MTH 930 HOMEWORK ASSIGNMENT 5

DUE NOV. 18 IN CLASS

(1) Let γ be a geodesic and J_1, J_2 Jacobi fields along γ . Show that $\langle J_1, J_2 \rangle$ –

$$\left\langle J_1, J_2 \right\rangle$$
 is constant.

- (2) Suppose (M^m, g) and (N^n, h) are two Riemannian manifolds. Consider $M \times N$ with the product metric g + h.
 - (Pythagorian theorem) Prove that for two points $p = (x,\xi), q = (y,\eta) \in M \times N$

$$d(p,q) = \sqrt{d_M (x,y)^2 + d_N (\xi,\eta)^2}$$

- A curve $(x(t), \xi(t))$ in $M \times N$ is a geodesic in $M \times N$ iff x(t) and $\xi(t)$ are geodesics in M and N, respectively. (A constant curve is viewed as a geodesic.)
- (3) A vector field X on a Riemannian manifold (M, g) is called Killing if $L_X g = 0$. Geometrically it means that the (local) flow generated by X consists of isometries. Prove
 - X is Killing iff $\langle \nabla_u X, v \rangle + \langle \nabla_v X, u \rangle = 0.$
 - If X is Killing and γ is a geoesic, then the restriction of X to γ is a Jacobi field along γ .
- (4) Let (M^n, g) be a complete Riemannian manifold of *positive* curvature and A, B two closed totally geodesic submanfolds. Show that A and B must intersect if dim A + dim $B \ge n$. (A submanifold is totally geodesic if its second fundamental form is zero. Hint: if A and B do not intersect, then there is a minimizing geodesic joining them. Get a contradiction by using the 2nd variation formula and ideas from the proof of the Synge theorem)
- (5) The following theorem was proved by Weintein:

Let (M^n, g) be a compact and orientable Riemannan manifold with positive sectional curvature and $\phi: M \to M$ an isometry. Suppose the dimension n is even, then ϕ must have a fixed point.

We will prove Weintein's theorem by controditcion in the following steps.Deduce Synge's theorem from Weinstein's theorem.

- Suppose $l = \inf_{q \in M} d(q, \phi q) > 0$ and is achieved at a point p. Let $\gamma : [0, l] \to M$ be a minimizing geodesic from p to $\phi(p)$. Prove that $\phi_*(\gamma'(0)) = \gamma'(l)$. (Apply 1st variation formula to appropriate variations).
- Construct a parallel vector field and apply the 2nd variation formula as in the proof of Synge's theorem to get a contradiction.