left|x + \frac{e_n}{2}\right|^{n-2}} u\left(\frac{x + \frac{e_n}{2}}{\left|x + \frac{e_n}{2}\right|^2} - e_n \right),$$

then $\phi^* \left(u^{\frac{4}{n-2}} g_{\mathbb{R}^n} \right) = v^{\frac{4}{n-2}} g_{\mathbb{R}^n}$. The inequality (1.3) becomes

$$|v|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} \leq n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} |v|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}. \tag{1.4}$$

Note that since v is the Poission integral of $v|_{\mathbb{R}^{n-1}}$, inequality (1.4) follows from Theorem 1.1 below. To state the results, let us fix some notations. For convenience, we use $x, y, \cdots$ for points in $\mathbb{R}^n$ and $\xi, \zeta, \cdots$ for points in $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$. For $x \in \mathbb{R}^n$, we let $x' = (x_1, \cdots, x_{n-1})$, $x = (x', x_n)$. The Poission kernel for the upper half space is given by (see [S, p61])

$$P(x, \xi) = \frac{2}{n\omega_n} \frac{x_n}{\left(|x' - \xi|^2 + x_n^2\right)^{n/2}} \quad \text{for } x \in \mathbb{R}^n_+,\ \xi \in \mathbb{R}^{n-1}.$$

Given a function f defined on $\mathbb{R}^{n-1}$, let

$$(Pf)(x)=\int_{\mathbb{R}^{n-1}}P(x,\xi)f(\xi)\,d\xi \quad \text{for } x\in\mathbb{R}^n_+.$$

We have the following sharp inequality for P (see Theorem 4.1):

Theorem 1.1. *Assume $n\geq 3$, then for any $f\in L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$,*

$$|Pf|_{L^{\frac{2n}{n-2}}\left(\mathbb{R}^n_+\right)}\leq n^{-\frac{n-2}{2(n-1)}}\omega_n^{-\frac{n-2}{2n(n-1)}}|f|_{L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)}. \tag{1.5}$$

Moreover, equality holds if and only if $f(\xi)=\frac{c}{\left(\lambda^2+|\xi-\xi_0|^2\right)^{\frac{n-2}{2}}}$ for some constant c, positive constant λ and $\xi_0\in\mathbb{R}^{n-1}$.

If we look at the variational problem

$$c_n=\sup\left\{|Pf|_{L^{\frac{2n}{n-2}}\left(\mathbb{R}^n_+\right)}:f\in L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right),|f|_{L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)}=1\right\}. \tag{1.6}$$

Then any nonnegative critical function f, after scaling must satisfy

$$f(\xi)^{\frac{n}{n-2}}=\int_{\mathbb{R}^n_+}P(x,\xi)(Pf)(x)^{\frac{n+2}{n-2}}\,dx. \tag{1.7}$$

We have the following Liouville type theorem (see Proposition 6.1) which is in the same spirit as those in [GNN, CGS, CLO1, L].

Theorem 1.2. *Assume $n\geq 3$, $f\in L_{loc}^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$ is nonnegative, not identically zero and it satisfies (1.7), then for some $\lambda>0$ and $\xi_0\in\mathbb{R}^{n-1}$,*

$$f(\xi)=c(n)\left(\frac{\lambda}{\lambda^2+|\xi-\xi_0|^2}\right)^{\frac{n-2}{2}}.$$

We note that the condition $f\in L_{loc}^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$ can not be dropped since $c(n)|\xi|^{-\frac{n-2}{2}}$ is a singular solution for (1.7). During the process of identifying maximizing functions in Theorem 1.1 and the critical functions in Theorem 1.2, we establish the following interesting fact (see Proposition 4.1):

Proposition 1.1. *Let $n\geq 2$, u be a function on $\mathbb{R}^n$ which is radial with respect to the origin, $0<u(x)<\infty$ for $x\neq 0$, $e_1=(1,0,\cdots,0)$, $\alpha\in\mathbb{R}$, $\alpha\neq 0$. If $v(x)=|x|^{\alpha}u\left(\frac{x}{|x|^2}-e_1\right)$ is radial with respect to some points, then either $u(x)=\left(c_1|x|^2+c_2\right)^{\alpha/2}$ for some $c_1\geq 0$, $c_2>0$ or*

$$u(x)=\begin{cases} c_1|x|^{\alpha}, \text{ if } x\neq 0,\\ c_2, \text{ if } x=0,\end{cases}$$

for some $c_1>0$ and c_2, an arbitrary number.

There are similar statements for the cases $\alpha=0$ or $n=1$ (see Remark 4.1 and Proposition 4.2). The crucial point of Proposition 1.1 is that we do not need any regularity assumption on the function u. This is very convenient when the regularity of extremal functions are hard to get apriorly. The radial symmetry property of function may come from symmetrization arguments or the method of moving planes etc. For example, Proposition 1.1 gives another way to determine the maximizing functions for those cases of Hardy-Littlewood-Sobolev inequalities studied in [Li2, section III]. The formulation of Proposition 1.1 is motivated from previous works in [CL, O], [CLO1, section 3] and [CLO3, section 6]. It is worth pointing out that Proposition 1.1 is the fact for method of moving planes which corresponds to the fact [LZ, lemma 2.5] or [L, lemma 5.8] for the method of moving spheres, a variant of the method of moving planes.

According to Proposition 2.1 below, for $n \geq 2$ and $1 < p < \infty$ the operator

$$P : L^p\left(\mathbb{R}^{n-1}\right) \to L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right) : f \mapsto Pf$$

is always a bounded linear map. From the analytical point view it is interesting to consider the variational problem

$$c_{n,p} = \sup\left\{ |Pf|_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)} : f \in L^p\left(\mathbb{R}^{n-1}\right), |f|_{L^p(\mathbb{R}^{n-1})} = 1 \right\} \tag{1.8}$$

for all such p's. Fix $1 < p < \infty$, for a function f defined on $\mathbb{R}^{n-1}$, $\lambda > 0$ and $\zeta \in \mathbb{R}^{n-1}$, we write

$$f^{\lambda,\zeta}(\xi) = \lambda^{-\frac{n-1}{p}} f\left(\frac{\xi - \zeta}{\lambda}\right) \quad \text{for } \xi \in \mathbb{R}^{n-1}.$$

Then we have (see Theorem 3.1 and Theorem 4.1):

Theorem 1.3. *Given $n \geq 2$ and $1 < p < \infty$.*

- *Let f_i be a maximizing sequence of functions for (1.8), then after passing to a subsequence there exists $\lambda_i > 0$ and $\zeta_i \in \mathbb{R}^{n-1}$ such that $f_i^{\lambda_i,\zeta_i} \to f$ in $L^p\left(\mathbb{R}^{n-1}\right)$. In particular, there exists at least one maximizing function.*
- *After multiplying by a nonzero constant, every maximizer f of (1.8) is non-negative, radial symmetric with respect to some points, strictly decreasing in the radial direction and it satisfies*

$$f(\xi)^{p-1} = \int_{\mathbb{R}^n_+} P(x,\xi)\,(Pf)(x)^{\frac{np}{n-1}-1}\,dx. \tag{1.9}$$

- *If $n \geq 3$ and $p = \frac{2(n-1)}{n-2}$, then any maximizer of (1.8) must be of the form*

$$f(\xi) = \pm c(n)\left(\frac{\lambda}{\lambda^2 + |\xi - \xi_0|^2}\right)^{\frac{n-2}{2}}$$

for some $\lambda > 0$, $\xi_0 \in \mathbb{R}^{n-1}$. In particular $c_{n,\frac{2(n-1)}{n-2}} = n^{-\frac{n-2}{2(n-1)}}\omega_n^{-\frac{n-2}{2n(n-1)}}$.

- *If $n \geq 3$ and $p = \frac{2(n-1)}{n}$, then any maximizer of (1.8) must be of the form*

$$f(\xi) = \pm c(n)\left(\frac{\lambda}{\lambda^2 + |\xi - \xi_0|^2}\right)^{n/2}$$

for some $\lambda > 0$, $\xi_0 \in \mathbb{R}^{n-1}$. In particular

$$c_{n,\frac{2(n-1)}{n}} = \frac{1}{\sqrt{2(n-2)\sqrt[4]{\pi}}}\left(\frac{(n-2)!}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{1}{2(n-1)}}.$$

It is interesting that the problem considered here demonstrates very similar structures to the sharp Hardy-Littlewood-Sobolev inequalities studied in [Li2]. Besides above properties of maximizing functions, we know they are smooth. This is a nontrivial fact since it does not follow from the usual bootstrap method. Indeed, we know all the nonnegative critical functions of (1.8) are smooth and radial symmetric with respect to some points (see Theorem 5.1 and Theorem 6.1). More precisely we have

Theorem 1.4. *Given $n \geq 2$ and $1 < p < \infty$. If $f \in L^p\left(\mathbb{R}^{n-1}\right)$ is nonnegative, not identically zero and it satisfies (1.9), then $f \in C^\infty\left(\mathbb{R}^{n-1}\right)$, moreover it is radial symmetric with respect to some point and strictly decreasing along the radial direction.*

In Section 2 below, we will collect some basic estimates for Poission integrals and show the operator P is bounded in suitable Lebesgue spaces and Lorentz spaces. In Section 3, we apply the general frame of concentration compactness principle ([Lion]) to show that every maximizing sequence of (1.8), after scaling and translation, must converge strongly. In Section 4, following Lieb we use the method of symmetrization based on the Riesz rearrangement inequalities ([LiL, section 3.7]) and its strong form ([Li1]) to show that all maximizing functions must be radial and give another approach to the existence of maximizing functions. In Section 5 we use the method in [Hn] to deduce the regularity of all nonnegative critical functions. Indeed what we will prove is a local regularity result. These results are similar in nature to those proved in [ChL, L]. In Section 6 we use the integral version of the method of moving planes ([GNN]), which was discovered in [CLO1], to deduce the symmetry property of the nonnegative critical functions. Here we will need some ideas from [Hn] again.

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2. Basic inequalities for Poission integrals

The main aim of this section is to derive some basic estimates associated with Poission kernel and harmonic extensions which we will use freely in the future. For $x_0 \in \mathbb{R}^n$ and $r > 0$, we write

$$B_r^n(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \quad B_r^n = B_r^n(0), \quad B_r^+ = B_r^n \cap \mathbb{R}_+^n$$

and $\overline{B_r^n}(x_0)$ to mean the closure of $B_r^n(x_0)$. Assume $n \geq 2$. For $t > 0$, $\xi \in \mathbb{R}^{n-1}$, we write

$$P_t(\xi) = \frac{2}{n\omega_n} \frac{t}{\left(|\xi|^2 + t^2\right)^{n/2}}.$$

Clearly we have

- $P_t(\xi) = \frac{1}{t^{n-1}} P_1\left(\frac{\xi}{t}\right)$.
- $P(x, \xi) = P_{x_n}(x' - \xi)$ for $x \in \mathbb{R}_+^n$, $\xi \in \mathbb{R}^{n-1}$.
- $(Pf)(x) = (P_{x_n} * f)(x')$ for $x \in \mathbb{R}_+^n$.
- $|P_t|_{L^1(\mathbb{R}^{n-1})} = 1$, $|P_t|_{L^\infty(\mathbb{R}^{n-1})} = \frac{2}{n\omega_n} \frac{1}{t^{n-1}}$.

- $|P_t|_{L^p(\mathbb{R}^{n-1})} = c(n,p)\, t^{-\frac{(n-1)(p-1)}{p}}$ for $\frac{n-1}{n} < p \le \infty$.

Recall if X is a measure space, $p > 0$ and u is a measurable function on X, then

$$|u|_{L^p_W(X)} = \sup_{t>0} t\, ||u| > t|^{1/p}.$$

The space $L^p_W(X) = \left\{ u : u \text{ is measurable and } |u|_{L^p_W(X)} < \infty \right\}$. More generally, for any $0 < p < \infty$ and $0 < q \le \infty$, we have the Lorentz norm $|\cdot|_{L^{p,q}(X)}$ and Lorentz space $L^{p,q}(X)$ (see [SW, p188]). $L^p_W(X) = L^{p,\infty}(X)$ is a special case of such spaces.

Proposition 2.1. *We have*

$$|Pf|_{L_W^{\frac{n}{n-1}}\left(\mathbb{R}^n_+\right)} \le c(n)\, |f|_{L^1(\mathbb{R}^{n-1})}$$

and

$$|Pf|_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)} \le c(n,p)\, |f|_{L^p(\mathbb{R}^{n-1})} \tag{2.1}$$

for $1 < p \le \infty$. Moreover for $1 < p < \infty$ we have

$$|Pf|_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)} \le c(n,p)\, |f|_{L^{p,\frac{np}{n-1}}(\mathbb{R}^{n-1})}. \tag{2.2}$$

Proof. We only need to prove the weak type estimate. The strong estimate follows from Marcinkiewicz interpolation theorem (see [SW, p197]) and the basic fact $|Pf|_{L^\infty\left(\mathbb{R}^n_+\right)} \le |f|_{L^\infty(\mathbb{R}^{n-1})}$. To prove the weak type estimate, we may assume $f \ge 0$ and $|f|_{L^1(\mathbb{R}^{n-1})} = 1$. First we observe that $(Pf)(x) \le \frac{c(n)}{x_n^{n-1}}$ for $x \in \mathbb{R}^n_+$ and

$$\int_{x\in\mathbb{R}^n_+, 0<x_n<a} (Pf)(x)\, dx = \int_{\mathbb{R}^{n-1}} d\xi \left(f(\xi) \int_0^a dx_n \int_{\mathbb{R}^{n-1}} P(x,\xi)\, dx' \right) = a$$

for $a > 0$. Hence for $t > 0$,

$$\begin{aligned} |Pf > t| &= \left| \left\{ x \in \mathbb{R}^n_+ : 0 < x_n < c(n)\, t^{-\frac{1}{n-1}}, (Pf)(x) > t \right\} \right| \\ &\le \frac{1}{t} \int_{0<x_n<c(n)t^{-\frac{1}{n-1}}, x'\in\mathbb{R}^{n-1}} (Pf)(x)\, dx = c(n)\, t^{-\frac{n}{n-1}}. \end{aligned}$$

The weak type inequality follows. □

In the future we will also need some elementary estimates for the harmonic extensions. They are listed below without proofs:

- For $1 \le p \le q \le \infty$, we have

$$\begin{aligned} |P_t * f|_{L^p(\mathbb{R}^{n-1})} &\le |f|_{L^p(\mathbb{R}^{n-1})}; \\ |P_t * f|_{L^\infty(\mathbb{R}^{n-1})} &\le c(n,p)\, t^{-\frac{n-1}{p}} |f|_{L^p(\mathbb{R}^{n-1})}; \\ |P_t * f|_{L^q(\mathbb{R}^{n-1})} &\le c(n,p,q)\, t^{-(n-1)\left(\frac{1}{p}-\frac{1}{q}\right)} |f|_{L^p(\mathbb{R}^{n-1})} \end{aligned}$$

  and

$$\begin{aligned} |(Pf)(\cdot, x_n)|_{L^p(\mathbb{R}^{n-1})} &\le |f|_{L^p(\mathbb{R}^{n-1})}; \\ |(Pf)(x)| &\le c(n,p)\, x_n^{-\frac{n-1}{p}} |f|_{L^p(\mathbb{R}^{n-1})}; \\ |(Pf)(\cdot, x_n)|_{L^q(\mathbb{R}^{n-1})} &\le c(n,p,q)\, x_n^{-(n-1)\left(\frac{1}{p}-\frac{1}{q}\right)} |f|_{L^p(\mathbb{R}^{n-1})}. \end{aligned}$$

- Assume $f(\xi)=0$ for $|\xi|\geq R$, then we have

$$\left|(P_t*f)(\xi)\right|\leq\frac{c(n)\,t}{\left[\left((|\xi|-R)^+\right)^2+t^2\right]^{n/2}}|f|_{L^1(\mathbb{R}^{n-1})}$$

and

$$\left|(Pf)(x)\right|\leq\frac{c(n)\,x_n}{\left[\left((|x'|-R)^+\right)^2+x_n^2\right]^{n/2}}|f|_{L^1(\mathbb{R}^{n-1})}.$$

- Assume $f(\xi)=0$ for $|\xi|<R$, $1\leq p\leq\infty$, then we have

$$|P_t*f|_{L^\infty\left(B_{R/2}^{n-1}\right)}\leq c(n,p)\,tR^{-\frac{n-1}{p}-1}|f|_{L^p(\mathbb{R}^{n-1})}$$

and

$$|Pf|_{L^\infty\left(B_{R/2}^+\right)}\leq c(n,p)\,R^{-\frac{n-1}{p}}|f|_{L^p(\mathbb{R}^{n-1})}.$$

For $t>0$, $\xi\in\mathbb{R}^{n-1}$, let

$$Q_t(\xi)=P_t(\xi)\cdot\frac{|\xi|}{t}=\frac{2}{n\omega_n}\frac{|\xi|}{\left(|\xi|^2+t^2\right)^{n/2}},$$

then

- $|Q_t|_{L^p(\mathbb{R}^{n-1})}=c(n,p)\,t^{-\frac{(n-1)(p-1)}{p}}$ for $1<p\leq\infty$.
- Let $\varphi\in L^\infty\left(\mathbb{R}^{n-1}\right)\cap\mathrm{Lip}\left(\mathbb{R}^{n-1}\right)$, then

$$|P_t*(\varphi f)-\varphi(P_t*f)|\leq[\varphi]_{\mathrm{Lip}(\mathbb{R}^{n-1})}\,tQ_t*f.$$

In particular, it follows from Hausdorff-Young's inequality that

$$\begin{aligned}&|P_t*(\varphi f)-\varphi(P_t*f)|_{L^q(\mathbb{R}^{n-1})}\\ \leq\;& c(n,p,q)\,[\varphi]_{\mathrm{Lip}(\mathbb{R}^{n-1})}\,t^{1-(n-1)\left(\frac{1}{p}-\frac{1}{q}\right)}|f|_{L^p(\mathbb{R}^{n-1})}\end{aligned}$$

for $1\leq p<q\leq\infty$.

As a simple application of these estimates, we derive the following compactness result.

Corollary 2.1. *For $1\leq p<\infty$, $1\leq q<\frac{np}{n-1}$, the operator*

$$P:L^p\left(\mathbb{R}^{n-1}\right)\to L^q_{loc}\left(\overline{\mathbb{R}^n_+}\right)$$

is compact.

Proof. Assume $f_i\in L^p\left(\mathbb{R}^{n-1}\right)$ such that $|f_i|_{L^p(\mathbb{R}^{n-1})}\leq1$, it follows that

$$\left|(Pf_i)(x)\right|\leq c(p,n)\,x_n^{-\frac{n-1}{p}}\text{ for }x\in\mathbb{R}^n_+.$$

By the gradient estimates of harmonic functions, after passing to a subsequence we have $Pf_i \to u$ in $C^\infty_{loc}\left(\mathbb{R}^n_+\right)$. For any $R>0$,

$$\begin{aligned} & \left|Pf_i - Pf_j\right|^q_{L^q\left(B^+_R\right)} \\ = & \int_{x\in B^+_R, x_n \geq \varepsilon} \left|Pf_i - Pf_j\right|^q dx + \int_{x\in B^+_R, x_n < \varepsilon} \left|Pf_i - Pf_j\right|^q dx \\ \leq & \int_{x\in B^+_R, x_n \geq \varepsilon} \left|Pf_i - Pf_j\right|^q dx + c\left(p,q,n\right) \int_0^\varepsilon t^{-(n-1)\left(\frac{q}{p}-1\right)} dt \\ \leq & \int_{x\in B^+_R, x_n \geq \varepsilon} \left|Pf_i - Pf_j\right|^q dx + c\left(p,q,n\right) \varepsilon^{1-(n-1)\left(\frac{q}{p}-1\right)}. \end{aligned}$$

Hence

$$\limsup_{i,j\to\infty} \left|Pf_i - Pf_j\right|^q_{L^q\left(B^+_R\right)} \leq c\left(p,q,n\right) \varepsilon^{1-(n-1)\left(\frac{q}{p}-1\right)}.$$

Let $\varepsilon \to 0^+$, we see Pf_i is a Cauchy sequence in $L^q_{loc}\left(\overline{\mathbb{R}^n_+}\right)$. □

Finally we derive a dual statement to Proposition 2.1. Let u be a function on $\mathbb{R}^n_+$, we write

$$\left(Tu\right)\left(\xi\right) = \int_{\mathbb{R}^n_+} P\left(x,\xi\right) u\left(x\right) dx.$$

Proposition 2.2. *For $1 \leq p < n$ we have*

$$\left|Tu\right|_{L^{\frac{(n-1)p}{n-p}}\left(\mathbb{R}^{n-1}\right)} \leq c\left(n,p\right) \left|u\right|_{L^p\left(\mathbb{R}^n_+\right)} \tag{2.3}$$

for any $u \in L^p\left(\mathbb{R}^n_+\right)$.

Proof. We may prove the inequality by a duality argument. Indeed, for any non-negative functions u on $\mathbb{R}^n_+$ and f on $\mathbb{R}^{n-1}$ we have

$$\begin{aligned} 0 & \leq \int_{\mathbb{R}^{n-1}} \left(Tu\right)\left(\xi\right) f\left(\xi\right) d\xi \\ & = \int_{\mathbb{R}^{n-1}} d\xi \int_{\mathbb{R}^n_+} P\left(x,\xi\right) u\left(x\right) f\left(\xi\right) dx \\ & = \int_{\mathbb{R}^n_+} \left(Pf\right)\left(x\right) u\left(x\right) dx \\ & \leq \left|Pf\right|_{L^{\frac{p}{p-1}}\left(\mathbb{R}^n_+\right)} \left|u\right|_{L^p\left(\mathbb{R}^n_+\right)} \\ & \leq c\left(n,p\right) \left|u\right|_{L^p\left(\mathbb{R}^n_+\right)} \left|f\right|_{L^{\frac{(n-1)p}{n(p-1)}}\left(\mathbb{R}^{n-1}\right)}. \end{aligned}$$

Inequality (2.3) follows. We may also prove such an inequality directly. Indeed, since

$$\left|P\left(\cdot,\xi\right)\right|_{L^{\frac{n}{n-1},\infty}\left(\mathbb{R}^n_+\right)} = \left|P\left(\cdot,0\right)\right|_{L^{\frac{n}{n-1},\infty}\left(\mathbb{R}^n_+\right)} = c\left(n\right) < \infty,$$

we see $T : L^{n,1}\left(\mathbb{R}^n_+\right) \to L^\infty\left(\mathbb{R}^{n-1}\right)$ is a bounded linear map. On the other hand, for $u \in L^1\left(\mathbb{R}^n_+\right)$, we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \left|(Tu)(\xi)\right| d\xi &\le \int_{\mathbb{R}^{n-1}} d\xi \int_{\mathbb{R}^n_+} P(x,\xi)\left|u(x)\right| dx \\ &= \int_{\mathbb{R}^n_+} dx \int_{\mathbb{R}^{n-1}} P(x,\xi)\left|u(x)\right| d\xi \\ &= \int_{\mathbb{R}^n_+} \left|u(x)\right| dx.\end{aligned}$$

Hence $T : L^1\left(\mathbb{R}^n_+\right) \to L^1\left(\mathbb{R}^{n-1}\right)$ is also bounded. The inequality (2.3) follows from the Marcinkiewicz interpolation theorem. □

3. The existence of maximizing functions for sharp inequalities by the concentration compactness principle

Assume $n \ge 2$ and $1 < p < \infty$. Let $c_{n,p}$ be the sharp constant in (2.1), then $c_{n,p} > 0$ and

$$c_{n,p}^{\frac{np}{n-1}} = \sup\left\{\int_{\mathbb{R}^n_+} \left|Pf\right|^{\frac{np}{n-1}} dx : f \in L^p\left(\mathbb{R}^{n-1}\right), \left|f\right|_{L^p(\mathbb{R}^{n-1})} = 1\right\}. \tag{3.1}$$

The aim of this section is to show $c_{n,p}^{\frac{np}{n-1}}$ is attained by some functions. Let f be a function defined on $\mathbb{R}^{n-1}$. For $\lambda > 0$ and $\zeta \in \mathbb{R}^{n-1}$ we write

$$f^{\lambda,\zeta}(\xi) = \lambda^{-\frac{n-1}{p}} f\left(\frac{\xi-\zeta}{\lambda}\right) \quad \text{for } \xi \in \mathbb{R}^{n-1},$$

then

$$\left|f^{\lambda,\zeta}\right|_{L^p(\mathbb{R}^{n-1})} = \left|f\right|_{L^p(\mathbb{R}^{n-1})}, \quad \left|Pf^{\lambda,\zeta}\right|_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)} = \left|Pf\right|_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)}.$$

In particular the variational problem (3.1) has both translation and dilation invariance. The problem fits in the general frame of concentration compactness principle of [Lion]. We will apply this principle to prove the following result.

Theorem 3.1. *Assume $n \ge 2$ and $1 < p < \infty$. Let f_i be a maximizing sequence of functions, then after passing to a subsequence there exists $\lambda_i > 0$ and $\zeta_i \in \mathbb{R}^{n-1}$ such that $f_i^{\lambda_i,\zeta_i} \to f$ in $L^p\left(\mathbb{R}^{n-1}\right)$. In particular, there exists at least one maximizing function for the variational problem (3.1).*

A basic ingredient in the proof of Theorem 3.1 is the following proposition corresponding to [Lion, lemma 2.1].

Proposition 3.1. *Assume $n \ge 2$, $1 < p < \infty$ and $f_i \in L^p\left(\mathbb{R}^{n-1}\right)$ such that $f_i \rightharpoonup f$ in $L^p\left(\mathbb{R}^{n-1}\right)$. After passing to a subsequence, assume*

$$\left|f_i\right|^p d\xi \rightharpoonup \mu \text{ in } \mathcal{M}\left(\mathbb{R}^{n-1}\right), \quad \left|Pf_i\right|^{\frac{np}{n-1}} dx \rightharpoonup \nu \text{ in } \mathcal{M}\left(\overline{\mathbb{R}^n_+}\right).$$

Here $\mathcal{M}\left(\mathbb{R}^{n-1}\right)$ denotes the space of all Radon measures on $\mathbb{R}^{n-1}$. Then we have

- $$\nu|_{\mathbb{R}^n_+} = \left|Pf\right|^{\frac{np}{n-1}} dx.$$

Moreover for any Borel set $E \subset \mathbb{R}^{n-1}$,

$$\nu(E)^{\frac{n-1}{np}} \leq c_{n,p}\mu(E)^{\frac{1}{p}}.$$

- *There exists a countable set of points $\zeta_j \in \mathbb{R}^{n-1}$ such that*

$$\nu = |Pf|^{\frac{np}{n-1}} dx + \sum_j \nu_j \delta_{\zeta_j}, \ \mu \geq |f|^p dx + \sum_j \mu_j \delta_{\zeta_j},$$

here $\mu_j = \mu(\{\zeta_j\})$ and

$$\nu_j^{\frac{n-1}{np}} \leq c_{n,p}\mu_j^{\frac{1}{p}}.$$

- *If $\nu(\mathbb{R}^{n-1})^{\frac{n-1}{np}} \geq c_{n,p}\mu(\mathbb{R}^{n-1})^{\frac{1}{p}}$, then ν is supported on a single point.*

Proof. Without losing of generality, we may assume $|f_i|_{L^p(\mathbb{R}^{n-1})} \leq 1$. Since

$$|(Pf_i)(x)| \leq c(n,p)\, x_n^{-\frac{n-1}{p}} \quad \text{for } x \in \mathbb{R}^n_+,$$

it follows from the gradient estimate of harmonic function that $Pf_i \to Pf$ in $C^\infty_{loc}(\mathbb{R}^n_+)$. In particular,

$$\nu|_{\mathbb{R}^n_+} = |Pf|^{\frac{np}{n-1}} dx.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n-1})$ and $\eta \in C_c^\infty([0,\infty))$ such that $0 \leq \eta \leq 1$, we have

$$\begin{aligned}
& |\varphi(x')\eta(x_n)(Pf_i)(x)|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \\
\leq\ & |\eta(x_n)P(\varphi f_i)(x)|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} + |\eta(x_n)(\varphi(x')(Pf_i)(x) - P(\varphi f_i)(x))|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \\
\leq\ & c_{n,p}|\varphi f_i|_{L^p(\mathbb{R}^{n-1})} + c(n,p)|\nabla\varphi|_{L^\infty(\mathbb{R}^{n-1})}\left(\int_0^\infty \eta(t)^{\frac{np}{n-1}} t^{\frac{np}{n-1}-1} dt\right)^{\frac{n-1}{np}}.
\end{aligned}$$

Now fix a $\eta \in C^\infty([0,\infty))$ such that $0 \leq \eta \leq 1$, $\eta(0) = 1$ and $\eta(t) = 0$ for $t \geq 1$. For $\varepsilon > 0$, denote $\eta_\varepsilon(t) = \eta(t/\varepsilon)$. Then

$$\begin{aligned}
& |\varphi(x')\eta_\varepsilon(x_n)(Pf_i)(x)|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \\
\leq\ & c_{n,p}|\varphi f_i|_{L^p(\mathbb{R}^{n-1})} + c(n,p)|\nabla\varphi|_{L^\infty(\mathbb{R}^{n-1})}\varepsilon.
\end{aligned}$$

Letting $i \to \infty$ and then $\varepsilon \to 0^+$, we see

$$\left(\int_{\mathbb{R}^{n-1}} |\varphi|^{\frac{np}{n-1}} d\nu\right)^{\frac{n-1}{np}} \leq c_{n,p}\left(\int_{\mathbb{R}^{n-1}} |\varphi|^p d\mu\right)^{1/p}.$$

A limit process shows for any Borel function h on $\mathbb{R}^{n-1}$,

$$\left(\int_{\mathbb{R}^{n-1}} |h|^{\frac{np}{n-1}} d\nu\right)^{\frac{n-1}{np}} \leq c_{n,p}\left(\int_{\mathbb{R}^{n-1}} |h|^p d\mu\right)^{1/p}.$$

This implies for any Borel set $E \subset \mathbb{R}^{n-1}$, $\nu(E)^{\frac{n-1}{np}} \leq c_{n,p}\mu(E)^{\frac{1}{p}}$. In particular, ν is absolutely continuous with respect to μ. By Radon-Nikydom theorem ([EG, section 1.6]) we have

$$\nu(E) = \int_E g\,d\mu.$$

Moreover for μ a.e. $\xi \in \mathbb{R}^{n-1}$

$$g(\xi) = \lim_{r \to 0^+} \frac{\nu\left(\overline{B_r^{n-1}}(\xi)\right)}{\mu\left(\overline{B_r^{n-1}}(\xi)\right)}.$$

Let $J = \left\{\xi \in \mathbb{R}^{n-1} : \mu(\{\xi\}) > 0\right\}$, then J is countable. Moreover, for $\xi \notin J$, we have

$$g(\xi) \le \liminf_{r \to 0^+} c_{n,p}^{\frac{np}{n-1}} \mu\left(\overline{B_r^{n-1}}(\xi)\right)^{\frac{1}{n-1}} = 0.$$

Hence $\nu = |Pf|^{\frac{np}{n-1}} dx + \sum_j \nu_j \delta_{\zeta_j}$. For the third assertion, if we know $\nu\left(\mathbb{R}^{n-1}\right)^{\frac{n-1}{np}} \ge c_{n,p}\mu\left(\mathbb{R}^{n-1}\right)^{\frac{1}{p}}$, then $\nu\left(\mathbb{R}^{n-1}\right)^{\frac{n-1}{np}} = c_{n,p}\mu\left(\mathbb{R}^{n-1}\right)^{\frac{1}{p}}$. In particular,

$$\left(\sum_j \nu_j\right)^{\frac{n-1}{np}} = c_{n,p}\left(\sum_j \mu_j\right)^{1/p} \ge \left(\sum_j \nu_j^{\frac{n-1}{n}}\right)^{1/p},$$

hence

$$\left(\sum_j \nu_j\right)^{\frac{n-1}{n}} \ge \sum_j \nu_j^{\frac{n-1}{n}}.$$

Since $0 < \frac{n-1}{n} < 1$, we see at most one ν_j is nonzero. $\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. For $r > 0$, let

$$\phi_i(r) = \sup_{\zeta \in \mathbb{R}^{n-1}} \int_{B_r^{n-1}(\zeta)} |f_i|^p \, d\xi.$$

Then $\phi_i : (0, \infty) \to [0, 1]$ is a continuous nondecreasing function with

$$\lim_{r \to 0^+} \phi_i(r) = 0, \quad \lim_{r \to \infty} \phi_i(r) = 1.$$

By introducing dilation factor λ_i and translation by ζ_i, we may assume

$$\phi_i(1) = 1/2 = \int_{B_1} |f_i|^p \, d\xi.$$

After passing to a subsequence, we may find $f \in L^p\left(\mathbb{R}^{n-1}\right)$ such that

$$f_i \rightharpoonup f \text{ in } L^p\left(\mathbb{R}^{n-1}\right), \; |f_i|^p \, d\xi \rightharpoonup \mu \text{ in } \mathcal{M}\left(\mathbb{R}^{n-1}\right), \; |Pf_i|^{\frac{np}{n-1}} dx \rightharpoonup \nu \text{ in } \mathcal{M}\left(\overline{\mathbb{R}^n_+}\right).$$

In particular, this implies $\mu\left(\overline{B_1^{n-1}}\right) \ge 1/2$. We claim $\mu\left(\mathbb{R}^{n-1}\right) = 1$. If not, then $\mu\left(\mathbb{R}^{n-1}\right) = \theta \in (0, 1)$. For $\varepsilon > 0$ small, we claim that after passing to a subsequence, we may find $r_0 > 0$ and a sequence $r_i \to \infty$ such that

$$\theta - \varepsilon < \int_{B_{r_0}^{n-1}} |f_i|^p \, d\xi \le \int_{B_{r_0+2r_i}^{n-1}} |f_i|^p \, d\xi < \theta + \varepsilon.$$

Indeed, fix $r_0 > 0$ such that $\mu\left(B_{r_0}^{n-1}\right) > \theta - \varepsilon$, then for i large enough, we have $\int_{B_{r_0}^{n-1}} |f_i|^p \, d\xi > \theta - \varepsilon$. On the other hand, since $\mu\left(\overline{B_{r_0+2i}^{n-1}}\right) \le \theta < \theta + \varepsilon$, we may inductively define $n_i > i$, $n_{i+1} > n_i$ such that $\int_{B^{n-1}_{r_0+2i}} |f_{n_i}|^p \, dx < \theta + \varepsilon$. Replacing f_i by f_{n_i} we get the needed claim. Let

$$g_i = f_i \chi_{B^{n-1}_{r_0}}, \quad h_i = f_i \chi_{\mathbb{R}^{n-1} \backslash B^{n-1}_{r_0+2r_i}}.$$

Since

$$\int_{B^{n-1}_{r_0+2r_i} \backslash B^{n-1}_{r_0}} |f_i|^p \, d\xi \le 2\varepsilon,$$

we see

$$|f_i - g_i - h_i|_{L^p(\mathbb{R}^{n-1})} \le c(n,p) \, \varepsilon^{1/p}.$$

Note that

$$\begin{aligned} & \left| |Pg_i + Ph_i|^{\frac{np}{n-1}} - |Pg_i|^{\frac{np}{n-1}} - |Ph_i|^{\frac{np}{n-1}} \right| \\ \le \; & c(n,p) \left(|Pg_i|^{\frac{np}{n-1}-1} |Ph_i| + |Pg_i| \, |Ph_i|^{\frac{np}{n-1}-1} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{\mathbb{R}^n_+} |Pg_i|^{\frac{np}{n-1}-1} |Ph_i| \, dx \\ = \; & \int_{B^+_R} |Pg_i|^{\frac{np}{n-1}-1} |Ph_i| \, dx + \int_{\mathbb{R}^n_+ \backslash B^+_R} |Pg_i|^{\frac{np}{n-1}-1} |Ph_i| \, dx \\ \le \; & |Pg_i|^{\frac{np}{n-1}-1}_{L^{\frac{np}{n-1}}(B^+_R)} |Ph_i|_{L^{\frac{np}{n-1}}(B^+_R)} + |Pg_i|^{\frac{np}{n-1}-1}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+ \backslash B^+_R)} |Ph_i|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+ \backslash B^+_R)} \\ \le \; & c(n,p) \, R^{\frac{n-1}{p}} r_i^{-\frac{n-1}{p}} + c(n,p) \, r_0^{\frac{(p-1)(np-n+1)}{p}} \left| \frac{x_n}{\left[\left((|x'| - r_0)^+ \right)^2 + x_n^2 \right]^{n/2}} \right|^{\frac{np}{n-1}-1}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+ \backslash B^+_R)}, \end{aligned}$$

this implies

$$\begin{aligned} & \limsup_{i \to \infty} \int_{\mathbb{R}^n_+} |Pg_i|^{\frac{np}{n-1}-1} |Ph_i| \, dx \\ \le \; & c(n,p) \, r_0^{\frac{(p-1)(np-n+1)}{p}} \left| \frac{x_n}{\left[\left((|x'| - r_0)^+ \right)^2 + x_n^2 \right]^{n/2}} \right|^{\frac{np}{n-1}-1}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+ \backslash B^+_R)}. \end{aligned}$$

Let $R \to \infty$, we see

$$\lim_{i \to \infty} \int_{\mathbb{R}^n_+} |Pg_i|^{\frac{np}{n-1}-1} |Ph_i| \, dx = 0.$$

Similarly,

$$\lim_{i \to \infty} \int_{\mathbb{R}^n_+} |Pg_i| \, |Ph_i|^{\frac{np}{n-1}-1} \, dx = 0.$$

Hence

$$\lim_{i \to \infty} \int_{\mathbb{R}^n_+} \left| |Pg_i + Ph_i|^{\frac{np}{n-1}} - |Pg_i|^{\frac{np}{n-1}} - |Ph_i|^{\frac{np}{n-1}} \right| dx = 0.$$

Since

$$\begin{aligned}\int_{\mathbb{R}_+^n}|Pg_i|^{\frac{np}{n-1}}\,dx &\le c_{n,p}^{\frac{np}{n-1}}\,|g_i|_{L^p(\mathbb{R}^{n-1})}^{\frac{np}{n-1}}\le c_{n,p}^{\frac{np}{n-1}}\,(\theta+\varepsilon)^{\frac{n}{n-1}}\,,\\ \int_{\mathbb{R}_+^n}|Ph_i|^{\frac{np}{n-1}}\,dx &\le c_{n,p}^{\frac{np}{n-1}}\,|h_i|_{L^p(\mathbb{R}^{n-1})}^{\frac{np}{n-1}}\le c_{n,p}^{\frac{np}{n-1}}\,(1-\theta+\varepsilon)^{\frac{n}{n-1}}\,,\end{aligned}$$

we see

$$\begin{aligned}&c_{n,p}^{\frac{np}{n-1}}+o(1)\\ =\;&\int_{\mathbb{R}_+^n}|Pf_i|^{\frac{np}{n-1}}\,dx\\ \le\;&\left(|Pg_i+Ph_i|_{L^{\frac{np}{n-1}}(\mathbb{R}_+^n)}+c(n,p)\,\varepsilon^{1/p}\right)^{\frac{np}{n-1}}\\ \le\;&\int_{\mathbb{R}_+^n}|Pg_i+Ph_i|^{\frac{np}{n-1}}\,dx+c(n,p)\,\varepsilon^{1/p}\\ \le\;&\int_{\mathbb{R}_+^n}\left(|Pg_i|^{\frac{np}{n-1}}+|Ph_i|^{\frac{np}{n-1}}\right)dx+c(n,p)\,\varepsilon^{1/p}+o(1)\\ \le\;&c_{n,p}^{\frac{np}{n-1}}\,(\theta+\varepsilon)^{\frac{n}{n-1}}+c_{n,p}^{\frac{np}{n-1}}\,(1-\theta+\varepsilon)^{\frac{n}{n-1}}+c(n,p)\,\varepsilon^{1/p}+o(1)\,.\end{aligned}$$

Letting $i\to\infty$ and then $\varepsilon\to 0^+$, we see

$$1\le\theta^{\frac{n}{n-1}}+(1-\theta)^{\frac{n}{n-1}}\,.$$

This gives us a contradiction since $\frac{n}{n-1}>1$. Hence $\mu\left(\mathbb{R}^{n-1}\right)=1$. Next we claim $\nu\left(\overline{\mathbb{R}_+^n}\right)=c_{n,p}^{\frac{np}{n-1}}$. Indeed, for any $\varepsilon>0$ small, we may find $r>0$ such that $\mu\left(B_r^{n-1}\right)>1-\varepsilon$, this implies $\int_{B_r^{n-1}}|f_i|^p\,d\xi>1-\varepsilon$ when i is large enough. Hence $\int_{\mathbb{R}^{n-1}\backslash B_r^{n-1}}|f_i|^p\,d\xi\le\varepsilon$. Let $g_i=f_i\chi_{B_{r_0}}$ and $h_i=f_i\chi_{\mathbb{R}^{n-1}\backslash B_{r_0}}$, then

$$\begin{aligned}|Ph_i|_{L^{\frac{np}{n-1}}(\mathbb{R}_+^n)} &\le c(n,p)\,\varepsilon^{1/p},\\ |(Pg_i)(x)| &\le c(n,p)\,r^{\frac{(n-1)(p-1)}{p}}\frac{x_n}{\left[\left((|x'|-r)^+\right)^2+x_n^2\right]^{n/2}}.\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\mathbb{R}^n\backslash B_R^+}|Pf_i|^{\frac{np}{n-1}}\,dx\\ \le\;&c(n,p)\,\varepsilon^{\frac{n}{n-1}}+c(n,p)\,r^{n(p-1)}\left|\frac{x_n}{\left[\left((|x'|-r)^+\right)^2+x_n^2\right]^{n/2}}\right|_{L^{\frac{np}{n-1}}(\mathbb{R}^n\backslash B_R^+)}^{\frac{np}{n-1}}.\end{aligned}$$

Taking a limit for $i \to \infty$, we see

$$\begin{aligned}\nu\left(\overline{\mathbb{R}^n_+}\right) &\geq \nu\left(\overline{B^+_R}\right)\\ &\geq c_{n,p}^{\frac{np}{n-1}} - c(n,p)\,\varepsilon^{\frac{n}{n-1}} - c(n,p)\,r^{n(p-1)}\left|\frac{x_n}{\left[\left(\left(|x'|-r\right)^+\right)^2 + x_n^2\right]^{n/2}}\right|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n\backslash B^+_R\right)}.\end{aligned}$$

let $R \to \infty$ then $\varepsilon \to 0^+$, we see $\nu\left(\overline{\mathbb{R}^n_+}\right) = c_{n,p}^{\frac{np}{n-1}}$.

By the Proposition 3.1 we know there exists a countable set of points $\zeta_j \in \mathbb{R}^{n-1}$ such that

$$\nu = |Pf|^{\frac{np}{n-1}}\,dx + \sum_j \nu_j \delta_{\zeta_j}, \quad \mu \geq |f|^p\,dx + \sum_j \mu_j \delta_{\zeta_j},$$

here $\mu_j = \mu\left(\left\{\zeta_j\right\}\right)$ and

$$\nu_j^{\frac{n-1}{np}} \leq c_{n,p}\mu_j^{\frac{1}{p}}.$$

If $f = 0$, then $\nu\left(\mathbb{R}^{n-1}\right) = c_{n,p}^{\frac{np}{n-1}}$ and hence $\nu\left(\mathbb{R}^{n-1}\right)^{\frac{n-1}{np}} = c_{n,p}\mu\left(\mathbb{R}^{n-1}\right)^{\frac{1}{p}}$. This implies for some $\zeta_1 \in \mathbb{R}^{n-1}$, $\nu = c_{n,p}^{\frac{np}{n-1}}\delta_{\zeta_1}$. In particular, $\mu\left(\{\zeta_1\}\right) \geq 1$ and this implies $\mu = \delta_{\zeta_1}$. But

$$\int_{B_1^{n-1}(\zeta_1)} |f_i|^p\,d\xi \leq 1/2$$

implies $\mu\left(B_1^{n-1}(\zeta_1)\right) \leq 1/2$. This gives us a contradiction. Hence $f \neq 0$. Now

$$c_{n,p}^{\frac{np}{n-1}} = |Pf|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)} + \sum_j \nu_j \leq c_{n,p}^{\frac{np}{n-1}}\,|f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + c_{n,p}^{\frac{np}{n-1}}\sum_j \mu_j^{\frac{n}{n-1}},$$

hence

$$1 \leq |f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + \sum_j \mu_j^{\frac{n}{n-1}}.$$

But since

$$1 \geq |f|^p_{L^p(\mathbb{R}^{n-1})} + \sum_j \mu_j$$

and $\frac{n}{n-1} > 1$, we see $\mu_j = 0$ and $|f|_{L^p(\mathbb{R}^{n-1})} = 1$. This implies $f_i \to f$ in $L^p\left(\mathbb{R}^{n-1}\right)$. $\square$

4. The existence of maximizing functions for sharp inequalities by symmetrization

Following Lieb ([Li2]), using the method of symmetrization we will show all the maximizers of variational problem (3.1) are radial symmetric with respect to some points and we will give another approach to the existence of maximizing functions.

Let u be a measurable function on $\mathbb{R}^n$, the symmetric rearrangement of u is the nonnegative lower semi-continuous radial decreasing function u^* which has the same distribution as u. It satisfies the following important Riesz rearrangement inequality ([LiL, p87]): for any nonnegative measurable functions u, v, w on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u(x)\,v(y-x)\,w(y)\,dy \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u^*(x)\,v^*(y-x)\,w^*(y)\,dy.$$

Using the fact $|w|_{L^p(\mathbb{R}^n)} = |w^*|_{L^p(\mathbb{R}^n)}$ for $p > 0$, we see for $1 \le p \le \infty$,

$$|u * v|_{L^p(\mathbb{R}^n)} \le |u^* * v^*|_{L^p(\mathbb{R}^n)} .$$

Moreover if u is nonnegative radial symmetric and strictly decreasing in the radial direction, v is nonnegative, $1 < p < \infty$ and

$$|u * v|_{L^p(\mathbb{R}^n)} = |u * v^*|_{L^p(\mathbb{R}^n)} < \infty,$$

then for some $x_0 \in \mathbb{R}^n$, we have $v(x) = v^*(x - x_0)$.

Indeed, we may assume v is not identically zero. Choose a nonnegative $w \in L^{p'}(\mathbb{R}^n)$ with $|w|_{L^{p'}(\mathbb{R}^n)} = 1$ such that

$$|u * v|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (u * v)(y) \, w(y) \, dy.$$

Then we have

$$\begin{aligned} |u * v|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u(x) \, v(y - x) \, w(y) \, dy \\ &\le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u(x) \, v^*(y - x) \, w^*(y) \, dy \\ &= \int_{\mathbb{R}^n} (u * v^*)(y) \, w^*(y) \, dy \\ &\le |u * v^*|_{L^p(\mathbb{R}^n)} = |u * v|_{L^p(\mathbb{R}^n)} , \end{aligned}$$

hence

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u(x) \, v(y - x) \, w(y) \, dy = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} u(x) \, v^*(y - x) \, w^*(y) \, dy.$$

It follows from the Lieb's strong version of Riesz rearrangement inequality ([Li1]) that for some $x_0 \in \mathbb{R}^n$, $v(x) = v^*(x - x_0)$.

Theorem 4.1. *Assume $n \ge 2$ and $1 < p < \infty$, then the value*

$$c_{n,p}^{\frac{np}{n-1}} = \sup \left\{ \int_{\mathbb{R}^n_+} |Pf|^{\frac{np}{n-1}} \, dx : f \in L^p\left(\mathbb{R}^{n-1}\right), |f|_{L^p(\mathbb{R}^{n-1})} = 1 \right\},$$

is attained by some functions. After multiplying by a nonzero constant, every maximizer f is nonnegative, radial symmetric with respect to some points, strictly decreasing in the radial direction and it satisfies

$$f(\xi)^{p-1} = \int_{\mathbb{R}^n_+} P(x, \xi) \, (Pf)(x)^{\frac{np}{n-1} - 1} \, dx. \tag{4.1}$$

If $n \ge 3$ and $p = \frac{2(n-1)}{n-2}$, then any maximizer must be of the form

$$f(\xi) = \pm c(n) \left(\frac{\lambda}{\lambda^2 + |\xi - \xi_0|^2} \right)^{\frac{n-2}{2}}$$

for some $\lambda > 0$, $\xi_0 \in \mathbb{R}^{n-1}$. In particular, $c_{n, \frac{2(n-1)}{n-2}} = n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}}$.

If $n \ge 3$ and $p = \frac{2(n-1)}{n}$, then any maximizer must be of the form

$$f(\xi) = \pm c(n) \left(\frac{\lambda}{\lambda^2 + |\xi - \xi_0|^2} \right)^{\frac{n}{2}}$$

for some $\lambda > 0$, $\xi_0 \in \mathbb{R}^{n-1}$. *In particular,* $c_{n,\frac{2(n-1)}{n}} = \frac{1}{\sqrt{2(n-2)\sqrt[4]{\pi}}} \left(\frac{(n-2)!}{\Gamma\left(\frac{n-1}{2}\right)} \right)^{\frac{1}{2(n-1)}}$.

Proof. Assume f_i is a maximizing sequence. Since $|f_i^*|_{L^p(\mathbb{R}^{n-1})} = |f_i|_{L^p(\mathbb{R}^{n-1})} = 1$ and

$$\begin{aligned} |Pf_i|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} &= \int_0^\infty |P_{x_n} * f_i|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})} dx_n \\ &\le \int_0^\infty |P_{x_n} * f_i^*|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})} dx_n = |Pf_i^*|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}, \end{aligned}$$

we see f_i^* is again a maximizing sequence. Hence we may assume f_i is a nonnegative radial decreasing function.

For any $f \in L^p\left(\mathbb{R}^{n-1}\right)$ and any $\lambda > 0$, we let $f^\lambda(\xi) = \lambda^{-\frac{n-1}{p}} f\left(\frac{\xi}{\lambda}\right)$, then it is clear that $\left(Pf^\lambda\right)(x) = \lambda^{-\frac{n-1}{p}} (Pf)\left(\frac{x}{\lambda}\right)$ and hence $\left|f^\lambda\right|_{L^p(\mathbb{R}^{n-1})} = |f|_{L^p(\mathbb{R}^{n-1})}$ and $\left|Pf^\lambda\right|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} = |Pf|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}$. For convenience, denote $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$ and

$$a_i = \sup_{\lambda>0} f_i^\lambda(e_1') = \sup_{\lambda>0} \lambda^{-\frac{n-1}{p}} f_i\left(\frac{e_1'}{\lambda}\right).$$

It follows that

$$0 \le f_i(\xi) \le a_i |\xi|^{-\frac{n-1}{p}}$$

and hence

$$|f_i|_{L^{p,\infty}(\mathbb{R}^{n-1})} \le \omega_{n-1}^{1/p} a_i.$$

Now

$$\begin{aligned} |Pf_i|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} &\le c(n,p) |f_i|_{L^{p,\frac{np}{n-1}}(\mathbb{R}^n_+)} \\ &\le c(n,p) |f_i|^{\frac{n-1}{n}}_{L^p(\mathbb{R}^n_+)} |f_i|^{\frac{1}{n}}_{L^{p,\infty}(\mathbb{R}^n_+)} \\ &\le c(n,p) a_i^{1/n}, \end{aligned}$$

this implies $a_i \ge c(n,p) > 0$. We may choose $\lambda_i > 0$ such that $f_i^{\lambda_i}(e_1') \ge c(n,p) > 0$. Replacing f_i by $f_i^{\lambda_i}$ we may assume $f(e_1') \ge c(n,p) > 0$. On the other hand, since f_i is nonnegative radial decreasing and $|f_i|_{L^p(\mathbb{R}^{n-1})} = 1$, we see

$$|f_i(\xi)| \le \omega_{n-1}^{-1/p} |\xi|^{-(n-1)/p}.$$

Hence after passing to a subsequence, we may find a nonnegative radial decreasing function f such that $f_i \to f$ a.e.. It follows that $f(\xi) \ge c(n,p) > 0$ for $|\xi| \le 1$, $f_i \rightharpoonup f$ in $L^p\left(\mathbb{R}^{n-1}\right)$ and $|f|_{L^p(\mathbb{R}^{n-1})} \le 1$. Since

$$\int_{\mathbb{R}^{n-1}} \left| |f_i|^p - |f|^p - |f_i - f|^p \right| d\xi \to 0,$$

we see

$$\begin{aligned} |f_i - f|^p_{L^p(\mathbb{R}^{n-1})} &= |f_i|^p_{L^p(\mathbb{R}^{n-1})} - |f|^p_{L^p(\mathbb{R}^{n-1})} + o(1) \\ &= 1 - |f|^p_{L^p(\mathbb{R}^{n-1})} + o(1). \end{aligned}$$

On the other hand, since $(Pf_i)(x) \to (Pf)(x)$ for $x \in \mathbb{R}^n_+$ and $|Pf_i|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c_{n,p}$, we see

$$\begin{aligned} |Pf_i|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} &= |Pf|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} + |Pf_i - Pf|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} + o(1) \\ &\leq c^{\frac{np}{n-1}}_{n,p} |f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + c^{\frac{np}{n-1}}_{n,p} |f_i - f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + o(1). \end{aligned}$$

Hence

$$1 \leq |f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + |f_i - f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + o(1).$$

Let $i \to \infty$, we see

$$1 \leq |f|^{\frac{np}{n-1}}_{L^p(\mathbb{R}^{n-1})} + \left(1 - |f|^p_{L^p(\mathbb{R}^{n-1})}\right)^{\frac{n}{n-1}}.$$

Since $\frac{n}{n-1} > 1$ and $f \neq 0$, we see $|f|_{L^p(\mathbb{R}^{n-1})} = 1$. Hence $f_i \to f$ in $L^p(\mathbb{R}^{n-1})$ and f is a maximizer. This implies the existence of an extremal function.

Assume $f \in L^p(\mathbb{R}^{n-1})$ is a maximizer, then so is $|f|$. Hence $|Pf|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} = |P|f||_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}$. On the other hand, since $|(Pf)(x)| \leq P(|f|)(x)$ for $x \in \mathbb{R}^n_+$, we see $|Pf| = P(|f|)$ and this implies either $f \geq 0$ or $f \leq 0$. Assume $f \geq 0$, then the Euler-Lagrange equation is given by

$$\int_{\mathbb{R}^n_+} P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1} dx = c \cdot f(\xi)^{p-1}.$$

Here c is a constant. Using the fact $|f|_{L^p(\mathbf{R}^{n-1})} = 1$, we see

$$c = |Pf|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} = c^{\frac{np}{n-1}}_{n,p}.$$

After scaling by a positive constant we get

$$\int_{\mathbb{R}^n_+} P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1} dx = f(\xi)^{p-1}.$$

On the other hand, we know for $x_n > 0$, $|P_{x_n} * f|_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})} = |P_{x_n} * f^*|_{L^{\frac{np}{n-1}}(\mathbb{R}^{n-1})}$, this implies $f(\xi) = f^*(\xi - \xi_0)$ for some ξ_0. It follows from the Euler-Lagrange equation that f must be strictly decreasing along the radial direction.

For the case when $p = \frac{2(n-1)}{n-2}$, we first observe that if $f \in L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})$, let $u = Pf$, $\widetilde{f}(\xi) = \frac{1}{|\xi|^{n-2}} f\left(\frac{\xi}{|\xi|^2}\right)$ and $\widetilde{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right)$, then we have $\widetilde{u} = P\widetilde{f}$, $\left|\widetilde{f}\right|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})} = |f|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}$ and $|\widetilde{u}|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} = |u|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$. This is the conformal invariance property for the particular power. As a consequence, if f is a maximizer which is nonnegative and radial, then $\frac{1}{|\xi|^{n-2}} f\left(\frac{\xi}{|\xi|^2} - e_1'\right)$ is a maximizer too. In particular, $\frac{1}{|\xi|^{n-2}} f\left(\frac{\xi}{|\xi|^2} - e_1'\right)$ is radial with respect to some points. To find such f, we prove the following facts. □

Proposition 4.1. *Let $n \geq 2$, u be a function on $\mathbb{R}^n$ which is radial with respect to the origin, $0 < u(x) < \infty$ for $x \neq 0$, $e_1 = (1, 0, \cdots, 0)$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$. If*

$v(x) = |x|^{\alpha} u\left(\frac{x}{|x|^2} - e_1\right)$ *is radial with respect to some point, then either* $u(x) = \left(c_1 |x|^2 + c_2\right)^{\alpha/2}$ *for some* $c_1 \geq 0$, $c_2 > 0$ *or*

$$u(x) = \begin{cases} c_1 |x|^{\alpha}, & \text{if } x \neq 0, \\ c_2, & \text{if } x = 0, \end{cases}$$

for some $c_1 > 0$ *and* c_2, *an arbitrary number.*

Proof. First we observe that $\left|\frac{x}{|x|^2} - e_1\right| = 1$ if and only if $x_1 = \frac{1}{2}$. For $r > 0$, $r \neq 1$, we have $\left|\frac{x}{|x|^2} - e_1\right| = r$ if and only if $x \in \partial B_{\frac{r}{|r^2-1|}}\left(\frac{e_1}{1-r^2}\right)$. By scaling, we may assume $u(e_1) = 1$. Then

$$v\left(\frac{1}{2}, x''\right) = \left(\frac{1}{4} + |x''|^2\right)^{\alpha/2} \cdot u(e_1) = \left(\frac{1}{4} + |x''|^2\right)^{\alpha/2}.$$

Assume v is symmetric with respect to $z = (z_1, z'')$. Then $v\left(\frac{1}{2}, \cdot\right)$ is symmetric with respect to z'', hence $z'' = 0$. Denote $z = ae_1$, we claim $0 \leq a \leq 1$. If this is not the case, then we may find a $r > 0$, $r \neq 1$ such that $a = \frac{1}{1-r^2}$. Now on $\partial B_{\frac{r}{|r^2-1|}}\left(\frac{e_1}{1-r^2}\right)$, $v(x) = |x|^{\alpha} u(re_1)$ and it is not a constant function, contradiction. For $x = \left(\frac{1}{2}, x''\right)$, we have

$$v(x) = \left(|x - ae_1|^2 + a - a^2\right)^{\alpha/2}.$$

Hence

$$v(x) = \left(|x - ae_1|^2 + a - a^2\right)^{\alpha/2} = \left(|x|^2 - 2ax_1 + a\right)^{\alpha/2}$$

for $|x - ae_1| \geq \left|\frac{1}{2} - a\right|$. When $a = \frac{1}{2}$, we see $v(x) = \left(|x|^2 - 2ax_1 + a\right)^{\alpha/2}$ for all x. This implies $u(x) = \left(\frac{1}{2}|x|^2 + \frac{1}{2}\right)^{\alpha/2}$. Hence we assume $a \neq \frac{1}{2}$ from now on. Without losing of generality, we assume $0 \leq a < \frac{1}{2}$. We claim that

$$v(x) = \left(|x|^2 - 2ax_1 + a\right)^{\alpha/2} \tag{4.2}$$

for all $x \neq 0$. To see this, we first make the following observation. Assume for some given $r > 0$, $r \neq 1$ and for some $y \in \partial B_{\frac{r}{|r^2-1|}}\left(\frac{e_1}{1-r^2}\right)$, (4.2) is true for y, then it is true for all $x \in \partial B_{\frac{r}{|r^2-1|}}\left(\frac{e_1}{1-r^2}\right)$. Indeed, for x on such a sphere, we have

$$\frac{1 - 2x_1}{|x|^2} = r^2 - 1.$$

Hence

$$\begin{aligned} v(x) &= |x|^{\alpha} u(re_1) = |x|^{\alpha} |y|^{-\alpha} \left(|y|^2 - 2ay_1 + a\right)^{\alpha/2} \\ &= |x|^{\alpha} \left(1 + \frac{a(1 - 2y_1)}{|y|^2}\right)^{\alpha/2} = |x|^{\alpha} \left(1 + \frac{a(1 - 2x_1)}{|x|^2}\right)^{\alpha/2} \\ &= \left(|x|^2 - 2ax_1 + a\right)^{\alpha/2}. \end{aligned}$$

Note that we know (4.2) is true for $x = te_1$ with $t \in \left(-\infty, -\frac{1}{2}\right]$. By the above observation we know it is also true for te_1 with $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$. This implies it is true for te_1 with $t \in \left(-\infty, -\frac{1}{4}\right]$. Go back we see it is true for te_1 with $t \in \left[\frac{1}{6}, \frac{1}{2}\right]$. Keep this procedure going, we see (4.2) is true for all te_1 with $t \neq 0$. Hence it is true for all $x \neq 0$. This implies $u(x) = \left(a|x|^2 + 1 - a\right)^{\alpha/2}$. □

Remark 4.1. *The case when $\alpha = 0$ is a little bit different. However one has: Let $n \geq 2$, u be a function on $\mathbb{R}^n$ which is radial with respect to the origin, $e_1 = (1, 0, \cdots, 0)$. If $v(x) = u\left(\frac{x}{|x|^2} - e_1\right)$ is radial with respect to some points, then either*

$$u(x) = \begin{cases} c_1, \text{ if } x = 0, \\ c_2, \text{ if } x \neq 0, \end{cases}$$

or there exists $r > 0$, $r \neq 1$ such that

$$u(x) = \begin{cases} c_1, & \text{if } |x| < r, \\ c_2, & \text{if } |x| = r, \\ c_3, & \text{if } |x| > r. \end{cases}$$

Here c_i's are arbitrary constants.

Proof of Theorem 4.1 continued. Since $|f|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})} = 1$ and it is strictly decreasing along the radial direction, we see $0 < f(\xi) < \infty$ for $\xi \neq 0$. Note that since f satisfies the Euler-Lagrange equation, it is defined everywhere instead of almost everywhere. It follows from Proposition 4.1 that $f(\xi) = \left(c_1|\xi|^2 + c_2\right)^{-\frac{n-2}{2}}$ for some $c_1, c_2 > 0$ (note that f can not be a constant function and the scalar multiple of $|\xi|^{2-n}$ is ruled out by the integrability). A simple change of variable shows

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} f(\xi)^{\frac{2(n-1)}{n-2}} d\xi &= \int_{\mathbb{R}^{n-1}} \left(c_1|\xi|^2 + c_2\right)^{-(n-1)} d\xi \\ &= (c_1 c_2)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \left(1 + |\xi|^2\right)^{-(n-1)} d\xi. \end{aligned}$$

Since $|f|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})} = 1$, we see $c_1 c_2 = c(n)$. Hence for some $\lambda > 0$, $f(\xi) = c(n)\left(\frac{\lambda}{\lambda^2 + |\xi|^2}\right)^{\frac{n-2}{2}}$. Let $e_n = (0, \cdots, 0, 1)$. Since $u(x) = |x + e_n|^{2-n}$ is a bounded harmonic function on $\overline{\mathbb{R}^n_+}$ and $u(\xi, 0) = \left(1 + |\xi|^2\right)^{-\frac{n-2}{2}}$, we see

$$P\left(\left(1 + |\xi|^2\right)^{-\frac{n-2}{2}}\right)(x) = |x + e_n|^{2-n}.$$

By the dilation invariance, we see

$$\begin{aligned} c_{n, \frac{2(n-1)}{n-2}} &= \frac{\left|P\left(\left(1 + |\xi|^2\right)^{-\frac{n-2}{2}}\right)\right|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}}{\left|\left(1 + |\xi|^2\right)^{-\frac{n-2}{2}}\right|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}} = \frac{\left||x + e_n|^{2-n}\right|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}}{\left|\left(1 + |\xi|^2\right)^{-\frac{n-2}{2}}\right|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}} \\ &= n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}}. \end{aligned}$$

For the case when $p=\frac{2(n-1)}{n}$, we know any maximizer after multiplying by a constant will be nonnegative and satisfy

$$\begin{aligned} f(\xi)^{\frac{n-2}{n}} &= \int_{\mathbb{R}^n_+} P(x,\xi)(Pf)(x)\,dx \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^n_+} P(x,\xi) P(x,\zeta)\,dx \right) f(\zeta)\,d\zeta \\ &= c(n) \int_{\mathbb{R}^{n-1}} \frac{f(\zeta)}{|\xi-\zeta|^{n-2}} d\zeta. \end{aligned}$$

Let $g(\xi)=f(\xi)^{\frac{n-2}{n}}$, then $g \in L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$ and

$$g(\xi)=c(n)\int_{\mathbb{R}^{n-1}} \frac{g(\zeta)^{\frac{n}{n-2}}}{|\xi-\zeta|^{n-2}} d\zeta = c(n)\int_{\mathbb{R}^{n-1}} \frac{g(\zeta)^{\frac{n-1+1}{n-1-1}}}{|\xi-\zeta|^{n-1-1}} d\zeta.$$

It follows from [CLO1, theorem 1] or [L] that for some $\lambda>0$ and $\xi_0 \in \mathbb{R}^{n-1}$, we have

$$g(\xi)=c(n)\left(\frac{\lambda}{\lambda^2+|\xi-\xi_0|^2}\right)^{\frac{n-2}{2}}.$$

Hence

$$f(\xi)=c(n)\left(\frac{\lambda}{\lambda^2+|\xi-\xi_0|^2}\right)^{\frac{n}{2}}.$$

Since $u(x)=\frac{x_n+1}{|x+e_n|^n}$ is a bounded harmonic function on $\overline{\mathbb{R}^n_+}$ and $u(\xi,0)=\left(1+|\xi|^2\right)^{-\frac{n}{2}}$, we see

$$P\left(\left(1+|\xi|^2\right)^{-\frac{n}{2}}\right)(x)=\frac{x_n+1}{|x+e_n|^n}.$$

By the dilation invariance, we see

$$\begin{aligned} c_{n,\frac{2(n-1)}{n}} &= \frac{\left| P\left(\left(1+|\xi|^2\right)^{-\frac{n}{2}}\right)\right|_{L^2\left(\mathbb{R}^n_+\right)}}{\left|\left(1+|\xi|^2\right)^{-\frac{n}{2}}\right|_{L^{\frac{2(n-1)}{n}}\left(\mathbb{R}^{n-1}\right)}} = \frac{\left|\frac{x_n+1}{|x+e_n|^n}\right|_{L^2\left(\mathbb{R}^n_+\right)}}{\left|\left(1+|\xi|^2\right)^{-\frac{n}{2}}\right|_{L^{\frac{2(n-1)}{n}}\left(\mathbb{R}^{n-1}\right)}} \\ &= \frac{1}{\sqrt{2(n-2)}\sqrt[4]{\pi}}\left(\frac{(n-2)!}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{1}{2(n-1)}}. \end{aligned}$$

□

As a final note, we point out the similar statement to Proposition 4.1 in dimension one.

Proposition 4.2. *Assume $u \in C^3(\mathbb{R})$, $u>0$, $\alpha \in \mathbb{R}$ such that for any $y \in \mathbb{R}$, $|x|^\alpha u\left(\frac{1}{x}+y\right)$ is symmetric with respect to some point, then for some $a \geq 0$, $b>0$ and $x_0 \in \mathbb{R}$, we have $u(x)=\left[a(x-x_0)^2+b\right]^{\alpha/2}$.*

Proof. Assume $|x|^\alpha u\left(\frac{1}{x}+y\right)$ is symmetric with respect to $z=z(y)$, then

$$|x|^\alpha u\left(\frac{1}{x}+y\right)=|2z-x|^\alpha u\left(\frac{1}{2z-x}+y\right).$$

Replace x by x^{-1}, we see

$$u(x+y)=|1-2zx|^{\alpha}u\left(y-\frac{x}{1-2zx}\right).$$

Calculation shows

$$\begin{aligned}
&|1-2zx|^{\alpha}u\left(y-\frac{x}{1-2zx}\right)\\
=\ & u(y)-(u'(y)+2\alpha zu(y))x+\left(\frac{u''(y)}{2}+2(\alpha-1)zu'(y)+2\alpha(\alpha-1)z^2u(y)\right)x^2\\
&-\left[\frac{u'''(y)}{6}+(\alpha-2)zu''(y)+2(\alpha-1)(\alpha-2)z^2u'(y)+\frac{4}{3}\alpha(\alpha-1)(\alpha-2)z^3u(y)\right]x^3\\
&+o\left(x^3\right).
\end{aligned}$$

Comparing the Taylor expansion coefficients, we see

$$u'(y)=-\alpha zu(y)$$

and

$$\frac{u'''(y)}{3}+(\alpha-2)zu''(y)+2(\alpha-1)(\alpha-2)z^2u'(y)+\frac{4}{3}\alpha(\alpha-1)(\alpha-2)z^3u(y)=0$$

If $\alpha=0$, then we see $u'=0$ and hence u must be a constant function and we are done. Assume $\alpha\neq 0$, then

$$z=-\frac{u'(y)}{\alpha u(y)}.$$

Plug this in the second equation, we get

$$u^2u'''+3\left(\frac{2}{\alpha}-1\right)uu'u''+\left(\frac{2}{\alpha}-1\right)\left(\frac{2}{\alpha}-2\right)u'^3=0.$$

Hence

$$\left(u^{2/\alpha}\right)'''=\frac{2}{\alpha}u^{\frac{2}{\alpha}-3}\left[u^2u'''+3\left(\frac{2}{\alpha}-1\right)uu'u''+\left(\frac{2}{\alpha}-1\right)\left(\frac{2}{\alpha}-2\right)u'^3\right]=0.$$

The proposition follows. □

5. Regularity of nonnegative critical functions

In this section we will study the regularity issue related to the Euler-Lagrange equation (1.9). Let f be a nonnegative function satisfying (1.9), define $u=Pf$, then the single equation becomes an integral system

$$\begin{aligned}
u(x) &= \int_{\mathbb{R}^{n-1}}P(x,\xi)f(\xi)\,d\xi,\\
f(\xi)^{p-1} &= \int_{\mathbb{R}^n_+}P(x,\xi)u(x)^{\frac{np}{n-1}-1}\,dx.
\end{aligned}$$

This system is very similar to the one appeared in the study of the sharp Hardy-Littlewood-Sobolev inequality ([Li2, part (ii) of theorem 2.3]). In [ChL, L] the regularity problem for some special cases of that system was resolved by a linear approach. In [Hn], a nonlinear approach was introduced to resolve the regularity issue for all the cases. We will apply the nonlinear approach to handle (1.9).

Theorem 5.1. *Assume $n \geq 2$, $1 < p < \infty$, $f \in L^p_{loc}\left(\mathbb{R}^{n-1}\right)$ is nonnegative, not identically zero and it satisfies*

$$f(\xi)^{p-1} = \int_{\mathbb{R}^n_+} P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1}\,dx,$$

then $f \in C^\infty\left(\mathbb{R}^{n-1}\right)$. If we know $f \in L^p\left(\mathbb{R}^{n-1}\right)$, then $f(\xi) \to 0$ as $|\xi| \to \infty$.

We note that the condition $f \in L^p_{loc}\left(\mathbb{R}^{n-1}\right)$ can not be dropped, since the above equation has $c(n,p)|\xi|^{-\frac{n-1}{p}}$ as a singular solution. To prove this theorem, we first derive some local regularity results for some integral inequalities. According to the range of p, we need two local results stated in Proposition 5.1 and Proposition 5.2 below, even though the arguments in both cases are similar. The two local regularity results are of the same nature as [Hn, proposition 2.1] and [L, theorem 1.3].

Proposition 5.1. *Given $n \geq 2$, $1 < a, b \leq \infty$, $1 \leq r < \infty$, $\frac{n}{n-1} < p < q < \infty$ such that*

$$\frac{1}{n} < \frac{r}{q} + \frac{1}{a} < \frac{r}{p} + \frac{1}{a} \leq 1$$

and

$$\frac{n}{ra} + \frac{n-1}{b} = \frac{1}{r}.$$

Denote $B_R = B_R^{n-1}$ and $B_R^+ = B_R^n \cap \mathbb{R}^n_+$. Assume $u, v \in L^p\left(B_R^+\right)$, $U \in L^a\left(B_R^+\right)$, $F \in L^b(B_R)$ are all nonnegative functions with $v|_{B^+_{R/2}} \in L^q\left(B^+_{R/2}\right)$,

$$|U|^{1/r}_{L^a\left(B_R^+\right)} |F|_{L^b(B_R)} \leq \varepsilon(n,p,q,r,a,b) \text{ small}$$

and

$$u(x) \leq \int_{B_R} P(x,\xi) F(\xi) \left[\int_{B_R^+} P(y,\xi) U(y) u(y)^r\,dy\right]^{1/r} d\xi + v(x)$$

for $x \in B_R^+$, then we have $u|_{B^+_{R/4}} \in L^q\left(B^+_{R/4}\right)$ and

$$|u|_{L^q\left(B^+_{R/4}\right)} \leq c(n,p,q,r,a,b)\left(R^{\frac{n}{q}-\frac{n}{p}} |u|_{L^p\left(B_R^+\right)} + |v|_{L^q\left(B^+_{R/2}\right)}\right).$$

Proof. By scaling we may assume $R = 1$. First assume we have $u, v \in L^q\left(B_1^+\right)$. Denote

$$f(\xi) = \int_{B_1^+} P(x,\xi) U(x) u(x)^r\,dx \text{ for } \xi \in B_1.$$

Let p_1 and q_1 be the numbers defined by

$$\frac{n-1}{p_1} = \frac{nr}{p} + \frac{n}{a} - 1, \quad \frac{n-1}{q_1} = \frac{nr}{q} + \frac{n}{a} - 1,$$

then it follows from Proposition 2.2 that

$$\begin{aligned} |f|_{L^{p_1}(B_1)} &\leq c(n,p,r,a)\,|U|_{L^a\left(B_1^+\right)} |u|^r_{L^p\left(B_1^+\right)}, \\ |f|_{L^{q_1}(B_1)} &\leq c(n,q,r,a)\,|U|_{L^a\left(B_1^+\right)} |u|^r_{L^q\left(B_1^+\right)}. \end{aligned}$$

Given $0 < s < t \le 1/2$. For $x \in B_s^+$, we have

$$\begin{aligned}
&u(x)\\
\le\ & \int_{B_{\frac{s+t}{2}}} P(x,\xi) F(\xi) f(\xi)^{1/r} d\xi + \int_{B_1 \setminus B_{\frac{s+t}{2}}} P(x,\xi) F(\xi) f(\xi)^{1/r} d\xi + v(x)\\
\le\ & \int_{B_{\frac{s+t}{2}}} P(x,\xi) F(\xi) f(\xi)^{1/r} d\xi + \frac{c(n)}{(t-s)^{n-1}} \int_{B_1 \setminus B_{\frac{s+t}{2}}} F(\xi) f(\xi)^{1/r} d\xi + v(x)\\
\le\ & \int_{B_{\frac{s+t}{2}}} P(x,\xi) F(\xi) f(\xi)^{1/r} d\xi + \frac{c(n,p)}{(t-s)^{n-1}} |F|_{L^b(B_1)} |f|_{L^{p_1}(B_1)}^{1/r} + v(x)\\
\le\ & \int_{B_{\frac{s+t}{2}}} P(x,\xi) F(\xi) f(\xi)^{1/r} d\xi + \frac{c(n,p,r,a)}{(t-s)^{n-1}} |u|_{L^p(B_1^+)} + v(x).
\end{aligned}$$

Hence

$$|u|_{L^q(B_s^+)} \le c(n,q) |F|_{L^b(B_1)} |f|_{L^{q_1}\left(B_{\frac{s+t}{2}}\right)}^{1/r} + \frac{c(n,p,q,r,a)}{(t-s)^{n-1}} |u|_{L^p(B_1^+)} + |v|_{L^q\left(B_{1/2}^+\right)}.$$

On the other hand, for $\xi \in B_{\frac{s+t}{2}}$, we have

$$\begin{aligned}
f(\xi) &= \int_{B_t^+} P(x,\xi) U(x) u(x)^r dx + \int_{B_1^+ \setminus B_t^+} P(x,\xi) U(x) u(x)^r dx\\
&\le \int_{B_t^+} P(x,\xi) U(x) u(x)^r dx + \frac{c(n)}{(t-s)^{n-1}} \int_{B_1^+ \setminus B_t^+} U(x) u(x)^r dx\\
&\le \int_{B_t^+} P(x,\xi) U(x) u(x)^r dx + \frac{c(n,p,r,a)}{(t-s)^{n-1}} |U|_{L^a(B_1^+)} |u|_{L^p(B_1^+)}^r.
\end{aligned}$$

It follows from Proposition 2.2 that

$$|f|_{L^{q_1}\left(B_{\frac{s+t}{2}}\right)} \le c(n,q,r,a) |U|_{L^a(B_1^+)} |u|_{L^q(B_t^+)}^r + \frac{c(n,p,q,r,a)}{(t-s)^{n-1}} |U|_{L^a(B_1^+)} |u|_{L^p(B_1^+)}^r.$$

Combine the two inequalities together, we see

$$\begin{aligned}
&|u|_{L^q(B_s^+)}\\
\le\ & c(n,q,r,a) |U|_{L^a(B_1^+)}^{1/r} |F|_{L^b(B_1)} |u|_{L^q(B_t^+)} + \frac{c(n,p,q,r,a)}{(t-s)^{n-1}} |u|_{L^p(B_1^+)} + |v|_{L^q\left(B_{1/2}^+\right)}\\
\le\ & \frac{1}{2} |u|_{L^q(B_t^+)} + \frac{c(n,p,q,r,a)}{(t-s)^{n-1}} |u|_{L^p(B_1^+)} + |v|_{L^q\left(B_{1/2}^+\right)}
\end{aligned}$$

if ε is small enough. It follows from the usual iteration procedure ([HL, lemma 4.3 on p.75]) that

$$|u|_{L^q\left(B_{1/4}^+\right)} \le c(n,p,q,r,a) \left(|u|_{L^p(B_1^+)} + |v|_{L^q\left(B_{1/2}^+\right)} \right).$$

To prove the full proposition, we note that there exists a function $0 \le \eta(x) \le 1$ such that

$$u(x) = \eta(x) \int_{B_1} P(x,\xi) F(\xi) \left[\int_{B_1^+} P(y,\xi) U(y) u(y)^r dy \right]^{1/r} d\xi + \eta(x) v(x).$$

We may define a map T by

$$T(\varphi)(x)=\eta(x)\int_{B_1}P(x,\xi)F(\xi)\left[\int_{B_1^+}P(y,\xi)U(y)|\varphi(y)|^r dy\right]^{1/r}d\xi.$$

Note that we have

$$\begin{aligned}|T(\varphi)|_{L^p(B_1^+)} &\leq c(n,p,r,a,b)|U|_{L^a(B_1^+)}^{1/r}|F|_{L^b(B_1)}|\varphi|_{L^p(B_1)}\leq\frac{1}{2}|\varphi|_{L^p(B_1)},\\ |T(\varphi)|_{L^q(B_1^+)} &\leq c(n,q,r,a,b)|U|_{L^a(B_1^+)}^{1/r}|F|_{L^b(B_1)}|\varphi|_{L^q(B_1)}\leq\frac{1}{2}|\varphi|_{L^q(B_1)}\end{aligned}$$

if ε is small enough. Moreover, for $\varphi,\psi\in L^p(B_1^+)$, it follows from Minkowski inequality that

$$|T(\varphi)(x)-T(\psi)(x)|\leq T(|\varphi-\psi|)(x)\text{ for }x\in B_1^+,$$

hence

$$|T(\varphi)-T(\psi)|_{L^p(B_1^+)}\leq|T(|\varphi-\psi|)|_{L^p(B_1^+)}\leq\frac{1}{2}|\varphi-\psi|_{L^p(B_1^+)}.$$

Similarly we have for any $\varphi,\psi\in L^q(B_1^+)$,

$$|T(\varphi)-T(\psi)|_{L^q(B_1^+)}\leq\frac{1}{2}|\varphi-\psi|_{L^q(B_1^+)}.$$

For $k\in\mathbb{N}$, let $v_k(x)=\min\{v(x),k\}$, then it follows from contraction mapping theorem that we may find a unique $u_k\in L^q(B_1^+)$ such that

$$\begin{aligned}&u_k(x)\\ =\;&T(u_k)(x)+\eta(x)v_k(x)\\ =\;&\eta(x)\int_{B_1}P(x,\xi)F(\xi)\left[\int_{B_1^+}P(y,\xi)U(y)|u_k(y)|^r dy\right]^{1/r}d\xi+\eta(x)v_k(x).\end{aligned}$$

Apply the apriori estimate to u_k, we see

$$|u_k|_{L^q(B_{1/4}^+)}\leq c(n,p,q,r,a)\left(|u_k|_{L^p(B_1^+)}+|v|_{L^q(B_{1/2}^+)}\right).$$

Observe that

$$u(x)=T(u)(x)+\eta(x)v(x),$$

we see

$$\begin{aligned}|u_k-u|_{L^p(B_1^+)} &\leq |T(u_k)-T(u)|_{L^p(B_1^+)}+|v_k-v|_{L^p(B_1^+)}\\ &\leq \frac{1}{2}|u_k-u|_{L^p(B_1^+)}+|v_k-v|_{L^p(B_1^+)}.\end{aligned}$$

Hence $|u_k-u|_{L^p(B_1^+)}\leq 2|v_k-v|_{L^p(B_1^+)}\to 0$ as $k\to\infty$. Take a limit process in the apriori estimate, we get the proposition. □

The other local regularity result is

Proposition 5.2. *Given $n\geq 2$, $1<a,b\leq\infty$, $1\leq r<\infty$, $1<p<q<\infty$ such that*

$$0<\frac{r}{p}+\frac{1}{a}<1$$

and

$$\frac{n-1}{ra}+\frac{n}{b}=1.$$

Denote $B_R=B_R^{n-1}$ *and* $B_R^+=B_R^n\cap\mathbb{R}_+^n$. *Assume* $f,g\in L^p(B_R)$, $F\in L^a(B_R)$, $U\in L^b\left(B_R^+\right)$ *are all nonnegative functions with* $\left.g\right|_{B_{R/2}}\in L^q\left(B_{R/2}\right)$,

$$|F|_{L^a(B_R)}^{1/r}|U|_{L^b\left(B_R^+\right)}\le\varepsilon(n,p,q,r,a,b)\text{ small}$$

and

$$f(\xi)\le\int_{B_R^+}P(x,\xi)U(x)\left[\int_{B_R}P(x,\zeta)F(\zeta)f(\zeta)^r\,d\zeta\right]^{1/r}dx+g(\xi)$$

for $\xi\in B_R$, *then we have* $\left.f\right|_{B_{R/4}}\in L^q\left(B_{R/4}\right)$ *and*

$$|f|_{L^q\left(B_{R/4}\right)}\le c(n,p,q,r,a,b)\left(R^{\frac{n-1}{q}-\frac{n-1}{p}}|f|_{L^p(B_R)}+|g|_{L^q\left(B_{R/2}\right)}\right).$$

Proof. By scaling, we may assume $R=1$. First assume we have $f,g\in L^q(B_1)$. Define

$$u(x)=\int_{B_1}P(x,\xi)F(\xi)f(\xi)^r\,d\xi$$

for $x\in B_1^+$. Let p_1 and q_1 be the numbers given by

$$\frac{n}{p_1}=\frac{n-1}{a}+\frac{(n-1)r}{p},\quad\frac{n}{q_1}=\frac{n-1}{a}+\frac{(n-1)r}{q}.$$

It follows from Proposition 2.1 that

$$\begin{aligned}|u|_{L^{p_1}\left(B_1^+\right)}&\le c(n,p,r,a)|F|_{L^a(B_1)}|f|_{L^p(B_1)}^r,\\|u|_{L^{q_1}\left(B_1^+\right)}&\le c(n,q,r,a)|F|_{L^a(B_1)}|f|_{L^q(B_1)}^r.\end{aligned}$$

Given $0<s<t\le 1/2$. For $\xi\in B_s$, we have

$$\begin{aligned}f(\xi)&\le\int_{B^+_{\frac{s+t}{2}}}P(x,\xi)U(x)u(x)^{1/r}dx+\int_{B_1^+\backslash B^+_{\frac{s+t}{2}}}P(x,\xi)U(x)u(x)^{1/r}dx+g(\xi)\\&\le\int_{B^+_{\frac{s+t}{2}}}P(x,\xi)U(x)u(x)^{1/r}dx+\frac{c(n)}{(t-s)^{n-1}}\int_{B_1^+\backslash B^+_{\frac{s+t}{2}}}U(x)u(x)^{1/r}dx+g(\xi)\\&\le\int_{B^+_{\frac{s+t}{2}}}P(x,\xi)U(x)u(x)^{1/r}dx+\frac{c(n,p)}{(t-s)^{n-1}}|U|_{L^b\left(B_1^+\right)}|u|^{1/r}_{L^{p_1}\left(B_1^+\right)}+g(\xi)\\&\le\int_{B^+_{\frac{s+t}{2}}}P(x,\xi)U(x)u(x)^{1/r}dx+\frac{c(n,p,r,a)}{(t-s)^{n-1}}|f|_{L^p(B_1)}+g(\xi).\end{aligned}$$

Hence

$$|f|_{L^q(B_s)}\le c(n,q,r,b)|U|_{L^b\left(B_1^+\right)}|u|^{1/r}_{L^{q_1}\left(B^+_{\frac{s+t}{2}}\right)}+\frac{c(n,p,q,r,a)}{(t-s)^{n-1}}|f|_{L^p(B_1)}+|g|_{L^q\left(B_{1/2}\right)}.$$

On the other hand, for $x\in B^+_{\frac{s+t}{2}}$, we have

$$\begin{aligned} u(x) &= \int_{B_t} P(x,\xi) F(\xi) f(\xi)^r d\xi + \int_{B_1\setminus B_t} P(x,\xi) F(\xi) f(\xi)^r d\xi \\ &\le \int_{B_t} P(x,\xi) F(\xi) f(\xi)^r d\xi + \frac{c(n)}{(t-s)^{n-1}} \int_{B_1\setminus B_t} F(\xi) f(\xi)^r d\xi \\ &\le \int_{B_t} P(x,\xi) F(\xi) f(\xi)^r d\xi + \frac{c(n,p,r,a)}{(t-s)^{n-1}} |F|_{L^a(B_1)} |f|^r_{L^p(B_1)}. \end{aligned}$$

Hence

$$|u|_{L^{q_1}\left(B^+_{\frac{s+t}{2}}\right)} \le c(n,q,r,a) |F|_{L^a(B_1)} |f|^r_{L^q(B_t)} + \frac{c(n,p,q,r,a)}{(t-s)^{n-1}} |F|_{L^a(B_1)} |f|^r_{L^p(B_1)}.$$

Combine the two inequalities together, we see

$$\begin{aligned} & |f|_{L^q(B_s)} \\ \le\ & c(n,q,r,a,b) |F|^{1/r}_{L^a(B_1)} |U|_{L^b\left(B_1^+\right)} |f|_{L^q(B_t)} + \frac{c(n,p,q,r,a,b)}{(t-s)^{n-1}} |f|_{L^p(B_1)} + |g|_{L^q\left(B_{1/2}\right)} \\ \le\ & \frac{1}{2} |f|_{L^q(B_t)} + \frac{c(n,p,q,r,a,b)}{(t-s)^{n-1}} |f|_{L^p(B_1)} + |g|_{L^q\left(B_{1/2}\right)} \end{aligned}$$

when ε is small enough. It follows from the usual iteration process ([HL, lemma 4.3 on p.75]) that

$$|f|_{L^q\left(B_{1/4}\right)} \le c(n,p,q,r,a,b) \left(|f|_{L^p(B_1)} + |g|_{L^q\left(B_{1/2}\right)} \right).$$

To prove the full proposition, we note that there exists a function $0\le \eta(\xi) \le 1$ such that

$$f(\xi) = \eta(\xi) \int_{B_1^+} P(x,\xi) U(x) \left[\int_{B_1} P(x,\zeta) F(\zeta) f(\zeta)^r d\zeta \right]^{1/r} dx + \eta(\xi) g(\xi).$$

We may define a map T by

$$T(\varphi)(\xi) = \eta(\xi) \int_{B_1^+} P(x,\xi) U(x) \left[\int_{B_1} P(x,\zeta) F(\zeta) |\varphi(\zeta)|^r d\zeta \right]^{1/r} dx.$$

Note that

$$\begin{aligned} |T(\varphi)|_{L^p(B_1)} &\le c(n,p,r,a,b) |F|^{1/r}_{L^a(B_1)} |U|_{L^b\left(B_1^+\right)} |\varphi|_{L^p(B_1)} \le \frac{1}{2} |\varphi|_{L^p(B_1)}, \\ |T(\varphi)|_{L^q(B_1)} &\le c(n,q,r,a,b) |F|^{1/r}_{L^a(B_1)} |U|_{L^b\left(B_1^+\right)} |\varphi|_{L^q(B_1)} \le \frac{1}{2} |\varphi|_{L^q(B_1)} \end{aligned}$$

if ε is small enough. Moreover we have

$$|T(\varphi) - T(\psi)|_{L^p(B_1)} \le |T(|\varphi-\psi|)|_{L^p(B_1)} \le \frac{1}{2} |\varphi-\psi|_{L^p(B_1)}$$

for $\varphi,\psi\in L^p(B_1)$ and

$$|T(\varphi) - T(\psi)|_{L^q(B_1)} \le |T(|\varphi-\psi|)|_{L^q(B_1)} \le \frac{1}{2} |\varphi-\psi|_{L^q(B_1)}$$

$\varphi, \psi \in L^q(B_1)$. For $k \in \mathbb{N}$, let $g_k(\xi) = \min\{g(\xi), k\}$, then it follows from contraction mapping theorem that we may find a unique $f_k \in L^q(B_1)$ such that

$$\begin{aligned} f_k(\xi) &= T(f_k)(\xi) + \eta(\xi) g_k(\xi) \\ &= \eta(\xi) \int_{B_1^+} P(x,\xi) U(x) \left[\int_{B_1} P(x,\zeta) F(\zeta) f(\zeta)^r d\zeta\right]^{1/r} dx + \eta(\xi) g_k(\xi). \end{aligned}$$

Apply the apriori estimate to f_k we see

$$|f_k|_{L^q(B_{1/4})} \le c(n,p,q,r,a,b) \left(|f_k|_{L^p(B_1^+)} + |g|_{L^q\left(B_{1/2}^+\right)} \right).$$

Observe that

$$f(\xi) = T(f)(\xi) + \eta(\xi) g(\xi),$$

we see

$$\begin{aligned} |f_k - f|_{L^p(B_1)} &\le |T(f_k) - T(f)|_{L^p(B_1)} + |g_k - g|_{L^p(B_1)} \\ &\le \frac{1}{2} |f_k - f|_{L^p(B_1)} + |g_k - g|_{L^p(B_1)}. \end{aligned}$$

Hence $|f_k - f|_{L^p(B_1)} \le 2 |g_k - g|_{L^p(B_1)} \to 0$ as $k \to \infty$. Take a limit process in the apriori estimate we get the proposition. □

Now we are ready to prove the main results of this section.

Proof of Theorem 5.1. Let $p_0 = \frac{1}{p-1}$, $f_0(\xi) = f(\xi)^{p-1}$, $u_0(x) = (Pf)(x)$, then $0 < p_0 < \infty$, $f_0 \in L_{loc}^{p_0+1}(\mathbb{R}^{n-1})$ and

$$u_0(x) = \int_{\mathbb{R}^{n-1}} P(x,\xi) f_0(\xi)^{p_0} d\xi, \quad f_0(\xi) = \int_{\mathbb{R}_+^n} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx.$$

For $R > 0$, denote $B_R = B_R^{n-1}$ and $B_R^+ = B_R^n \cap \mathbb{R}_+^n$, let

$$\begin{aligned} u_R(x) &= \int_{\mathbb{R}^{n-1} \setminus B_R} P(x,\xi) f_0(\xi)^{p_0} d\xi, \\ f_R(\xi) &= \int_{\mathbb{R}_+^n \setminus B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx, \end{aligned}$$

then

$$\begin{aligned} u_0(x) &= \int_{B_R} P(x,\xi) f_0(\xi)^{p_0} d\xi + u_R(x), \\ f_0(\xi) &= \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx + f_R(\xi). \end{aligned}$$

First we want to show $u_0 \in L_{loc}^{\frac{n(p_0+1)}{(n-1)p_0}}\left(\overline{\mathbb{R}}_+^n\right)$ and $u_R \in L^{\frac{n(p_0+1)}{(n-1)p_0}}\left(B_R^+\right) \cap L_{loc}^\infty\left(B_R^+ \cup B_R^{n-1}\right)$. Indeed, since $f_0 \in L_{loc}^{p_0+1}(\mathbb{R}^{n-1})$, we see $f_0 < \infty$ a.e. on $\mathbb{R}^{n-1}$. This implies $u_0 < \infty$ a.e. on $\mathbb{R}_+^n$. Hence there exists a $x_0 \in B_R^+$ such that $u_0(x_0) < \infty$. It follows that $\int_{\mathbb{R}^{n-1} \setminus B_R} \frac{f_0(\xi)^{p_0}}{\left(|x_0' - \xi|^2 + x_{0,n}^2\right)^{n/2}} d\xi < \infty$ and $\int_{\mathbb{R}^{n-1} \setminus B_R} \frac{f_0(\xi)^{p_0}}{|\xi|^n} d\xi < \infty$. For $0 < \theta < 1$, $x \in B_{\theta R}^+$, we have

$$u_R(x) = \int_{\mathbb{R}^{n-1} \setminus B_R} P(x,\xi) f_0(\xi)^{p_0} d\xi \le \frac{c(n) R}{(1-\theta)^n} \int_{\mathbb{R}^{n-1} \setminus B_R} \frac{f_0(\xi)^{p_0}}{|\xi|^n} d\xi.$$

It follows that $u_R \in L^\infty_{loc}\left(B_R^+ \cup B_R^{n-1}\right)$. Since $\int_{B_R} P(\cdot,\xi) f_0(\xi)^{p_0} d\xi \in L^{\frac{n(p_0+1)}{(n-1)p_0}}\left(\mathbb{R}^n_+\right)$, we know $u_0 \in L_{loc}^{\frac{n(p_0+1)}{(n-1)p_0}}\left(B_R^+ \cup B_R^{n-1}\right)$. By choosing R arbitrarily large, we deduce that $u_0 \in L_{loc}^{\frac{n(p_0+1)}{(n-1)p_0}}\left(\overline{\mathbb{R}}^n_+\right)$ and hence $u_R \in L^{\frac{n(p_0+1)}{(n-1)p_0}}\left(B_R^+\right)$.

Next we want to show $f_R \in L^{p_0+1}(B_R) \cap L^\infty_{loc}(B_R)$. Indeed we may find $\xi_0 \in B_R$ such that $\int_{\mathbb{R}^n_+} P(x,\xi_0) u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx < \infty$. This implies

$$\int_{\mathbb{R}^n_+ \backslash B_R^+} \frac{x_n}{\left(\left|x'-\xi_0\right|^2 + x_n^2\right)^{n/2}} u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx < \infty$$

and hence $\int_{\mathbb{R}^n_+ \backslash B_R^+} \frac{x_n}{|x|^n} u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx < \infty$. For $0<\theta<1$, $\xi \in B_{\theta R}$, we have

$$f_R(\xi) = \int_{\mathbb{R}^n_+ \backslash B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx \le \frac{c(n)}{(1-\theta)^n} \int_{\mathbb{R}^n_+ \backslash B_R^+} \frac{x_n}{|x|^n} u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx$$

and hence $f_R \in L^\infty_{loc}(B_R)$. To prove the regularity of f, we discuss two cases.

Case 5.1. $0 < p_0 \le \frac{n}{n-1}$.

In this case, we have $\frac{p_0+n}{(n-1)p_0} > 1$. Fix a number r such that

$$1 \le r < \frac{p_0+n}{(n-1)p_0} \text{ and } r > \frac{1}{p_0},$$

then

$$f_0(\xi)^{1/r} \le \left(\int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}} dx\right)^{1/r} + f_R(\xi)^{1/r}.$$

Hence

$$\begin{aligned} & u_0(x) \\ = & \int_{B_R} P(x,\xi) f_0(\xi)^{p_0-r^{-1}} f_0(\xi)^{1/r} d\xi + u_R(x) \\ \le & \int_{B_R} P(x,\xi) f_0(\xi)^{p_0-r^{-1}} \left(\int_{B_R^+} P(y,\xi) u_0(y)^{\frac{p_0+n}{(n-1)p_0}-r} u_0(y)^r dy\right)^{1/r} d\xi + v_R(x), \end{aligned}$$

here

$$v_R(x) = \int_{B_R} P(x,\xi) f_0(\xi)^{p_0-r^{-1}} f_R(\xi)^{1/r} d\xi + u_R(x).$$

Since $f_R \in L^{p_0+1}(B_R)$, we see $v_R \in L^{\frac{n(p_0+1)}{(n-1)p_0}}\left(B_R^+\right)$. On the other hand, for $0 < \theta < 1$, $x \in B_{\theta R}^+$, we have

$$\begin{aligned}
& \int_{B_R} P(x,\xi)\, f_0(\xi)^{p_0-r^{-1}}\, f_R(\xi)^{1/r}\, d\xi \\
\leq\ & |f_R|^{1/r}_{L^\infty\left(B_{\frac{1+\theta}{2}R}\right)} \int_{B_{\frac{1+\theta}{2}R}} P(x,\xi)\, f_0(\xi)^{p_0-r^{-1}}\, d\xi \\
& + \frac{c(n)}{(1-\theta)^n R^{n-1}} \int_{B_R \backslash B_{\frac{1+\theta}{2}R}} f_0(\xi)^{p_0-r^{-1}}\, f_R(\xi)^{1/r}\, d\xi \\
\leq\ & |f_R|^{1/r}_{L^\infty\left(B_{\frac{1+\theta}{2}R}\right)} \int_{B_{\frac{1+\theta}{2}R}} P(x,\xi)\, f_0(\xi)^{p_0-r^{-1}}\, d\xi + \frac{c(n,p_0)}{(1-\theta)^n R^{\frac{(n-1)p_0}{p_0+1}}} |f_0|^{p_0}_{L^{p_0+1}(B_R)},
\end{aligned}$$

hence $v_R \in L_{loc}^{\frac{n(p_0+1)}{(n-1)(p_0-r^{-1})}}\left(B_R^+ \cup B_R^{n-1}\right)$. Let

$$a = \frac{n(p_0+1)}{p_0+n-(n-1)p_0 r}, \quad b = \frac{(p_0+1)r}{p_0 r - 1}.$$

Then $\frac{n}{ra} + \frac{n-1}{b} = \frac{1}{r}$ and

$$\frac{r}{\frac{n(p_0+1)}{(n-1)p_0}} + \frac{1}{a} = \frac{p_0+n}{n(p_0+1)} < 1.$$

For $\frac{n(p_0+1)}{(n-1)p_0} < q < \frac{n(p_0+1)}{(n-1)(p_0-r^{-1})}$, we have $\frac{r}{q} + \frac{1}{a} > \frac{1}{n}$. It follows from Proposition 5.1 that $u_0|_{B_{R/4}^+} \in L^q\left(B_{R/4}^+\right)$. This implies

$$f_0(\xi) = \int_{B_{R/4}^+} P(x,\xi)\, u_0(x)^{\frac{p_0+n}{(n-1)p_0}}\, dx + f_{R/4}(\xi) \leq c(n,q)\, |u|^{\frac{p_0+n}{(n-1)p_0}}_{L^q\left(B_{R/4}^+\right)} + f_{R/4}(\xi)$$

when $q > \frac{n(p_0+n)}{(n-1)p_0}$. Such a choice of q is possible since $\frac{n(p_0+1)}{(n-1)(p_0-r^{-1})} > \frac{n(p_0+n)}{(n-1)p_0}$. In particular, we see $f_0|_{B_{R/8}} \in L^\infty\left(B_{R/8}\right)$. Since every point may be viewed as a center, we see $f_0 \in L^\infty_{loc}\left(\mathbb{R}^{n-1}\right)$ and hence $u_0 \in L^\infty_{loc}\left(\overline{\mathbb{R}}^n_+\right)$. For any $R > 0$, since

$$\int_{\mathbb{R}^{n-1} \backslash B_R} \frac{f_0(\xi)^{p_0}}{|\xi|^n} d\xi < \infty \text{ and } \int_{\mathbb{R}^n_+ \backslash B_R^+} \frac{x_n}{|x|^n} u_0(x)^{\frac{p_0+n}{(n-1)p_0}}\, dx < \infty,$$

we see $u_R \in C^\infty\left(B_R^+ \cup B_R^{n-1}\right)$ and $f_R \in C^\infty(B_R)$. It follows that $f_0 \in C^\alpha_{loc}\left(\mathbb{R}^{n-1}\right)$ for $0 < \alpha < 1$. In particular, $f_0(\xi) > 0$ for any $\xi \in \mathbb{R}^{n-1}$. This implies $u_0 \in C^\alpha_{loc}\left(\overline{\mathbb{R}}^n_+\right)$ for any $0 < \alpha < 1$. Using the fact $\partial_2 \log|x| = x_2 |x|^{-2}$ when $n = 2$, $\partial_n |x|^{2-n} = (2-n)\, x_n |x|^{-n}$ when $n \geq 3$ and the standard potential theory in [GT, chapter 4], it follows from bootstrap method that both u_0 and f_0 are smooth. If $f \in L^p\left(\mathbb{R}^{n-1}\right)$, then $f_0 \in L^{p_0+1}\left(\mathbb{R}^{n-1}\right)$ and $u_0 \in L^{\frac{n(p_0+1)}{(n-1)p_0}}\left(\mathbb{R}^n_+\right)$. If we go back to the proof with this fact and apply Holder inequality when necessary, we will get $f_0 \in L^\infty\left(\mathbb{R}^{n-1}\right)$ and $u_0 \in L^\infty\left(\mathbb{R}^n_+\right)$. This implies $u_0^{\frac{p_0+n}{(n-1)p_0}} \in L^s\left(\mathbb{R}^n_+\right)$ for

$\frac{n(p_0+1)}{p_0+n} \le s \le \infty$. Note that

$$\begin{aligned} U &= \frac{x_n}{|x|^n} * \left(u_0^{\frac{p_0+n}{(n-1)p_0}} \chi_{\mathbb{R}^n_+} \right) \\ &= \left(\frac{x_n}{|x|^n} \chi_{B_1^n} \right) * \left(u_0^{\frac{p_0+n}{(n-1)p_0}} \chi_{\mathbb{R}^n_+} \right) + \left(\frac{x_n}{|x|^n} \chi_{\mathbb{R}^n \backslash B_1^n} \right) * \left(u_0^{\frac{p_0+n}{(n-1)p_0}} \chi_{\mathbb{R}^n_+} \right), \end{aligned}$$

since $\frac{x_n}{|x|^n} \chi_{B_1^n} \in L^{\frac{n}{n-1}-\varepsilon}(\mathbb{R}^n)$, $\frac{x_n}{|x|^n} \chi_{\mathbb{R}^n \backslash B_1^n} \in L^{\frac{n}{n-1}+\varepsilon}(\mathbb{R}^n)$ and $\frac{n(p_0+1)}{p_0+n} < n$, we see U is continuous and $U(x) \to 0$ as $|x| \to \infty$. Since $f_0 = c(n) \left. U \right|_{\mathbb{R}^{n-1}}$, we see $f_0(\xi) \to 0$ as $|\xi| \to \infty$.

Case 5.2. $\frac{n}{n-1} \le p_0 < \infty$.

In this case, we fix a number r such that

$$1 \le r \le p_0 \text{ and } r \ge \frac{(n-1)p_0}{p_0+n},$$

then

$$u_0(x)^{1/r} \le \left(\int_{B_R} P(x,\xi) f_0(\xi)^{p_0} d\xi \right)^{1/r} + u_R(x)^{1/r}.$$

Hence

$$\begin{aligned} & f_0(\xi) \\ = & \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} u_0(x)^{1/r} dx + f_R(\xi) \\ \le & \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} \left(\int_{B_R} P(x,\zeta) f_0(\zeta)^{p_0-r} f_0(\zeta)^r d\zeta \right)^{1/r} dx + g_R(\xi), \end{aligned}$$

here

$$g_R(\xi) = \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} u_R(x)^{1/r} dx + f_R(\xi).$$

Since $u_R \in L^{\frac{n(p_0+1)}{(n-1)p_0}}(B_R^+)$, we see $g_R \in L^{p_0+1}(B_R)$. On the other hand, for $0 < \theta < 1$, $\xi \in B_{\theta R}$, we have

$$\begin{aligned} & \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} u_R(x)^{1/r} dx \\ \le & |u_R|^{1/r}_{L^\infty\left(B^+_{\frac{1+\theta}{2}R}\right)} \int_{B^+_{\frac{1+\theta}{2}R}} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} dx \\ & + \frac{c(n)}{(1-\theta)^n R^{n-1}} \int_{B_R^+ \backslash B^+_{\frac{1+\theta}{2}R}} u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} u_R(x)^{1/r} dx \\ \le & |u_R|^{1/r}_{L^\infty\left(B^+_{\frac{1+\theta}{2}R}\right)} \int_{B^+_{\frac{1+\theta}{2}R}} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0} - r^{-1}} dx \\ & + \frac{c(n,p_0)}{(1-\theta)^n R^{\frac{n-1}{p_0+1}}} |u_0|^{\frac{p_0+n}{(n-1)p_0}}_{L^{\frac{n(p_0+1)}{(n-1)p_0}}(B_R^+)}, \end{aligned}$$

hence $g_R \in L^q_{loc}(B_R)$ for any $q < \infty$. Let

$$a = \frac{p_0+1}{p_0-r}, \quad b = \frac{n(p_0+1)r}{(p_0+n)r - (n-1)p_0},$$

then $\frac{n-1}{ra}+\frac{n}{b}=1$, $\frac{r}{p_0+1}+\frac{1}{a}=\frac{p_0}{p_0+1}\in(0,1)$. For any $p_0+1<q<\infty$, it follows from Proposition 5.2 that when R is small enough, we have $f_0\in L^q\left(B_{R/4}\right)$. Since every point can be viewed as a center, we see $f_0\in L^q_{loc}\left(\mathbb{R}^{n-1}\right)$ and hence $u_0\in L^{\frac{nq}{n-1}}_{loc}\left(\overline{\mathbb{R}}^n_+\right)$. Using the equations of f_0 and u_0, we see $f_0\in L^\infty_{loc}\left(\mathbb{R}^{n-1}\right)$ and $u_0\in L^\infty_{loc}\left(\overline{\mathbb{R}}^n_+\right)$. Now the arguments in Case 5.1 tell us $f_0\in C^\infty\left(\mathbb{R}^{n-1}\right)$ and $u_0\in C^\infty\left(\overline{\mathbb{R}}^n_+\right)$, moreover, $f_0(\xi)\to 0$ as $|\xi|\to\infty$ under the assumption $f\in L^p\left(\mathbb{R}^{n-1}\right)$. □

6. Radial symmetry of nonnegative critical functions

In this section we will study the symmetry property of the nonnegative critical functions of the variational problem (1.8). We will show any nonnegative critical functions are radial symmetric with respect to some points. As explained at the beginning of Section 5, (1.9) may be viewed as an integral system which is very similar to the integral systems related to the Hardy-Littlewood-Sobolev inequalities. For the latter one, the radial symmetry of nonnegative solution for some special cases were solved in [CLO1, CLO2, L]. In particular, in [CLO1] an integral version of the method of moving planes ([GNN]) was introduced and later applied in [CLO2] to resolve the symmetry problems for some cases of the integral systems related to Hardy-Littlewood-Sobolev inequalities. In [Hn], some new observations were added and all the cases for the symmetry of the solutions to the systems were resolved. We will apply these new observations to (1.9).

Theorem 6.1. *Assume $1<p<\infty$, $n\geq 2$, $f\in L^p\left(\mathbb{R}^{n-1}\right)$ is nonnegative, not identically zero and it satisfies*

$$f(\xi)^{p-1}=\int_{\mathbb{R}^n_+}P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1}\,dx,$$

then $f\in C^\infty\left(\mathbb{R}^{n-1}\right)$, moreover f is radial symmetric with respect to some point and strictly decreasing along the radial direction.

For the case $n\geq 3$, $p=\frac{2(n-1)}{n-2}$, the Euler-Lagrange equation has conformal invariance property and we may weaken the assumption a little bit.

Proposition 6.1. *Assume $n\geq 3$, $f\in L^{\frac{2(n-1)}{n-2}}_{loc}\left(\mathbb{R}^{n-1}\right)$ is nonnegative, not identically zero and it satisfies*

$$f(\xi)^{\frac{n}{n-2}}=\int_{\mathbb{R}^n_+}P(x,\xi)(Pf)(x)^{\frac{n+2}{n-2}}\,dx,$$

then for some $\lambda>0$ and $\xi_0\in\mathbb{R}^{n-1}$, we have

$$f(\xi)=c(n)\left(\frac{\lambda}{\lambda^2+|\xi-\xi_0|^2}\right)^{\frac{n-2}{2}}.$$

During the proofs of these symmetry results, we will need the following basic inequality: assume $0<\theta\leq 1$, $a\geq b\geq 0$, $c\geq 0$, then

$$(a+c)^\theta-(b+c)^\theta\leq a^\theta-b^\theta.$$

Indeed, for $t \geq 0$, let $\phi(t) = (a+t)^\theta - (b+t)^\theta$, then for $t > 0$, $\phi'(t) = \theta (a+t)^{\theta-1} - \theta (b+t)^{\theta-1} \leq 0$. The inequality follows.

For $\sigma \in \mathbb{R}^m$ and $s > 0$, we denote

$$|\sigma|_{l^s} = \left(\sum_{i=1}^m |\sigma_i|^s \right)^{1/s}.$$

Proof of Theorem 6.1. By Theorem 5.1 we know $f \in C^\infty(\mathbb{R}^{n-1})$ and $f(\xi) \to 0$ as $|\xi| \to \infty$. Let $q = \frac{1}{p-1}$, $g(\xi) = f(\xi)^{p-1}$, $v(x) = (Pf)(x)$, then $0 < q < \infty$, $g \in L^{q+1}(\mathbb{R}^{n-1})$, $v \in L^{\frac{n(q+1)}{(n-1)q}}(\mathbb{R}^n_+)$ and

$$v(x) = \int_{\mathbb{R}^{n-1}} P(x,\xi) g(\xi)^q d\xi, \quad g(\xi) = \int_{\mathbb{R}^n_+} P(x,\xi) v(x)^{\frac{q+n}{(n-1)q}} dx.$$

For $\lambda \in \mathbb{R}$, denote

$$H_\lambda = \left\{ \xi \in \mathbb{R}^{n-1} : \xi_1 < \lambda \right\}, \quad Q_\lambda = \left\{ x \in \mathbb{R}^n_+ : x_1 < \lambda \right\}.$$

For $\xi \in \mathbb{R}^{n-1}$, $\xi = (\xi_1, \xi'')$, denote $\xi_\lambda = (2\lambda - \xi_1, \xi'')$. For $x \in \mathbb{R}^n$, $x = (x_1, x'')$, denote $x_\lambda = (2\lambda - x_1, x'')$. Define $g_\lambda(\xi) = g(\xi_\lambda)$, $v_\lambda(x) = v(x_\lambda)$ and

$$\mathcal{B}^g_\lambda = \{\xi \in H_\lambda : g_\lambda(\xi) > g(\xi)\}, \quad \mathcal{B}^v_\lambda = \{x \in Q_\lambda : v_\lambda(x) > v(x)\}.$$

By a simple change of variable, we see

$$\begin{aligned}
v(x) &= \int_{H_\lambda} P(x,\xi) g(\xi)^q d\xi + \int_{H_\lambda} P(x_\lambda,\xi) g(\xi_\lambda)^q d\xi, \\
v(x_\lambda) &= \int_{H_\lambda} P(x,\xi) g(\xi_\lambda)^q d\xi + \int_{H_\lambda} P(x_\lambda,\xi) g(\xi)^q d\xi, \\
g(\xi) &= \int_{Q_\lambda} P(x,\xi) v(x)^{\frac{q+n}{(n-1)q}} dx + \int_{Q_\lambda} P(x_\lambda,\xi) v(x_\lambda)^{\frac{q+n}{(n-1)q}} dx, \\
g(\xi_\lambda) &= \int_{Q_\lambda} P(x,\xi) v(x_\lambda)^{\frac{q+n}{(n-1)q}} dx + \int_{Q_\lambda} P(x_\lambda,\xi) v(x)^{\frac{q+n}{(n-1)q}} dx.
\end{aligned}$$

Case 6.1. $0 < q \leq \frac{n}{n-1}$.

In this case, we choose a number r such that

$$1 \leq r \leq \frac{q+n}{(n-1)q} \text{ and } q^{-1} < r.$$

We have

$$v(x_\lambda) - v(x) = \int_{H_\lambda} (P(x,\xi) - P(x_\lambda,\xi)) (g(\xi_\lambda)^q - g(\xi)^q) d\xi.$$

Hence for $x \in \mathcal{B}^v_\lambda$,

$$\begin{aligned}
0 &\leq v(x_\lambda) - v(x) \\
&\leq \int_{\mathcal{B}^g_\lambda} (P(x,\xi) - P(x_\lambda,\xi)) (g(\xi_\lambda)^q - g(\xi)^q) d\xi \\
&\leq \int_{\mathcal{B}^g_\lambda} P(x,\xi) \left(\left(g(\xi_\lambda)^{1/r} \right)^{qr} - \left(g(\xi)^{1/r} \right)^{qr} \right) d\xi \\
&\leq qr \int_{\mathcal{B}^g_\lambda} P(x,\xi) g(\xi_\lambda)^{q-r^{-1}} \left(g(\xi_\lambda)^{1/r} - g(\xi)^{1/r} \right) d\xi.
\end{aligned}$$

It follows that

$$\begin{aligned}
& |v_\lambda - v|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)} \\
\leq \quad & c(n,q,r)\left|g_\lambda^{q-r^{-1}}\left(g_\lambda^{1/r} - g^{1/r}\right)\right|_{L^{\frac{q+1}{q}}\left(\mathcal{B}_\lambda^g\right)} \\
\leq \quad & c(n,q,r)\left|g_\lambda^{q-r^{-1}}\right|_{L^{\frac{q+1}{q-r^{-1}}}\left(\mathcal{B}_\lambda^g\right)}\left|g_\lambda^{1/r} - g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)} \\
= \quad & c(n,q,r)\left|g_\lambda\right|^{q-r^{-1}}_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}\left|g_\lambda^{1/r} - g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}
\end{aligned}$$

On the other hand, for $\xi \in \mathcal{B}_\lambda^g$, we have

$$\begin{aligned}
g(\xi_\lambda) \quad = \quad & \int_{\mathcal{B}_\lambda^v} P(x,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q}}\, dx + \int_{\mathcal{B}_\lambda^v} P(x_\lambda,\xi)\, v(x)^{\frac{q+n}{(n-1)q}}\, dx \\
& + \int_{Q_\lambda \setminus \mathcal{B}_\lambda^v} P(x,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q}}\, dx + \int_{Q_\lambda \setminus \mathcal{B}_\lambda^v} P(x_\lambda,\xi)\, v(x)^{\frac{q+n}{(n-1)q}}\, dx \\
\leq \quad & \int_{\mathcal{B}_\lambda^v} P(x,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q}}\, dx + \int_{\mathcal{B}_\lambda^v} P(x_\lambda,\xi)\, v(x)^{\frac{q+n}{(n-1)q}}\, dx \\
& + \int_{Q_\lambda \setminus \mathcal{B}_\lambda^v} P(x,\xi)\, v(x)^{\frac{q+n}{(n-1)q}}\, dx + \int_{Q_\lambda \setminus \mathcal{B}_\lambda^v} P(x_\lambda,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q}}\, dx.
\end{aligned}$$

Since

$$\begin{aligned}
g(\xi) \quad = \quad & \int_{\mathcal{B}_\lambda^v} P(x,\xi)\, v(x)^{\frac{q+n}{(n-1)q}}\, dx + \int_{\mathcal{B}_\lambda^v} P(x_\lambda,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q}}\, dx \\
& + \int_{Q_\lambda \setminus \mathcal{B}_\lambda^v} P(x,\xi)\, v(x)^{\frac{q+n}{(n-1)q}}\, dx + \int_{Q_\lambda \setminus \mathcal{B}_\lambda^v} P(x_\lambda,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q}}\, dx,
\end{aligned}$$

we see

$$\begin{aligned}
& g(\xi_\lambda)^{1/r} - g(\xi)^{1/r} \\
\leq & \left(\int_{\mathcal{B}_\lambda^v} P(x,\xi) v(x_\lambda)^{\frac{q+n}{(n-1)q}} dx + \int_{\mathcal{B}_\lambda^v} P(x_\lambda,\xi) v(x)^{\frac{q+n}{(n-1)q}} dx\right)^{1/r} \\
& - \left(\int_{\mathcal{B}_\lambda^v} P(x,\xi) v(x)^{\frac{q+n}{(n-1)q}} dx + \int_{\mathcal{B}_\lambda^v} P(x_\lambda,\xi) v(x_\lambda)^{\frac{q+n}{(n-1)q}} dx\right)^{1/r} \\
= & \left(\int_{\mathcal{B}_\lambda^v} \left|\left(P(x,\xi)^{1/r} v(x_\lambda)^{\frac{q+n}{(n-1)qr}}, P(x_\lambda,\xi)^{1/r} v(x)^{\frac{q+n}{(n-1)qr}}\right)\right|_{l^r}^r dx\right)^{1/r} \\
& - \left(\int_{\mathcal{B}_\lambda^v} \left|\left(P(x,\xi)^{1/r} v(x)^{\frac{q+n}{(n-1)qr}}, P(x_\lambda,\xi)^{1/r} v(x_\lambda)^{\frac{q+n}{(n-1)qr}}\right)\right|_{l^r}^r dx\right)^{1/r} \\
\leq & \left(\int_{\mathcal{B}_\lambda^v} \left|\left(P(x,\xi)^{1/r}\left(v(x_\lambda)^{\frac{q+n}{(n-1)qr}} - v(x)^{\frac{q+n}{(n-1)qr}}\right), P(x_\lambda,\xi)^{1/r}\left(v(x)^{\frac{q+n}{(n-1)qr}} - v(x_\lambda)^{\frac{q+n}{(n-1)qr}}\right)\right)\right|_{l^r}^r dx\right)^{1/r} \\
\leq & 2\left(\int_{\mathcal{B}_\lambda^v} P(x,\xi)\left(v(x_\lambda)^{\frac{q+n}{(n-1)qr}} - v(x)^{\frac{q+n}{(n-1)qr}}\right)^r dx\right)^{1/r} \\
\leq & \frac{2(q+n)}{(n-1)qr}\left(\int_{\mathcal{B}_\lambda^v} P(x,\xi) v(x_\lambda)^{\frac{q+n}{(n-1)q}-r} (v(x_\lambda)-v(x))^r dx\right)^{1/r}.
\end{aligned}$$

This implies

$$\begin{aligned}
& \left|g_\lambda^{1/r} - g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)} \\
\leq & \frac{2(q+n)}{(n-1)qr}\left|\int_{\mathcal{B}_\lambda^v} P(x,\xi) v(x_\lambda)^{\frac{q+n}{(n-1)q}-r} (v(x_\lambda)-v(x))^r dx\right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}^{1/r} \\
\leq & c(n,q,r)\left|v_\lambda^{\frac{q+n}{(n-1)q}-r}(v_\lambda - v)^r\right|_{L^{\frac{n(q+1)}{q+n}}\left(\mathcal{B}_\lambda^v\right)}^{1/r} \\
\leq & c(n,q,r)\left|v_\lambda^{\frac{q+n}{(n-1)q}-r}\right|_{L^{\frac{\frac{n(q+1)}{(n-1)q}}{\frac{q+n}{(n-1)q}-r}}\left(\mathcal{B}_\lambda^v\right)}^{1/r} \left|(v_\lambda - v)^r\right|_{L^{\frac{n(q+1)}{(n-1)qr}}\left(\mathcal{B}_\lambda^v\right)}^{1/r} \\
= & c(n,q,r)\left|v_\lambda\right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)}^{\frac{q+n}{(n-1)qr}-1} \left|v_\lambda - v\right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)}.
\end{aligned}$$

It follows from the two inequalities that

$$\begin{aligned} & \left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)} \\ \leq\ & c\left(n,q,r\right)\left|v_\lambda\right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)}^{\frac{q+n}{(n-1)qr}-1}\left|g_\lambda\right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}^{q-r^{-1}}\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)} \\ =\ & c\left(n,q,r\right)\left|v\right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(2\lambda e_1-\mathcal{B}_\lambda^v\right)}^{\frac{q+n}{(n-1)qr}-1}\left|g\right|_{L^{q+1}\left(2\lambda e_1'-\mathcal{B}_\lambda^g\right)}^{q-r^{-1}}\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)} \\ \leq\ & c\left(n,q,r\right)\left|v\right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathbb{R}_+^n\right)}^{\frac{q+n}{(n-1)qr}-1}\left|g\right|_{L^{q+1}\left(2\lambda e_1'-\mathcal{B}_\lambda^g\right)}^{q-r^{-1}}\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}. \end{aligned}$$

Here $e_1=(1,0,\cdots,0)$. After these preparations, we will use the method of moving planes to prove the radial symmetry of g and hence f.

First, we have to show it is possible to start. Indeed, for λ large enough, we have $|g|_{L^{q+1}\left(2\lambda e_1'-\mathcal{B}_\lambda^v\right)}$ can be arbitrarily small, this implies

$$\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}\leq\frac{1}{2}\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}$$

and hence $\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}=0$. It follows that $\mathcal{B}_\lambda^g=\emptyset$ when λ is large enough.

Next we let $\lambda_0=\inf\left\{\lambda\in\mathbb{R}:\mathcal{B}_{\lambda'}^g=\emptyset\text{ for all }\lambda'\geq\lambda\right\}$. It follows from the fact $g\left(\xi\right)\to 0$ as $|\xi|\to\infty$ and $g\left(\xi\right)>0$ for all $\xi\in\mathbb{R}^{n-1}$ that λ_0 must be a finite number. By the definition of λ_0 we know $g_{\lambda_0}\left(\xi\right)\leq g\left(\xi\right)$ for $\xi\in H_{\lambda_0}$. We claim that $g_{\lambda_0}=g$. Indeed if this is not the case, then since

$$v_{\lambda_0}\left(x\right)-v\left(x\right)=\int_{H_{\lambda_0}}\left(P\left(x,\xi\right)-P\left(x_{\lambda_0},\xi\right)\right)\left(g_{\lambda_0}\left(\xi\right)^q-g\left(\xi\right)^q\right)d\xi$$

and

$$g_{\lambda_0}\left(\xi\right)-g\left(\xi\right)=\int_{Q_{\lambda_0}}\left(P\left(x,\xi\right)-P\left(x_{\lambda_0},\xi\right)\right)\left(v_{\lambda_0}\left(x\right)^{\frac{q+n}{(n-1)q}}-v\left(x\right)^{\frac{q+n}{(n-1)q}}\right)dx,$$

we get $g_{\lambda_0}\left(\xi\right)<g\left(\xi\right)$ for $\xi\in H_{\lambda_0}$. It follows that $\chi_{2\lambda e_1'-\mathcal{B}_{\lambda_0}^g}\to 0$ a.e. as $\lambda\uparrow\lambda_0$. By dominated convergence theorem we have $\left|g_\lambda\right|_{L^{q+1}\left(2\lambda e_1'-\mathcal{B}_\lambda^g\right)}\to 0$ as $\lambda\uparrow\lambda_0$. It implies

$$\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}\leq\frac{1}{2}\left|g_\lambda^{1/r}-g^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^g\right)}$$

when λ is very close to λ_0 and hence $\mathcal{B}_\lambda^g=\emptyset$. This contradicts with the choice of λ_0. Hence when the moving process stops, we must have symmetry. Moreover we claim that $g_\lambda\left(\xi\right)<g\left(\xi\right)$ for $\xi\in H_\lambda$ when $\lambda>\lambda_0$. Indeed for any $\lambda>\lambda_0$ we can not have $g_\lambda=g$ because otherwise g is periodic in the first direction and can not lie in $L^{q+1}\left(\mathbb{R}^{n-1}\right)$. Hence $g_\lambda<g$ in H_λ.

By translation, we may assume $g\left(0\right)=\max_{\xi\in\mathbb{R}^{n-1}}g\left(\xi\right)$, then it follows that the moving plane process from any direction must stop at the origin. Hence g must be radial symmetric and strictly decreasing in the radial direction.

Case 6.2. $\frac{n}{n-1}\leq q<\infty$.

In this case, we choose a number r such that

$$1 \leq r < q \text{ and } \frac{(n-1)\,q}{q+n} \leq r.$$

We have

$$g(\xi_\lambda) - g(\xi) = \int_{Q_\lambda} \left(P(x,\xi) - P(x_\lambda,\xi)\right)\left(v(x_\lambda)^{\frac{q+n}{(n-1)q}} - v(x)^{\frac{q+n}{(n-1)q}}\right) dx.$$

Hence for $\xi \in \mathcal{B}^g_\lambda$,

$$\begin{aligned}
0 &\leq g(\xi_\lambda) - g(\xi) \\
&\leq \int_{\mathcal{B}^v_\lambda} \left(P(x,\xi) - P(x,\xi_\lambda)\right)\left(v(x_\lambda)^{\frac{q+n}{(n-1)q}} - v(x)^{\frac{q+n}{(n-1)q}}\right) dx \\
&\leq \int_{\mathcal{B}^v_\lambda} P(x,\xi)\left(\left(v(x_\lambda)^{1/r}\right)^{\frac{(q+n)r}{(n-1)q}} - \left(v(x)^{1/r}\right)^{\frac{(q+n)r}{(n-1)q}}\right) dx \\
&\leq \frac{(q+n)\,r}{(n-1)\,q} \int_{\mathcal{B}^v_\lambda} P(x,\xi)\, v(x_\lambda)^{\frac{q+n}{(n-1)q} - r^{-1}} \left(v(x_\lambda)^{1/r} - v(x)^{1/r}\right) dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
& |g_\lambda - g|_{L^{q+1}(\mathcal{B}^g_\lambda)} \\
\leq\ & c(n,q,r) \left| v_\lambda^{\frac{q+n}{(n-1)q} - r^{-1}} \left(v_\lambda^{1/r} - v^{1/r}\right)\right|_{L^{\frac{n(q+1)}{q+n}}(\mathcal{B}^v_\lambda)} \\
\leq\ & c(n,q,r) \left| v_\lambda^{\frac{q+n}{(n-1)q} - r^{-1}} \right|_{L^{\frac{\frac{n(q+1)}{(n-1)q}}{\frac{q+n}{(n-1)q} - r^{-1}}}(\mathcal{B}^v_\lambda)} \left| v_\lambda^{1/r} - v^{1/r}\right|_{L^{\frac{n(q+1)r}{(n-1)q}}(\mathcal{B}^v_\lambda)} \\
=\ & c(n,q,r)\, |v_\lambda|^{\frac{q+n}{(n-1)q} - r^{-1}}_{L^{\frac{n(q+1)}{(n-1)q}}(\mathcal{B}^v_\lambda)} \left| v_\lambda^{1/r} - v^{1/r}\right|_{L^{\frac{n(q+1)r}{(n-1)q}}(\mathcal{B}^v_\lambda)}.
\end{aligned}$$

On the other hand, for $x \in \mathcal{B}^v_\lambda$, we have

$$\begin{aligned}
v(x_\lambda) &= \int_{\mathcal{B}^g_\lambda} P(x,\xi)\, g(\xi_\lambda)^q\, d\xi + \int_{\mathcal{B}^g_\lambda} P(x_\lambda,\xi)\, g(\xi)^q\, d\xi \\
&\quad + \int_{H_\lambda \setminus \mathcal{B}^g_\lambda} P(x,\xi)\, g(\xi_\lambda)^q\, d\xi + \int_{H_\lambda \setminus \mathcal{B}^g_\lambda} P(x_\lambda,\xi)\, g(\xi)^q\, d\xi \\
&\leq \int_{\mathcal{B}^g_\lambda} P(x,\xi)\, g(\xi_\lambda)^q\, d\xi + \int_{\mathcal{B}^g_\lambda} P(x_\lambda,\xi)\, g(\xi)^q\, d\xi \\
&\quad + \int_{H_\lambda \setminus \mathcal{B}^g_\lambda} P(x,\xi)\, g(\xi)^q\, d\xi + \int_{H_\lambda \setminus \mathcal{B}^g_\lambda} P(x_\lambda,\xi)\, g(\xi_\lambda)^q\, d\xi.
\end{aligned}$$

Since

$$\begin{aligned}
v(x) &= \int_{\mathcal{B}^g_\lambda} P(x,\xi)\, g(\xi)^q\, d\xi + \int_{\mathcal{B}^g_\lambda} P(x_\lambda,\xi)\, g(\xi_\lambda)^q\, d\xi \\
&\quad + \int_{H_\lambda \setminus \mathcal{B}^g_\lambda} P(x,\xi)\, g(\xi)^q\, d\xi + \int_{H_\lambda \setminus \mathcal{B}^g_\lambda} P(x_\lambda,\xi)\, g(\xi_\lambda)^q\, d\xi,
\end{aligned}$$

we see

$$\begin{aligned}
0 &\leq v(x_\lambda)^{1/r} - v(x)^{1/r} \\
&\leq \left(\int_{\mathcal{B}_\lambda^g} P(x,\xi) g(\xi_\lambda)^q \, d\xi + \int_{\mathcal{B}_\lambda^g} P(x_\lambda,\xi) g(\xi)^q \, d\xi \right)^{1/r} \\
&\quad - \left(\int_{\mathcal{B}_\lambda^g} P(x,\xi) g(\xi)^q \, d\xi + \int_{\mathcal{B}_\lambda^g} P(x_\lambda,\xi) g(\xi_\lambda)^q \, d\xi \right)^{1/r} \\
&= \left(\int_{\mathcal{B}_\lambda^g} \left| \left(P(x,\xi)^{1/r} g(\xi_\lambda)^{q/r}, P(x_\lambda,\xi)^{1/r} g(\xi)^{q/r} \right) \right|_{l^r}^r d\xi \right)^{1/r} \\
&\quad - \left(\int_{\mathcal{B}_\lambda^g} \left| \left(P(x,\xi)^{1/r} g(\xi)^{q/r}, P(x_\lambda,\xi)^{1/r} g(\xi_\lambda)^{q/r} \right) \right|_{l^r}^r d\xi \right)^{1/r} \\
&\leq \left(\int_{\mathcal{B}_\lambda^g} \left| \left(P(x,\xi)^{1/r} \left(g(\xi_\lambda)^{q/r} - g(\xi)^{q/r} \right), P(x_\lambda,\xi)^{1/r} \left(g(\xi)^{q/r} - g(\xi_\lambda)^{q/r} \right) \right) \right|_{l^r}^r d\xi \right)^{1/r} \\
&\leq 2 \left(\int_{\mathcal{B}_\lambda^g} P(x,\xi) \left(g(\xi_\lambda)^{q/r} - g(\xi)^{q/r} \right)^r d\xi \right)^{1/r} \\
&\leq \frac{2q}{r} \left(\int_{\mathcal{B}_\lambda^g} P(x,\xi) g(\xi_\lambda)^{q-r} \left(g(\xi_\lambda) - g(\xi) \right)^r d\xi \right)^{1/r}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left| v_\lambda^{1/r} - v^{1/r} \right|_{L^{\frac{n(q+1)r}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)} \\
\leq\ & \frac{2q}{r} \left| \int_{\mathcal{B}_\lambda^g} P(x,\xi) g(\xi_\lambda)^{q-r} \left(g(\xi_\lambda) - g(\xi) \right)^r d\xi \right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)}^{1/r} \\
\leq\ & c(n,q,r) \left| g_\lambda^{q-r} (g_\lambda - g)^r \right|_{L^{\frac{q+1}{q}}\left(\mathcal{B}_\lambda^g\right)}^{1/r} \\
\leq\ & c(n,q,r) \left| g_\lambda^{q-r} \right|_{L^{\frac{q+1}{q-r}}\left(\mathcal{B}_\lambda^g\right)}^{1/r} \left| (g_\lambda - g)^r \right|_{L^{\frac{q+1}{r}}\left(\mathcal{B}_\lambda^g\right)}^{1/r} \\
=\ & c(n,q,r) \left| g_\lambda \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}^{\frac{q-r}{r}} \left| g_\lambda - g \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}.
\end{aligned}$$

Combine the two inequalities together we see

$$\begin{aligned}
\left| g_\lambda - g \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)} &\leq c(n,q,r) \left| v_\lambda \right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathcal{B}_\lambda^v\right)}^{\frac{q+n}{(n-1)q} - r^{-1}} \left| g_\lambda \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}^{\frac{q-r}{r}} \left| g_\lambda - g \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)} \\
&= c(n,q,r) \left| v \right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(2\lambda e_1 - \mathcal{B}_\lambda^v\right)}^{\frac{q+n}{(n-1)q} - r^{-1}} \left| g \right|_{L^{q+1}\left(2\lambda e_1' - \mathcal{B}_\lambda^g\right)}^{\frac{q-r}{r}} \left| g_\lambda - g \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)} \\
&\leq c(n,q,r) \left| v \right|_{L^{\frac{n(q+1)}{(n-1)q}}\left(\mathbb{R}_+^n\right)}^{\frac{q+n}{(n-1)q} - r^{-1}} \left| g \right|_{L^{q+1}\left(2\lambda e_1' - \mathcal{B}_\lambda^g\right)}^{\frac{q-r}{r}} \left| g_\lambda - g \right|_{L^{q+1}\left(\mathcal{B}_\lambda^g\right)}.
\end{aligned}$$

With this inequality at hand, we may proceed in the same way as in the Case 6.1 to get the conclusion that g is radial symmetric with respect to some point and strictly decreasing along the radial direction. □

Next we look at the special power $p=\frac{2(n-1)}{n-2}$.

Proof of Proposition 6.1. If we know $f\in L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$, then it follows from Theorem 6.1 that $f\in C^{\infty}\left(\mathbb{R}^{n-1}\right)$, it is strictly positive and radial symmetric with respect to some point. By translation we may assume f is radial symmetric with respect to 0.

On the other hand, if f is a solution to the equation, let $u(x)=(Pf)(x)$, $\widetilde{f}(\xi)=\frac{1}{|\xi|^{n-2}}f\left(\frac{\xi}{|\xi|^2}\right)$ and $\widetilde{u}(x)=\frac{1}{|x|^{n-2}}u\left(\frac{x}{|x|^2}\right)$, by change of variable we know

$$\widetilde{u}(x)=\left(P\widetilde{f}\right)(x),\quad \widetilde{f}(\xi)^{\frac{n}{n-2}}=\int_{\mathbb{R}^n_+}P(x,\xi)\,\widetilde{u}(x)^{\frac{n+2}{n-2}}\,dx$$

and $\left|\widetilde{f}\right|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}=|f|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}$. In particular, $\widetilde{f}\in L^{\frac{2(n-1)}{n-2}}$ and satisfies the same equation.

Let $e_1=(1,0,\cdots,0)\in\mathbb{R}^n$, then it follows from Theorem 6.1 that $f_1(\xi)=\frac{1}{|\xi|^{n-2}}f\left(\frac{\xi}{|\xi|^2}-e_1'\right)$ is smooth and radial symmetric with respect to some point. It follows from Proposition 4.1 and the fact that $f\in L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$ that for some $c_1>0$ and $c_2>0$, $f(\xi)=\left(c_1|\xi|^2+c_2\right)^{-\frac{n-2}{2}}$. Since f satisfies the equation, it follows that for some $\lambda>0$, $f(\xi)=c(n)\left(\frac{\lambda}{\lambda^2+|\xi|^2}\right)^{\frac{n-2}{2}}$.

Next we want to show under the assumption of the Proposition 6.1, f always lies in $L^{\frac{2(n-1)}{n-2}}\left(\mathbb{R}^{n-1}\right)$. This will be proved by contradiction. Indeed, if this is not the case, then $\int_{\mathbb{R}^{n-1}}f(\xi)^{\frac{2(n-1)}{n-2}}\,d\xi=\infty$. Let $g_0(\xi)=f(\xi)^{\frac{n}{n-2}}$, $v_0(x)=(Pf)(x)$, then $g_0\in L_{loc}^{\frac{2(n-1)}{n}}\left(\mathbb{R}^{n-1}\right)$, $\int_{\mathbb{R}^{n-1}}g_0(\xi)^{\frac{2(n-1)}{n}}\,d\xi=\infty$ and

$$v_0(x)=\int_{\mathbb{R}^{n-1}}P(x,\xi)\,g_0(\xi)^{\frac{n-2}{n}}\,d\xi,\quad g_0(\xi)=\int_{\mathbb{R}^n_+}P(x,\xi)\,v_0(x)^{\frac{n+2}{n-2}}\,dx.$$

It follows from the proof of Theorem 5.1 that $g_0\in C^{\infty}\left(\mathbb{R}^{n-1}\right)$ and $v_0\in C^{\infty}\left(\overline{\mathbb{R}}^n_+\right)$.

Let $g(\xi)=\frac{1}{|\xi|^n}g_0\left(\frac{\xi}{|\xi|^2}\right)$, $v(x)=\frac{1}{|x|^{n-2}}v_0\left(\frac{x}{|x|^2}\right)$, then

$$v(x)=\int_{\mathbb{R}^{n-1}}P(x,\xi)\,g(\xi)^{\frac{n-2}{n}}\,d\xi,\quad g(\xi)=\int_{\mathbb{R}^n_+}P(x,\xi)\,v(x)^{\frac{n+2}{n-2}}\,dx.$$

Moreover for any $R>0$, $\int_{\mathbb{R}^{n-1}\backslash B_R}g(\xi)^{\frac{2(n-1)}{n}}\,d\xi<\infty$ and $\int_{\mathbb{R}^{n-1}}g(\xi)^{\frac{2(n-1)}{n}}\,d\xi=\infty$. For $\lambda>0$, we define H_λ, g_λ as in the Case 6.1 of the proof of Theorem 6.1, but let $\mathcal{B}^g_\lambda=\{\xi\in H_\lambda\backslash\{0\}:g_\lambda(\xi)>g(\xi)\}$. Put the number in the proof of Theorem 6.1 $r=\frac{n+2}{n-2}$, then the same argument shows

$$\left|g_\lambda^{\frac{n-2}{n+2}}-g^{\frac{n-2}{n+2}}\right|_{L^{\frac{2(n-1)(n+2)}{n(n-2)}}(\mathcal{B}^g_\lambda)}\le c(n)\,|g|^{\frac{2(n-2)}{n(n+2)}}_{L^{\frac{2(n-1)}{n}}\left(2\lambda e_1'-\mathcal{B}^g_\lambda\right)}\left|g_\lambda^{\frac{n-2}{n+2}}-g^{\frac{n-2}{n+2}}\right|_{L^{\frac{2(n-1)(n+2)}{n(n-2)}}(\mathcal{B}^g_\lambda)}.$$

Note that for $\xi \in \mathcal{B}_\lambda^g$, $g_\lambda(\xi) > g(\xi)$, hence

$$\int_{\mathcal{B}_\lambda^g} g(\xi)^{\frac{2(n-1)}{n}} d\xi \le \int_{\mathcal{B}_\lambda^g} g_\lambda(\xi)^{\frac{2(n-1)}{n}} d\xi \le \int_{\mathbb{R}^{n-1}\setminus H_\lambda} g(\xi)^{\frac{2(n-1)}{n}} d\xi < \infty.$$

When λ is large enough, it implies

$$\left| g_\lambda^{\frac{n-2}{n+2}} - g^{\frac{n-2}{n+2}} \right|_{L^{\frac{2(n-1)(n+2)}{n(n-2)}}\left(\mathcal{B}_\lambda^g\right)} \le \frac{1}{2} \left| g_\lambda^{\frac{n-2}{n+2}} - g^{\frac{n-2}{n+2}} \right|_{L^{\frac{2(n-1)(n+2)}{n(n-2)}}\left(\mathcal{B}_\lambda^g\right)}$$

and hence $\left| g_\lambda^{\frac{n-2}{n+2}} - g^{\frac{n-2}{n+2}} \right|_{L^{\frac{2(n-1)(n+2)}{n(n-2)}}\left(\mathcal{B}_\lambda^g\right)} = 0$, $\mathcal{B}_\lambda^g = \emptyset$. Let

$$\lambda_0 = \inf \left\{ \lambda > 0 : \mathcal{B}_{\lambda'}^g = \emptyset \text{ for all } \lambda' \ge \lambda \right\}.$$

We claim $\lambda_0 = 0$. Indeed if this is not the case, then $\lambda_0 > 0$. We may argue as in the Case 6.1 of the proof of Theorem 6.1 and get $g_{\lambda_0} = g$. In particular, this would imply $\int_{\mathbb{R}^{n-1}} g(\xi)^{\frac{2(n-1)}{n}} d\xi < \infty$, a contradiction. It follows that $\lambda_0 = 0$ and $g(\xi_1, \xi'') \ge g(-\xi_1, \xi'')$ for $\xi_1 < 0$. Since we may perform this process along any direction, we see g must be radial symmetric with respect to 0. Hence g_0 must be radial symmetric with respect to 0. For any $\zeta \in \mathbb{R}^{n-1}$, we may apply the argument to $g_0(\cdot + \zeta)$ and deduce that g_0 is also radial symmetric with respect to ζ, hence g_0 must be a constant function, so if f. But this contradicts with the fact that f satisfies the equation. □

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