MTH 930 HOMEWORK ASSIGNMENT 1

DUE SEPT. 13 IN CLASS

(1) Let $(\mathbb{S}^n, g_0)$ be the standard sphere. Let $\phi : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection from the north pole $N = (0, \cdots, 0, 1)$, i.e. for $\xi \in \mathbb{S}^n \backslash \{N\}$ the line joining N and ξ intersects the equator hyperplane $\xi_{n+1} = 0$ at $\phi(\xi)$.
- Find explicitly ϕ and ϕ^{-1}.
- Show that
$$\left(\phi^{-1}\right)^* g_0 = \frac{4}{\left(1 + |x|^2\right)^2} dx^2.$$

(2) In class we defined the hyperbolic space as
$$\mathbb{H}^n = \left\{ \xi \in \mathbb{R}^{n+1} : \xi_{n+1} = \sqrt{1 + \sum_{i=1}^n \xi_i^2} \right\}$$
in the Minkowski space $\mathbb{R}^{n,1}$. Define the stereographic projection $\phi : \mathbb{H}^n \to B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ as following: $\phi(\xi)$ is the intersection point of the line joining ξ and $-e_{n+1}$ with the hyperplane $\xi_{n+1} = 0$.
- Find explicitly ϕ and ϕ^{-1} and show that $\phi : \mathbb{H}^n \to B^n$ is a diffeomorphism.
- Show that
$$\left(\phi^{-1}\right)^* g_0 = \frac{4}{\left(1 - |x|^2\right)^2} dx^2.$$

B^n with the metric $\frac{4}{\left(1-|x|^2\right)^2} dx^2$ is called the conformal ball model of the hyperbolic space.

(3) Let $\pi : E \to M$ be a vector bundle over a smooth manifold. Recall that a connection on E is a map
$$\nabla : \Gamma(M) \times \Gamma(E) \to \Gamma(E)$$
satisfying
- for any $X_1, X_2 \in \Gamma(M), f_1, f_2 \in C^\infty(M)$ and $\sigma \in \Gamma(E)$
$$\nabla_{f_1 X_1 + f_2 X_2} \sigma = f_1 \nabla_{X_1} \sigma + f_2 \nabla_{X_2} \sigma.$$
- $\nabla_X (f\sigma) = X f \sigma + f \nabla_X \sigma$.
- $\nabla_X (\sigma_1 + \sigma_2) = \nabla_X \sigma_1 + \nabla_X \sigma_2$.

Prove

(a) If X vanishes at a point p, so does $\nabla_X \sigma$. Therefore for $v \in T_p M, \sigma \in \Gamma(E)$, we can define $\nabla_v \sigma \in E_p$.

(b) If σ vanishes in a neighborhood of p, then $\nabla_v \sigma = 0$ for any $v \in T_pM$. Therefore we can define $\nabla_v \sigma$ if σ is only a smooth section on a neighborhood of p.

(4) Exercise 1.6.24 on Page 38 of Petersen's book.

(5) Let g be a Riemannian metric on M and $\widetilde{g} = e^{2f} g$ another metric conformal to g, where f is a smooth function on M. Given the relation between the Levi-Civita connection ∇ of g and the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$.