

# AN INTEGRAL FORMULA FOR THE VOLUME ENTROPY WITH AN APPLICATION TO RIGIDITY

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## 1. INTRODUCTION

Let  $M^n$  be a compact Riemannian manifold and  $\pi : \widetilde{M} \rightarrow M$  its universal covering. The fundamental group  $G = \pi_1(M)$  acts on  $\widetilde{M}$  as isometries such that  $M = \widetilde{M}/G$ . Associated to  $\widetilde{M}$  are several asymptotic invariants. In this paper we are primarily concerned with the volume entropy  $v$  defined by

$$v = \lim_{r \rightarrow \infty} \frac{\ln \text{vol} B_{\widetilde{M}}(x, r)}{r},$$

where  $B_{\widetilde{M}}(x, r)$  is the ball of radius  $r$  centered at  $x$  in  $\widetilde{M}$ . It is proved by Manning [M] and Freire-Mañé [FM] that

- the limit exists and is independent of the center  $x \in \widetilde{M}$ ,
- $v \leq H$ , the topological entropy of the geodesic flow on  $M$ ,
- $v = H$  if  $M$  has no conjugate points.

There has been a lot of work on understanding the volume entropy of which we only mention the celebrated work of Besson, Courtois and Gallot [BCG1] where one can find other results and references. But the volume entropy still remains a subtle invariant. If  $M$  is negatively curved, it is better understood due to the existence of the so called Patterson-Sullivan measure on the ideal boundary. Let  $\partial\widetilde{M}$  be the ideal boundary of  $\widetilde{M}$  defined as equivalence classes of geodesic rays. We fix a base point  $o \in \widetilde{M}$  and for  $\xi \in \partial\widetilde{M}$  we denote  $B_\xi$  the associated Busemann function, i.e.

$$B_\xi(x) = \lim_{t \rightarrow \infty} d(x, \gamma(t)) - t,$$

where  $\gamma$  is the geodesic ray initiating from  $o$  representing  $\xi$ . It is well known that  $B_\xi$  is smooth and its gradient is of length one. The Patterson-Sullivan measure [P, S, K] is a family  $\{\nu_x : x \in \widetilde{M}\}$  of measures on  $\partial\widetilde{M}$  s.t.:

- for any pair  $x, y \in \widetilde{M}$ , the two measures  $\nu_x, \nu_y$  are equivalent with

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-h(B_\xi(x) - B_\xi(y))};$$

- for any  $g \in G$

$$g_*\nu_x = \nu_{gx}.$$

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The Patterson-Sullivan measure contains a lot of information and plays an important role in [BCG2]. Moreover, it is proved by Knieper, Ledrappier and Yue ([K, L2, Y1]) that the following integral formula for the volume entropy holds in terms of the Patterson-Sullivan measure:

$$(1.1) \quad v = \frac{1}{C} \int_M \left( \int_{\partial \widetilde{M}} \Delta B_\xi(x) d\nu_x(\xi) \right) dx,$$

where  $C = \int_M \nu_x(\partial \widetilde{M}) dx$ . (To interpret the formula properly, notice after integrating over  $\partial \widetilde{M}$  we get a function on  $\widetilde{M}$  which is  $G$ -invariant and hence descends to  $M$ .) This formula shows how  $v$  interacts with local geometry.

In this paper, we will extend the theory of Patterson-Sullivan measure to any manifold without the negative curvature assumption. More generally, let  $\pi : \widetilde{M} \rightarrow M$  be a regular Riemannian covering of a compact manifold  $M$  and  $G$  the discrete group of deck transformations. We will consider the Busemann compactification of  $\widetilde{M}$ , denoted by  $\widehat{M}$ . On the Busemann boundary  $\partial \widehat{M}$  we will construct Patterson-Sullivan measure which retains the essential features of the classical theory. Namely

**Theorem 1.** *There exists a probability measure  $\nu$  on the laminated space  $X_M = (\widetilde{M} \times \partial \widehat{M})/G$  such that for any continuous vector field  $Y$  on  $X_M$  which is  $C^1$  along the leaves,*

$$\int \operatorname{div}^{\mathcal{W}} Y d\nu = \nu \int \langle Y, \nabla^{\mathcal{W}} \xi \rangle d\nu,$$

where  $\operatorname{div}^{\mathcal{W}}$  and  $\nabla^{\mathcal{W}}$  are laminated divergence and gradient, respectively.

As an application of the above theorem, we will prove the following rigidity theorem.

**Theorem 2.** *Let  $M^n$  be a compact Riemannian manifold with  $\operatorname{Ric} \geq -(n-1)$  and  $\pi : \widetilde{M} \rightarrow M$  a regular covering. Then the volume entropy of  $\widetilde{M}$  satisfies  $v \leq (n-1)$  and equality holds iff  $M$  is hyperbolic.*

The inequality  $v \leq (n-1)$  is of course well known and follows easily from the volume comparison theorem. What is new is the rigidity part. To have some perspective on this result, recall another invariant: the bottom spectrum of the Laplacian on  $\widetilde{M}$ , denoted by  $\lambda_0$  and defined as

$$\lambda_0 = \inf_{f \in C_c^1(\widetilde{M})} \frac{\int_{\widetilde{M}} |\nabla f|^2}{\int_{\widetilde{M}} f^2}.$$

It is a well known fact that  $\lambda_0 \leq v^2/4$ . Therefore as an immediate corollary of Theorem 2 we have the following result previously proved by the second author [W].

**Corollary 1.** *Let  $(M^n, g)$  be a compact Riemannian manifold with  $\operatorname{Ric} \geq -(n-1)$  and  $\pi : \widetilde{M} \rightarrow M$  a regular covering. If  $\lambda_0 = (n-1)^2/4$ , then  $\widetilde{M}$  is isometric to the hyperbolic space  $\mathbb{H}^n$ .*

Clearly the asymptotic invariant  $v$  is much weaker than  $\lambda_0$ . It is somewhat surprising that we still have a rigidity theorem for  $v$ . If  $M$  is negatively curved, Theorem 2 is proved by Knieper [K] using (1.1). The proof in the general case is more subtle due to the fact the Busemann functions are only Lipschitz. In fact,

it is partly to prove this rigidity result that we are led to the construction of the measure  $\nu$  and the formula in Theorem 1.

We will also discuss the Kähler analogue of Theorem 2. Our method yields the following

**Theorem 3.** *Let  $M$  be a compact Kähler manifold with  $\dim_{\mathbb{C}} M = m$  and  $\pi : \widetilde{M} \rightarrow M$  a regular covering. If the bisectional curvature  $K_{\mathbb{C}} \geq -2$ , then the volume entropy  $v$  satisfies  $v \leq 2m$ . Moreover equality holds iff  $M$  is complex hyperbolic (normalized to have constant holomorphic sectional curvature  $-4$ ).*

To clarify the statement, the condition  $K_{\mathbb{C}} \geq -2$  means that for any two vectors  $X, Y$

$$R(X, Y, X, Y) + R(X, JY, X, JY) \geq -2 \left( |X|^2 |Y|^2 + \langle X, Y \rangle^2 + \langle X, JY \rangle^2 \right),$$

where  $J$  is the complex structure.

The paper is organized as follows. In Section 2, we discuss the Busemann compactification and construct the Patterson-Sullivan measure and prove Theorem 1. Theorem 2 will be proved in Section 3. We will discuss the Kähler case in Section 4.

## 2. CONSTRUCTION OF THE MEASURE

Let  $\pi : \widetilde{M} \rightarrow M$  be a regular Riemannian covering of a compact manifold:  $\widetilde{M}$  is a Riemannian manifold and there is a discrete group  $G$  of isometries of  $\widetilde{M}$  acting freely and such that the quotient  $M = \widetilde{M}/G$  is a compact manifold. The quotient metric makes  $M$  a compact Riemannian manifold. We recall the construction of the laminated space  $X_M$  ([L1]). Fix a point  $o \in \widetilde{M}$  and define, for  $x \in \widetilde{M}$  the function  $\xi_x(z)$  on  $\widetilde{M}$  by:

$$\xi_x(z) = d(x, z) - d(x, o).$$

The assignment  $x \mapsto \xi_x$  is continuous, one-to-one and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of  $\widetilde{M}$ . The Busemann compactification  $\widehat{M}$  of  $\widetilde{M}$  is the closure of  $\widetilde{M}$  for that topology. The space  $\widehat{M}$  is a compact separable space. The *Busemann boundary*  $\partial\widehat{M} := \widehat{M} \setminus \widetilde{M}$  is made of Lipschitz continuous functions  $\xi$  on  $\widetilde{M}$  such that  $\xi(o) = 0$ . Elements of  $\partial\widehat{M}$  are called *horofunctions*. Observe that we may extend by continuity the action of  $G$  from  $\widetilde{M}$  to  $\widehat{M}$ , in such a way that for  $\xi$  in  $\widehat{M}$  and  $g$  in  $G$ ,

$$g.\xi(z) = \xi(g^{-1}z) - \xi(g^{-1}o).$$

We define now the *horospheric suspension*  $X_M$  of  $M$  as the quotient of the space  $\widehat{M} \times \widehat{M}$  by the diagonal action of  $G$ . The projection onto the first component in  $\widehat{M} \times \widehat{M}$  factors into a projection from  $X_M$  to  $M$  so that the fibers are isometric to  $\widehat{M}$ . It is clear that the space  $X_M$  is metric compact. If  $M_0 \subset \widetilde{M}$  is a fundamental domain for  $M$ , one can represent  $X_M$  as  $M_0 \times \widehat{M}$  in a natural way.

To each point  $\xi \in \widehat{M}$  is associated the projection  $W_{\xi}$  of  $\widehat{M} \times \{\xi\}$ . As a subgroup of  $G$ , the stabilizer  $G_{\xi}$  of the point  $\xi$  acts discretely on  $\widehat{M}$  and the space  $W_{\xi}$  is homeomorphic to the quotient of  $\widehat{M}$  by  $G_{\xi}$ . We put on each  $W_{\xi}$  the smooth structure and the metric inherited from  $\widehat{M}$ . The manifold  $W_{\xi}$  and its metric vary

continuously on  $X_M$ . The collection of all  $W_\xi, \xi \in \widetilde{M}$  form a continuous lamination  $\mathcal{W}_M$  with leaves which are manifolds locally modeled on  $\widetilde{M}$ . In particular, it makes sense to differentiate along the leaves of the lamination and we denote  $\nabla^{\mathcal{W}}$  and  $\operatorname{div}^{\mathcal{W}}$  the associated gradient and divergence operators:  $\nabla^{\mathcal{W}}$  acts on continuous functions which are  $C^1$  along the leaves of  $\mathcal{W}$ ,  $\operatorname{div}^{\mathcal{W}}$  on continuous vector fields in  $T\mathcal{W}$  which are of class  $C^1$  along the leaves of  $\mathcal{W}$ . We want to construct a measure on  $X_M$  which would behave as the Knieper measure  $\nu = \int_M (\int_{\partial\widetilde{M}} d\nu_x(\xi)) dx$  in the negatively curved case. The construction follows Patterson's in the Fuchsian case.

Let  $v$  be the volume entropy of  $\widetilde{M}$

$$v = \lim_{r \rightarrow \infty} \frac{\ln \operatorname{vol} B_{\widetilde{M}}(x, r)}{r},$$

where  $B_{\widetilde{M}}(x, r)$  is the ball of radius  $r$  centered at  $x$  in  $\widetilde{M}$ .

Let us consider the Poincaré series of  $\widetilde{M}$ :

$$P(s) := \sum_{g \in G} e^{-sd(o, go)}.$$

**Proposition 1.** *The series  $P(s)$  converges for  $s > v$ , diverges to  $+\infty$  for  $s < v$ .*

*Proof.* This is classical and for completeness we recall the proof. Take  $M_0$  a fundamental domain in  $\widetilde{M}$  containing  $o$  in its interior, and positive constants  $d, D$  such that  $B(o, d) \subset M_0 \subset B(o, D)$ .

Define  $\pi(R) := \#\{g \in G : d(o, go) \leq R\}$ . We have  $\pi(R+S) \leq \pi(R+D)\pi(S+D)$ , which implies  $\pi(R+S+2D) \leq \pi(R+2D)\pi(S+2D)$ . It follows that the following limit exists

$$\lim_{R \rightarrow \infty} \frac{1}{R} \ln \pi(R+2D) = \inf \frac{1}{R} \ln \pi(R+2D).$$

The above limit is the critical exponent of the Poincaré series. Since

$$\pi(R) \operatorname{vol} B(o, d) \leq \operatorname{vol} B(o, R+d) \leq \pi(R+D) \operatorname{vol} M,$$

the above limit is also  $\lim_{R \rightarrow \infty} \frac{\ln \operatorname{vol} B_{\widetilde{M}}(x, R)}{R} = v$ .  $\square$

As in the classical case, a distinction has to be made between the case that  $P(s)$  diverges at  $v$  and the case that it converges. The following lemma is due to Patterson [P] (see also [N]).

**Lemma 1.** *There exists a function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is continuous, non-decreasing, and*

- (1) *the series  $P^*(s) := \sum_{g \in G} e^{-sd(o, go)} h(e^{d(o, go)})$  converges for  $s > v$  and diverges for  $s \leq v$ ;*
- (2) *if  $\varepsilon > 0$  is given there exists  $r_0$  s.t. for  $r > r_0, t > 1, h(rt) \leq t^\varepsilon h(r)$ .*

If  $P(s)$  diverges at  $v$  we will simply take  $h$  to be identically 1. As a consequence of property (2) above we note that for  $t$  in a bounded interval

$$\frac{h(e^{r+t})}{h(e^r)} \rightarrow 1$$

uniformly as  $r \rightarrow \infty$ .

For  $x \in \widetilde{M}, s > v$ , we define a finite measure  $\nu_{x,s}$  by setting, for all  $f$  continuous on  $\widetilde{M}$ ,

$$\int f(\xi) d\nu_{x,s}(\xi) := \frac{1}{P^*(s)} \sum_{g \in G} e^{-sd(x,go)} h\left(e^{d(x,go)}\right) f(\xi_{go}).$$

Clearly, for  $g \in G$ ,  $g_*\nu_{x,s} = \nu_{gx,s}$ , so that the measure  $\tilde{\nu}_s := \int \nu_{x,s} dx$  is  $G$ -invariant on  $\widetilde{M} \times \widehat{M}$ . We write  $\nu_s$  for the corresponding measure on  $X_M = \widetilde{M} \times \widehat{M}/G$ . Choose a sequence  $s_k > v$  and  $s_k \rightarrow v$  as  $k \rightarrow \infty$  such that the probability measures  $\nu_{o,s_k}$  converge towards some probability measure  $\nu_o$ . Since  $\lim_{k \rightarrow \infty} P^*(s_k) = \infty$ , the measure  $\nu_o$  is supported on  $\partial\widehat{M}$ .

**Proposition 2.** *For any  $x \in \widetilde{M}$ , the measures  $\nu_{x,s_k}$  converge to a measure  $\nu_x$  on  $\partial\widehat{M}$ . Moreover*

$$d\nu_x(\xi) = e^{-v\xi(x)} d\nu_o(\xi).$$

In particular, for any  $g \in G$  we have

$$d(g_*\nu_o)(\xi) = d\nu_{go}(\xi) = e^{-v\xi(go)} d\nu_o(\xi), \quad d(g_*\nu_x)(\xi) = d\nu_{gx}(\xi)$$

and the limit of the measures  $\nu_{s_k}$  on  $X_M$  is a measure  $\nu$  on  $X_M$  which can be written, in the  $M_0 \times \widehat{M}$  representation of  $X_M$ , as

$$(2.1) \quad \nu = e^{-v\xi(x)} d\nu_o(\xi) dx$$

*Proof.* Observe first that for a fixed  $x$ ,  $\nu_{x,s}(\widehat{M}) \leq e^{(v+s)d(o,x)}$ , so that the  $\nu_{x,s}$  form a bounded family of measures on  $\widehat{M}$ . Let  $f$  be a continuous function on  $\widehat{M}$ . We may write:

$$\begin{aligned} & \int f(\xi) e^{-v\xi(x)} d\nu_o(\xi) \\ &= \lim_{k \rightarrow \infty} \int f(\xi) e^{-v\xi(x)} d\nu_{o,s_k}(\xi) \\ &= \lim_{k \rightarrow \infty} \frac{1}{P^*(s_k)} \sum_{g \in G} f(\xi_{go}) e^{-v(d(x,go)-d(o,go))} e^{-s_k d(o,go)} h\left(e^{d(o,go)}\right) \\ &= \lim_{k \rightarrow \infty} \frac{1}{P^*(s_k)} \sum_{g \in G} f(\xi_{go}) e^{(s_k-v)\xi_{go}(x)} \frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right) \\ &= \lim_{k \rightarrow \infty} \left( \int f(\xi) e^{(s_k-v)\xi(x)} d\nu_{x,s_k}(\xi) + \varepsilon_k \right), \end{aligned}$$

where

$$\varepsilon_k = \frac{1}{P^*(s_k)} \sum_{g \in G} f(x, \xi_{go}) e^{(s_k-v)\xi_{go}(x)} \left( \frac{h\left(e^{d(o,go)}\right)}{h\left(e^{d(x,go)}\right)} - 1 \right) e^{-s_k d(x,go)} h\left(e^{d(x,go)}\right).$$

Suppose  $\lim \varepsilon_k = 0$ . Then, for a fixed  $x$ ,  $e^{(s_k-v)\xi(x)}$  converges to 1 and therefore, the limit exists and is  $\int f e^{-v\xi(x)} d\nu_o$ , as claimed. It only remains to show that

$\lim \varepsilon_k = 0$ . Indeed, for any  $\delta > 0$  and any  $x$ , there exists a finite set  $E \subset G$  s.t. for any  $g \in G \setminus E$

$$\left| \frac{h(e^{d(o,go)})}{h(e^{d(x,go)})} - 1 \right| < \delta.$$

Then

$$\begin{aligned} |\varepsilon_k| &\leq \frac{1}{P^*(s_k)} \sum_{g \in E} f(\xi_{go}) e^{(s_k-v)\xi_{go}(x)} \left| \frac{h(e^{d(o,go)})}{h(e^{d(x,go)})} - 1 \right| e^{-s_k d(x,go)} h(e^{d(x,go)}) \\ &\quad + \delta \frac{1}{P^*(s_k)} \sum_{g \in G \setminus E} f(\xi_{go}) e^{(s_k-v)\xi_{go}(x)} e^{-s_k d(x,go)} h(e^{d(x,go)}) \\ &\leq \frac{1}{P^*(s_k)} \sum_{g \in E} f(\xi_{go}) e^{(s_k-v)\xi_{go}(x)} \left| \frac{h(e^{d(o,go)})}{h(e^{d(x,go)})} - 1 \right| e^{-s_k d(x,go)} h(e^{d(x,go)}) \\ &\quad + \delta \int f(\xi) e^{(s_k-v)\xi(x)} d\nu_{x, s_k}(\xi). \end{aligned}$$

Taking limit yields

$$\lim_{k \rightarrow \infty} |\varepsilon_k| \leq \delta \|f\|_{\infty} e^{v d(o,x)}.$$

Therefore  $\lim \varepsilon_k = 0$ .  $\square$

We can integrate by parts along each  $M_0 \times \{\xi\}$ , for  $\nu_o$ -almost every  $\xi$ , and get for any function  $f$  which is  $C^2$  along the leaves of the lamination  $\mathcal{W}$  and has a support contained in  $M_0 \times \widehat{M}$ :

$$\begin{aligned} \int \Delta f d\nu &= \int \left( \int_{M_0} \Delta^{\mathcal{W}} f e^{-v\xi(x)} dx \right) d\nu_o(\xi) \\ &= v \int \left( \int_{M_0} \langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \xi \rangle e^{-v\xi(x)} dx \right) d\nu_o(\xi) \\ &= v \int \langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \xi \rangle d\nu. \end{aligned}$$

The integral makes sense because  $\nabla^{\mathcal{W}} \xi$  is defined Lebesgue almost everywhere on the leaves and because, by (2.1), the measure  $\nu$  has absolutely continuous conditional measures along the leaves  $\mathcal{W}$ . By choosing the fundamental domain  $M_0$ , we get the same formula for any function which is  $C^2$  along the leaves of the lamination  $\mathcal{W}$  and has a small support. Using a partition of unity on  $M$ , we see that for all functions on  $X_M$  which are  $C^2$  along the leaves of the lamination  $\mathcal{W}$ , we have:

$$\int \Delta f d\nu = v \int \langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \xi \rangle d\nu.$$

In the same way, one gets for all continuous functions  $f_1, f_2$  which are smooth along the leaves of the lamination  $\mathcal{W}$ :

$$\int \operatorname{div}^{\mathcal{W}}(f_1 \nabla^{\mathcal{W}} f_2) d\nu = v \int f_1 \langle \nabla^{\mathcal{W}} f_2, \nabla^{\mathcal{W}} \xi \rangle d\nu.$$

By approximation, we have for all  $\mathcal{W}$  vector field  $Y$  which is  $C^1$  along the leaves and globally continuous,

$$(2.2) \quad \int \operatorname{div}^{\mathcal{W}} Y d\nu = v \int \langle Y, \nabla^{\mathcal{W}} \xi \rangle d\nu.$$

Since the measure  $\nu$  gives full measure to  $\widetilde{M} \times \partial\widetilde{M}$ , Theorem 1 is proven.

### 3. THE RIGIDITY THEOREM

In this section we prove the rigidity theorem.

**Theorem 4.** *Let  $M^n$  be a compact Riemannian manifold with  $\operatorname{Ric} \geq -(n-1)$  and  $\pi : \widetilde{M} \rightarrow M$  its universal covering. Then the volume entropy of  $\widetilde{M}$  satisfies  $v \leq (n-1)$  and equality holds iff  $M$  is hyperbolic.*

Observe that this proves Theorem 2, since the volume entropy of the universal covering is not smaller than the volume entropy of an intermediate covering space.

First we collect some general facts about Busemann functions, see e.g. [SY, Pe]. Suppose  $\xi \in \partial\widetilde{M}$  is the limit of  $\{a_k\} \subset \widetilde{M}$ , i.e.

$$(3.1) \quad \xi(x) = \lim_{k \rightarrow \infty} f_k(x),$$

where  $f_k(x) = d(x, a_k) - d(o, a_k)$ . The convergence is uniform over compact sets.

We fix  $p$  and for each  $k$  let  $\gamma_k$  be a minimizing geodesic from  $p$  to  $a_k$ . Passing to a subsequence, we can assume that  $\gamma_k$  converges to a geodesic ray  $\gamma$  starting from  $p$ .

**Lemma 2.** *We have*

- (1)  $\xi \circ \gamma(s) = \xi(p) - s$  for  $s \geq 0$ ;
- (2)  $\xi(x) \leq \xi(p) + d(x, \gamma(s)) - s$  for  $s \geq 0$ ;
- (3)  $\xi(x) \leq \xi(p) + b_\gamma(x)$ .

*Proof.* For any  $s > 0$  and  $\varepsilon > 0$  we have  $d(\gamma_k(s), \gamma(s)) \leq \varepsilon$  for  $k$  sufficiently large. Then

$$\begin{aligned} f_k \circ \gamma(s) - f_k(p) &= d(\gamma(s), a_k) - d(p, a_k) \\ &= d(\gamma(s), a_k) - d(\gamma_k(s), a_k) + d(\gamma_k(s), a_k) - d(p, a_k) \\ &\leq d(\gamma(s), \gamma_k(s)) + d(\gamma_k(s), a_k) - d(p, a_k) \\ &= d(\gamma(s), \gamma_k(s)) - s \\ &\leq \varepsilon - s. \end{aligned}$$

Taking limit yields  $\xi \circ \gamma(s) - \xi(p) \leq \varepsilon - s$ . Hence  $\xi \circ \gamma(s) \leq \xi(p) - s$ . On the other hand we have the reversed inequality  $\xi \circ \gamma(s) \geq \xi(p) - s$  as  $\xi$  is Lipschitz with Lipschitz constant 1.

To prove the second part, we have for  $s \geq 0$

$$\begin{aligned} f_k(x) &= d(x, a_k) - d(o, a_k) \\ &\leq d(x, \gamma(s)) + d(a_k, \gamma(s)) - d(o, a_k). \end{aligned}$$

Letting  $k \rightarrow \infty$  yields

$$\begin{aligned} \xi(x) &\leq d(x, \gamma(s)) + \xi \circ \gamma(s) \\ &= d(x, \gamma(s)) - s + \xi(p). \end{aligned}$$

Taking limit as  $s \rightarrow \infty$  yields the third part.  $\square$

It follows that if  $\xi$  is differentiable at  $x$ , then  $|\nabla\xi(x)| = 1$ . Therefore  $|\nabla\xi| = 1$  almost everywhere on  $\widetilde{M}$ . Moreover we have

**Proposition 3.** *For any  $\xi \in \partial\widehat{M}$  we have  $\Delta(e^{-(n-1)\xi}) \geq 0$  in the sense of distribution.*

*Proof.* Firstly, it is well known that  $\Delta\xi \leq n - 1$  in the distribution sense for any  $\xi \in \partial\widehat{M}$ . Indeed, suppose  $\xi$  is given as in formula (3.1). By the Laplacian comparison theorem

$$\Delta f_k(x) \leq (n-1) \frac{\cosh(d(x, a_k))}{\sinh(d(x, a_k))}$$

in the distribution sense. Taking limit then yields  $\Delta\xi \leq n - 1$ . Therefore

$$\begin{aligned} \Delta(e^{-(n-1)\xi}) &= -(n-1)e^{-(n-1)\xi} (\Delta\xi - (n-1)|\nabla\xi|^2) \\ &= -(n-1)e^{-(n-1)\xi} (\Delta\xi - (n-1)) \\ &\geq 0, \end{aligned}$$

all understood in the sense of distribution.  $\square$

Let  $p_t(x, y)$  be the heat kernel on  $\widetilde{M}$ . For any function  $f$  on  $\widetilde{M}$  we define

$$P_t f(x) = \int_{\widetilde{M}} p_t(x, y) f(y) dy.$$

We have  $P_t(g \cdot f) = g \cdot P_t f$  for any  $g \in G$ .

We now proceed to prove Theorem 4. We consider the following vector field on  $\widetilde{M} \times \widehat{M}$

$$Y_t(x, \xi) = \nabla(P_t \xi)(x).$$

It is easy to see that  $Y_t$  descends to  $X_M$ , i.e. for any  $g \in G$  we have  $Y_t(gx, g \cdot \xi) = g_* Y_t(x, \xi)$ . By Theorem 1

$$\begin{aligned} v \int_{X_M} \langle \nabla^w \xi, Y_t \rangle d\nu &= \int_{X_M} \operatorname{div}^w Y_t d\nu \\ &= \int_M \left( \int_{\partial\widehat{M}} \operatorname{div}^w Y_t e^{-v\xi(x)} d\nu_o(\xi) \right) dx \\ &= \int_M \left( \int_{\partial\widehat{M}} \left( \operatorname{div}^w (Y_t e^{-v\xi(x)}) + v \langle \nabla^w \xi, Y_t \rangle e^{-v\xi(x)} \right) d\nu_o(\xi) \right) dx \\ &= \int_M \left( \int_{\partial\widehat{M}} \operatorname{div}^w (Y_t e^{-v\xi(x)}) d\nu_o(\xi) \right) dx + v \int_{X_M} \langle \nabla^w \xi, Y_t \rangle d\nu, \end{aligned}$$

whence

$$\int_M \left( \int_{\partial\widehat{M}} \operatorname{div}^w (Y_t e^{-v\xi(x)}) d\nu_o(\xi) \right) dx = 0.$$

We now cover  $M$  by finitely many open sets  $\{U_i : 1 \leq i \leq k\}$  s.t. each  $U_i$  is so small that  $\pi^{-1}(U_i)$  is the disjoint union of open sets each diffeomorphic to  $U_i$  via  $\pi$ . Let

$\{\chi_i\}$  be a partition of unity subordinating to  $\{U_i\}$ . For each  $U_i$  let  $\tilde{U}_i$  be one of the components of  $\pi^{-1}(U_i)$  and let  $\tilde{\chi}_i$  be the lifting of  $\chi_i$  to  $\tilde{U}_i$ . Then

$$\begin{aligned}
0 &= \int_M \left( \int_{\partial\widehat{M}} \operatorname{div}^w \left( Y_t e^{-v\xi(x)} \right) d\nu_o(\xi) \right) dx \\
&= \sum_i \int_M \left( \int_{\partial\widehat{M}} \operatorname{div}^w \left( Y_t e^{-v\xi(x)} \right) \chi_i \circ \pi(x) d\nu_o(\xi) \right) dx \\
&= \sum_i \int_{U_i} \left( \int_{\partial\widehat{M}} \operatorname{div}^w \left( Y_t e^{-v\xi(x)} \right) \chi_i \circ \pi(x) d\nu_o(\xi) \right) dx \\
&= \sum_i \int_{\tilde{U}_i} \left( \int_{\partial\widehat{M}} \operatorname{div}^w \left( Y_t e^{-v\xi(x)} \right) \chi_i \circ \pi(x) d\nu_o(\xi) \right) dx \\
&= \sum_i \int_{\partial\widehat{M}} \left( \int_{\tilde{U}_i} \operatorname{div}^w \left( Y_t e^{-v\xi(x)} \right) \tilde{\chi}_i dx \right) d\nu_o(\xi) \\
&= - \sum_i \int_{\partial\widehat{M}} \left( \int_{\tilde{U}_i} \langle Y_t, \nabla \tilde{\chi}_i \rangle e^{-v\xi(x)} dx \right) d\nu_o(\xi).
\end{aligned}$$

Letting  $t \rightarrow 0$  yields

$$\sum_i \int_{\partial\widehat{M}} \left( \int_{\tilde{U}_i} \langle \nabla \xi, \nabla \tilde{\chi}_i \rangle e^{-v\xi(x)} dx \right) d\nu_o(\xi) = 0.$$

Integrating by parts again, we obtain

$$\begin{aligned}
&\sum_i \int_{\partial\widehat{M}} \left( \int_{\tilde{U}_i} e^{-v\xi(x)} \Delta \tilde{\chi}_i dx \right) d\nu_o(\xi) \\
&= - \sum_i \int_{\partial\widehat{M}} \left( \int_{\tilde{U}_i} \langle \nabla \left( e^{-v\xi(x)} \right), \nabla \tilde{\chi}_i \rangle dx \right) d\nu_o(\xi) \\
&= v \sum_i \int_{\partial\widehat{M}} \left( \int_{\tilde{U}_i} \langle \nabla \xi, \nabla \tilde{\chi}_i \rangle e^{-v\xi(x)} dx \right) d\nu_o(\xi).
\end{aligned}$$

Therefore

$$\int_{\partial\widehat{M}} \sum_i \left( \int_{\tilde{U}_i} e^{-v\xi(x)} \Delta \tilde{\chi}_i dx \right) d\nu_o(\xi) = 0.$$

We now assume  $v = n-1$ . By Proposition 3  $\Delta e^{-v\xi(x)} \geq 0$  in the sense of distribution for all  $\xi \in \partial\widehat{M}$  and hence  $\int_{\tilde{U}_i} e^{-v\xi(x)} \Delta \tilde{\chi}_i dx \geq 0$  for all  $i$ . Therefore we conclude for  $\nu_o$ -a.e.  $\xi \in \partial\widehat{M}$

$$\int_{\tilde{U}_i} e^{-v\xi(x)} \Delta \tilde{\chi}_i dx = 0$$

for all  $i$ . In this discussion we can replace  $\tilde{U}_i$  by  $g\tilde{U}_i$  and  $\tilde{\chi}_i$  by  $g \cdot \tilde{\chi}_i$  for any  $g \in G$ . Since  $G$  is countable we conclude for  $\nu_o$ -a.e.  $\xi \in \widehat{M}$

$$\int_{g\tilde{U}_i} e^{-v\xi(x)} \Delta (g \cdot \tilde{\chi}_i) dx = 0$$

for all  $i$  and  $g \in G$ .

We claim that  $\Delta e^{-v\xi(x)} = 0$  in the sense of distribution. Indeed, denote the distribution  $\Delta e^{-v\xi(x)}$  simply by  $T$ , i.e.  $T(f) = \int_{\widehat{M}} e^{-v\xi(x)} \Delta f(x) dx$  for  $f \in C_c^\infty(\widehat{M})$ .

We know that  $T(f) \geq 0$  if  $f \geq 0$ . We observe that  $\{g \cdot \tilde{\chi}_i : 1 \leq i \leq k, g \in G\}$  is a partition of unity on  $\widetilde{M}$  subordinating to the open cover  $\{g\tilde{U}_i : 1 \leq i \leq k, g \in G\}$ .

Hence for any  $f \in C_c^\infty(\widetilde{M})$  with  $f \geq 0$  we have

$$0 \leq f \leq C \sum_{g\tilde{U}_i \cap \text{spt} f \neq \emptyset} g \cdot \tilde{\chi}_i,$$

with  $C = \sup f$ . Notice that the right hand side is a finite sum as the support  $\text{spt} f$  is compact. Then

$$\begin{aligned} 0 \leq T(f) &\leq T\left(C \sum_{g\tilde{U}_i \cap \text{spt} f} g \cdot \tilde{\chi}_i\right) \\ &= C \sum_{g\tilde{U}_i \cap \text{spt} f \neq \emptyset} T(g \cdot \tilde{\chi}_i) \\ &= 0. \end{aligned}$$

Hence  $T(f) = 0$ , i.e.  $\Delta e^{-v\xi(x)} = 0$  in the sense of distribution. By elliptic regularity  $\phi = e^{-v\xi(x)}$  is then a smooth harmonic function and obviously  $|\nabla \log \phi| = n - 1$ . The rigidity now follows from the following result.

**Theorem 5.** *Let  $N^n$  be a complete, simply connected Riemannian manifold s.t.*

- (1)  $\text{Ric} \geq -(n - 1)$ ;
- (2) *the sectional curvature is bounded.*

*Suppose that there is a positive harmonic function  $\phi$  on  $N$  s.t.  $|\nabla \log \phi| = n - 1$ . Then  $N$  is isometric to the hyperbolic space  $\mathbb{H}^n$ .*

**Remark 1.** *Without assuming bounded sectional curvature, the second author [W] proved that  $N$  is isometric to the hyperbolic space  $\mathbb{H}^n$  provided that there are two such special harmonic functions. We thank Ovidiu Munteanu for pointing out that one such special harmonic function is enough if the sectional curvature is bounded.*

*Proof.* The first step is to show that  $\phi$  satisfies an over-determined system which then leads to the splitting of  $N$  as a warped product. This is standard and we outline the argument. Let  $f = \log \phi$ . We have  $\Delta f = -|\nabla f|^2 = -(n - 1)^2$ . Since

$$D^2 f(\nabla f, \nabla f) = \frac{1}{2} \left\langle \nabla f, \nabla |\nabla f|^2 \right\rangle = 0,$$

we have by Cauchy-Schwarz

$$(3.2) \quad |D^2 f|^2 \geq \frac{(\Delta f)^2}{n - 1} = (n - 1)^3$$

with equality iff

$$D^2 f = -(n - 1) \left[ g - \frac{1}{(n - 1)^2} df \otimes df \right]$$

On the other hand, by the Bochner formula, we have

$$\begin{aligned} 0 &= \frac{1}{2} \Delta |\nabla f|^2 \\ &= |D^2 f|^2 + \langle \nabla f, \nabla \Delta f \rangle + Ric(\nabla f, \nabla f) \\ &\geq |D^2 f|^2 - (n-1) |\nabla f|^2 \\ &= |D^2 f|^2 - (n-1)^3, \end{aligned}$$

i.e.  $|D^2 f|^2 \leq (n-1)^3$ . This show that (3.2) is in fact an equality. Therefore we have  $D^2 f = -(n-1) \left[ g - \frac{1}{(n-1)^2} df \otimes df \right]$ . From this one can show that  $N = \mathbb{R} \times \Sigma^{n-1}$  with the metric  $g = dt^2 + e^{2t} h$ , where  $h$  is a Riemannian metric on  $\Sigma$ . For more detail, cf. [LW3].

For any  $p \in \Sigma$  let  $\{e_i\}$  be an orthogonal basis on  $(T_p \Sigma, h)$ . By a simple calculation using the Gauss equation the curvature of  $N$  is given by

$$R(e^{-t} e_i, e^{-t} e_j, e^{-t} e_i, e^{-t} e_j) = e^{-4t} R^h(e_i, e_j, e_i, e_j) - (\delta_{ii} \delta_{jj} - \delta_{ij}^2),$$

where  $R^h$  is the curvature tensor of  $(\Sigma, h)$ . Since  $N$  has bounded sectional curvature, the left hand side is bounded in  $t$ . Therefore  $R^h = 0$ , i.e.  $\Sigma$  is flat. Since  $\Sigma$  is also connected as  $N$  is simply connected, it is isometric to  $\mathbb{R}^{n-1}$ . It follows that  $N$  is the hyperbolic space.  $\square$

#### 4. THE KÄHLER CASE

In this Section, we discuss the Kähler case.

**Theorem 6.** *Let  $M$  be a compact Kähler manifold with  $\dim_{\mathbb{C}} M = m$  and  $\pi : \widetilde{M} \rightarrow M$  the universal covering. If the bisectional curvature  $K_{\mathbb{C}} \geq -2$ , then the volume entropy  $v$  satisfies  $v \leq 2m$ . Moreover equality holds iff  $M$  is complex hyperbolic (normalized to have constant holomorphic sectional curvature  $-4$ ).*

The inequality follows from the comparison theorem in [LW2]. Indeed, under the curvature assumption  $K_{\mathbb{C}} \geq -2$ , Li and J. Wang [LW2] proved

$$\text{vol} B_{\widetilde{M}}(x, r) \leq V_{\mathbb{C}\mathbb{H}^m}(r) = \tau_{2m-1} \int_0^r \sinh(2t) \sinh^{2(m-1)}(t) dt,$$

where  $\tau_{2m-1}$  is the volume of the unit sphere in  $\mathbb{R}^{2m}$ . It follows that  $v \leq 2m$ . Another consequence of the comparison theorem is that for  $\xi \in \partial \widehat{M}$

$$\Delta \xi \leq 2m$$

in the distribution sense. It follows as in the Riemannian case that  $\Delta e^{-2m\xi} \geq 0$  in the distribution sense.

We now assume that  $v = 2m$ . By the argument in Section 3 we conclude that  $\xi$  is smooth and

$$\Delta \xi = 2m, |\nabla \xi| = 1$$

for  $\nu_o$ -a.e.  $\xi \in \partial \widehat{M}$ . Take such a function  $\xi$ . We choose a local unitary frame  $\{X_i, \overline{X}_i\}$ .

**Lemma 3.** *We have*

$$\xi_{i\bar{j}} = \delta_{ij}, \xi_{ij} = -2\xi_i \xi_{\bar{j}}.$$

*Proof.* Without loss of generality we can assume that  $X_1 = (\nabla\xi - \sqrt{-1}J\nabla\xi) / \sqrt{2}$ . Therefore

$$\xi_1 = \frac{1}{\sqrt{2}}, \xi_i = 0 \text{ for } i \geq 2.$$

Suppose  $\xi$  is given as in (3.1). Let  $p \in \widetilde{M}$  and we use the construction preceding Lemma 2. By the second part of that Lemma we see that for any  $s > 0$  the function  $u_s(x) = \xi(p) + d(x, \gamma(s)) - s$  is a support function for  $\xi$  from above at  $p$ . Moreover  $u_s$  is clearly smooth at  $p$ . Therefore we have at  $p$

$$D^2\xi \leq D^2u_s.$$

By the comparison theorem in [LW2] we have

$$\begin{aligned} \xi_{1\bar{1}} &\leq (u_s)_{1\bar{1}} \leq \frac{\cosh 2s}{\sinh 2s}, \\ \xi_{i\bar{i}} &\leq (u_s)_{i\bar{i}} \leq \frac{\cosh s}{\sinh s} \text{ for } i \geq 2. \end{aligned}$$

Taking limit as  $s \rightarrow \infty$  yields  $\xi_{i\bar{i}} \leq 1$ . On the other hand we have  $\sum_{i=1}^m \xi_{i\bar{i}} = \frac{1}{2}\Delta\xi = m$ . Therefore we must have

$$(4.1) \quad \xi_{i\bar{i}} = 1.$$

By the Bochner formula we have

$$\begin{aligned} 0 &= \frac{1}{2}\Delta|\nabla\xi|^2 = |D^2\xi|^2 + \langle \nabla\xi, \nabla\Delta\xi \rangle + Ric(\nabla\xi, \nabla\xi) \\ &\geq |D^2\xi|^2 - 2(m+1). \end{aligned}$$

Therefore

$$(4.2) \quad \left| \xi_{i\bar{j}} \right|^2 + |\xi_{ij}|^2 \leq m+1.$$

We have

$$0 \leq |\xi_{ij} + 2\xi_i\xi_j|^2 = |\xi_{ij}|^2 + 2\xi_{ij}\xi_{i\bar{j}}\xi_j + 2\xi_{i\bar{j}}\xi_i\xi_j + 1.$$

By differentiating  $\xi_j\xi_{\bar{j}} = \frac{1}{2}$  we obtain  $\xi_{ij}\xi_{\bar{j}} + \xi_j\xi_{i\bar{j}} = 0$ ,  $\xi_{i\bar{j}}\xi_{\bar{j}} + \xi_j\xi_{i\bar{j}} = 0$ . Hence from the previous inequality we obtain

$$(4.3) \quad |\xi_{ij}|^2 \geq 4\xi_{i\bar{j}}\xi_{i\bar{i}}\xi_j - 1 = 2\xi_{1\bar{1}} - 1.$$

On the other hand

$$\left| \xi_{i\bar{j}} \right|^2 \geq \frac{1}{m} \left( \frac{1}{2}\Delta\xi \right)^2 = m$$

Combining this inequality with (4.2) and (4.3) yields

$$\xi_{1\bar{1}} \leq 1.$$

However we already proved that equality holds (4.1). By inspecting the argument we conclude that  $\xi$  satisfies the following over-determined system

$$\xi_{i\bar{j}} = \delta_{ij}, \xi_{ij} = -2\xi_i\xi_j.$$

□

With such a function, it is proved by Li and J. Wang [LW1] that  $\widetilde{M}$  is isometric to  $\mathbb{R} \times N^{2m-1}$  with the metric

$$g = dt^2 + e^{-4t}\theta_0^2 + e^{-2t} \sum_{i=1}^{2(m-1)} \theta_i^2,$$

where  $\{\theta_0, \theta_1, \dots, \theta_{2(m-1)}\}$  is an orthonormal frame for  $T^*N$ . Moreover, since our  $\widetilde{M}$  is simply connected and has bounded curvature,  $N$  is isometric to the Heisenberg group by their theorem. Therefore  $\widetilde{M}$  is isometric to the complex hyperbolic space  $\mathbb{C}\mathbb{H}^m$ .

We close with some remarks. An obvious question is whether this theorem remains true if the curvature condition is relaxed to  $Ric \geq -2(m+1)$ . This seems a very subtle question. It is quite unlikely that the comparison theorem for Kähler manifolds could still hold in this case. On the other hand it is conceivable Theorem 6 will remain valid due to some global reason. This hope is partly based on the recent work of Munteanu [Mu] in which a sharp estimate for the Kaimanovich entropy is derived under the condition  $Ric \geq -2(m+1)$ .

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