## MTH 930 HOMEWORK ASSIGNMENT 5

DUE NOV. 18 IN CLASS

(1) Let $\gamma$ be a geodesic and $J_{1}, J_{2}$ Jacobi fields along $\gamma$. Show that $\left\langle\dot{J}_{1}, J_{2}\right\rangle$ $\left\langle J_{1}, \dot{J}_{2}\right\rangle$ is constant.
(2) Suppose $\left(M^{m}, g\right)$ and $\left(N^{n}, h\right)$ are two Riemannian manifolds. Consider $M \times N$ with the product metric $g+h$.

- (Pythagorian theorem) Prove that for two points $p=(x, \xi), q=$ $(y, \eta) \in M \times N$

$$
d(p, q)=\sqrt{d_{M}(x, y)^{2}+d_{N}(\xi, \eta)^{2}}
$$

- A curve $(x(t), \xi(t))$ in $M \times N$ is a geodesic in $M \times N$ iff $x(t)$ and $\xi(t)$ are geodesics in $M$ and $N$, respectively. (A constant curve is viewed as a geodesic.)
(3) A vector field $X$ on a Riemannian manifold $(M, g)$ is called Killing if $L_{X} g=$ 0 . Geometrically it means that the (local) flow generated by $X$ consists of isometries. Prove
- $X$ is Killing iff $\left\langle\nabla_{u} X, v\right\rangle+\left\langle\nabla_{v} X, u\right\rangle=0$.
- If $X$ is Killing and $\gamma$ is a geoesic, then the restriction of $X$ to $\gamma$ is a Jacobi field along $\gamma$.
(4) Let $\left(M^{n}, g\right)$ be a complete Riemannian manifold of positive curvature and $A, B$ two closed totally geodesic submanfolds. Show that $A$ and $B$ must intersect if $\operatorname{dim} A+\operatorname{dim} B \geq n$. (A submanifold is totally geodesic if its second fundamental form is zero. Hint: if $A$ and $B$ do not intersect, then there is a minimizing geodesic joining them. Get a contradiction by using the 2 nd variation formula and ideas from the proof of the Synge theorem )
(5) The following theorem was proved by Weintein:

Let $\left(M^{n}, g\right)$ be a compact and orientable Riemannan manifold with positive sectional curvature and $\phi: M \rightarrow M$ an isometry. Suppose the dimension $n$ is even, then $\phi$ must have a fixed point.

We will prove Weintein's theorem by controditcion in the following steps.

- Deduce Synge's theorem from Weinstein's theorem.
- Suppose $l=\inf _{q \in M} d(q, \phi q)>0$ and is achieved at a point $p$. Let $\gamma:[0, l] \rightarrow M$ be a minimizing geodesic from $p$ to $\phi(p)$. Prove that $\phi_{*}\left(\gamma^{\prime}(0)\right)=\gamma^{\prime}(l)$. (Apply 1st variation formula to appropriate variations).
- Construct a parallel vector field and apply the 2 nd variation formula as in the proof of Synge's theorem to get a contradiction.

