MTH 930 HOMEWORK ASSIGNMENT 2

DUE SEPT. 22 IN CLASS

In all the problems, we work on a Riemannian manifold (M^n, g).

(1) Let X be a vector field. Its divergence is defined by $\mathrm{div}X = tr\nabla X$. For a smooth function f its gradient is the vector field ∇f s.t. $\langle \nabla f, \cdot \rangle = df$. The Laplace operator is defind by $\Delta f = \mathrm{div}\nabla f$. Prove the following formulas in local coordinates.
   - $\nabla f = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$.
   - If we write $X = a^i \frac{\partial}{\partial x_i}$, then
   $$\mathrm{div}X = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x_i} \left(\sqrt{G} a^i \right),$$
   where $G = \det [g_{ij}]$.
   - $$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x_i} \left(\sqrt{G} g^{ij} \frac{\partial f}{\partial x_j} \right).$$

(2) Exercises 2.65 on P75 of Gallot-Hulin-Lafontaine.

(3) Suppose M is oriented. Then there is a natural volume form Ω on M s.t. for any positive orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM
$$\Omega (e_1, \cdots, e_n) = 1.$$
   - Prove that in a positive local chart
   $$\Omega = \sqrt{G} dx_1 \wedge \cdots \wedge dx_n.$$
   - Prove that for any C^1 vector field X
   $$\mathrm{div}X = di_X\Omega / \Omega,$$

(4) Consider the Riemannian metric $g = f(r)^2 dx^2$ on some open set $U \subset \mathbb{R}^n \setminus \{0\}$, where f is a smooth and positive function of $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Calculate the connection and curvature explicitly. Check your anwer on the following examples: $f = \frac{2}{1+r^2}$ (the sphere) and $f = \frac{2}{1-r^2}$ (the hyperbolic space).

(5) Suppose $\phi : (M, g) \to (M', g')$ is an isometry. For vector fields $X, Y, \cdots$ on M, let $X' = \phi_* X, Y' = \phi_* Y, \cdots$ be the corresponding vector fields on M'. Prove
   - $\phi_* (\nabla_X Y) = \nabla'_{X'} Y'$.
   - $R' (X', Y', Z', W') = R (X, Y, Z, W) \circ \phi$.