MTH 930 HOMEWORK ASSIGNMENT 3

DUE SEPT. 29 IN CLASS

In all the problems, we work on a Riemannian manifold (M^n, g).

(1) Exercises 2.65 on P75 of Gallot-Hulin-Lafontaine. (If you didn't do it on HW2)

(2) Let $f \in C^\infty(M)$ and $\gamma : [0, l] \to M$ a geodesic. Show that
$$\frac{d^2}{dt^2} f \circ \gamma(t) = D^2 f\left(\dot{\gamma}(t), \dot{\gamma}(t)\right).$$

(3) Let Σ^k be a smooth manifold and $\phi : \Sigma^k \to M$ a smooth map. Let $\Gamma(\phi^* TM)$ be the space of vector fields along ϕ. We defined the induced connection ∇ on $\Gamma(\phi^* TM)$ (vector fields along ϕ) satisfying

- for any vector field V along ϕ and any $f \in C^\infty(\Sigma)$
$$\nabla(fV) = df \otimes V + f \nabla V.$$
- for any vector fields V, W along ϕ and for any $\xi \in T\Sigma$
$$\xi \langle V, W \rangle = \langle \nabla_\xi V, W \rangle + \langle V, \nabla_\xi W \rangle.$$
- for any $X, Y \in \Gamma(T\Sigma)$
$$\nabla_X(\phi_* Y) - \nabla_Y(\phi_* X) = \phi_*[X, Y].$$

Prove that for $X, Y \in \Gamma(T\Sigma)$ and for any vector field V along ϕ
$$-\nabla_X \nabla_Y V + \nabla_Y \nabla_X V + \nabla_{[X,Y]} V = R(\phi_* X, \phi_* Y) V,$$
where R is the curvature tensor of (M^n, g).

(4) (Gauss equation) Suppose $\Sigma^k \subset M$ is a submanifold. Recall that for $X, Y \in \Gamma(T\Sigma)$ we have the orthogonal decomposition $\nabla_X Y = \nabla^\Sigma_X Y + \Pi(X, Y)$, where $\nabla^\Sigma_X Y$ is the Levi-Civita connection of $(\Sigma, g|_\Sigma)$ and Π is the second fundamental form of Σ in M. Prove that for $X, Y, Z, W \in \Gamma(T\Sigma)$
$$R(X, Y, Z, W) = R^\Sigma(X, Y, Z, W) - \langle \Pi(X, Z), \Pi(Y, W) \rangle + \langle \Pi(X, W), \Pi(Y, Z) \rangle.$$
Calculate the curvature of S^n using the above formula.

(5) Prove the hyperbolic space is complete. (You can use any of the equivalent models.)

(6) Let (M^n, g) be a complete Riemannian manifold and X a bounded vector field (i.e. there exists $C > 0$ s.t. $|X| \leq C$ on M). Prove that X generates a global flow, i.e. every integral curve of X is defined on $\mathbb{R}$.