MTH 930 HOMEWORK ASSIGNMENT 4

DUE OCT. 11 IN CLASS

(1) Let (M^n, g) be a Riemannian manifold. For $p \in M$, pick $\delta > 0$ s.t. $\exp_p : B_\delta \to B(p, \delta)$ is a diffeomorphism, where

$$B_\delta = \{v \in T_pM : |v| < \delta\}, B(p, \delta) = \{q \in M : d(p, q) < \delta\}.$$

Let $\{e_i\}$ be an orthonormal basis for T_pM and $\phi : \mathbb{R}^n \to T_pM$ the isometry $\phi(x) = \sum_i x_i e_i$. We can introduce a local chart $\Phi : B(p, \delta) \to \mathbb{R}^n$ by $x = \Phi(q) = \phi^{-1} \circ \exp_p^{-1}(q)$. Write $g = g_{ij}(x)\, dx_i \otimes dx_j$ in these coordinates. Prove

- $g_{ij}(0) = \delta_{ij}, \Gamma_{ij}^k(0) = 0$;
- $g_{ij}(x)\, x_j = x_i$.

(Hint: $t \to (tx_1, \cdots, tx_n)$ is a geodesic.)

(2) Let (M^n, g) be a complete Riemannian manifold $\widetilde{g}$ another metric on M s.t. $\widetilde{g} \geq g$, i.e. for any $X \in TM$, $\widetilde{g}(X, X) \geq g(X, X)$. Show that $(M, \widetilde{g})$ is also complete.

(3) Let (M^n, g) be a complete Riemannian manifold and $K \subset M$ a closed subset. The distance from K to $p \in M$ is defined as

$$d(p, K) := \inf\{d(p, q) : q \in K\}.$$

- Prove the infimum is achieved at some point $\overline{q} \in K$.
- Further assume that K is a submanifold. Let $\gamma : [0, l] \to M$ be a minimizing geodesic from p to $\overline{q}$. Prove that $\gamma'(l) \perp T_{\overline{q}}K$

(4) Consider the conformal ball model of the hyperbolic space: B^n with $g = \frac{4}{(1-|x|^2)^2} dx^2$. Prove

$$d(0, x) = \log \frac{1 + |x|}{1 - |x|}.$$

(5) Let (M^n, g) be a Riemannan manifold with $\sec \leq \kappa$. Let $\gamma : [0, l] \to M$ be a unit-speed geodesic and J a normal Jacobi field along γ with $J(0) = 0, \left|\dot{J}(0)\right| = 1$. Suppose $J(t) \neq 0$ for all $t \in (0, l]$. Prove

$$|J(t)| \geq \begin{cases} \frac{\sin\sqrt{\kappa}t}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \frac{\sinh\sqrt{-\kappa}t}{\sqrt{-\kappa}} & \text{if } \kappa < 0. \end{cases}$$

(Hint: inspect the proof of the Cartan -Hardamard theorem where the case $\kappa = 0$ is proved.)