MTH 930 HOMEWORK ASSIGNMENT 6

1. Find the distance on the sphere $\mathbb{S}^n$ with the standard metric.

2. Let (M, g) be a complete Riemannian manifold. A ray is a unit-speed geodesic $\gamma : [0, \infty) \to M$ s.t. for all $t > 0$, $\gamma|_{[0,t]}$ is mininimizing. Prove that M is noncompact iff $\forall p \in M$ there is a ray starting from p.

3. Let (M, g) be a complete Riemannian manifold with sec ≤ 0. Then for a Jacobi field $Y(t)$ along a geodesic γ with $Y(0) = 0$, show that

$$\left|\dot{Y}(0)\right| \leq \frac{|Y(t)|}{t}, t \geq 0.$$

4. In class we have focused on the distance function $\rho(x) = d(x, p)$ to a fixed point p. Much of the theory can be extended to the case when p is replaced by a submanifold. You will work out the details of the fundation in this problem. Let Σ^k be a closed submanifold of the Riemannian manifold M^n. Consider a unit-speed geodesic $\gamma : [0, L] \to M$ with $p = \gamma(0) \in \Sigma$ and $\xi = \dot{\gamma}(0) \in (T_p\Sigma)^{\perp}$. A Jacobi field J along γ is called a Σ-Jacobi field if

$$J(0) \in T_p\Sigma, \dot{J}(0) - A_\xi J(0) \in (T_p\Sigma)^{\perp},$$

where A_ξ denotes the shape operator of Σ with respect to the normal vector ξ, i.e. for $X \in T_p\Sigma$, $A_\xi X = (\nabla_X \xi)^T$ (it is the second fundamental form of Σ in disguise). Define

$$\mathfrak{V} = \left\{\text{piecewise smooth vector fields } X \text{ along } \gamma \text{ s.t. } X \perp \dot{\gamma}, X(0) \in T_p\Sigma\right\}.$$

On $\mathfrak{V}$ we consider the the following symmetric, bilinear form, called the index form

$$I(X, Y) = \langle A_\xi X(0), Y(0)\rangle + \int_0^L \left[\left\langle \dot{X}, \dot{Y}\right\rangle - R\left(\dot{\gamma}, X, \dot{\gamma}, Y\right)\right] dt.$$

For $a > 0$, $\gamma(a)$ is called a focal point along γ if there is a Σ-Jacobi field J along γ with $J(a) = 0$. Show that

- $\gamma(a)$ is a focal point iff it is a singular value of $\exp : (T\Sigma)^{\perp} \to M$.
- Suppose there is no focal point on γ. Then for any $X \in \mathfrak{V}$, there is a unique Σ-Jacobi field J s.t. $J(L) = X(L)$. Moreover

  $$I(J, J) \leq I(X, X),$$

  and equality holds iff $X = J$.

Date: Due on Nov. 8.