MTH 930 HOMEWORK ASSIGNMENT 7

1. Let (M, g) be a Riemannian manifold and $f \in C^{\infty}(M)$. As explained in class, we always work with a local orthonormal frame.
   - Prove the following formula
   $$f_{i,jk} = f_{i,kj} + R(e_j, e_k, \nabla f, e_i).$$
   - Derive a similar formula for a $(0,2)$-tensor field.
2. Compute the divergence of the Ricci curvature.
3. Show that a complete Riemannian manifold (M^n, g) with the property that
$$Ric \geq 0,$$
$$\lim_{r\to\infty} \frac{\mathrm{vol} B(p,r)}{\omega_n r^n} = 1,$$
for some $p \in M$, must be isometric to the Euclidean space $\mathbb{R}^n$. (Here ω_n is the volume of the unit ball in $\mathbb{R}^n$.)
4. Let M^n be a closed submanifold in $\mathbb{R}^{n+1}$and ν its outer unit normal. The second fundamental form is defined by
$$h(X,Y) = \left\langle \overline{\nabla}_X \nu, Y \right\rangle$$
for $X, Y \in TM$. Its trace is the mean curvature H. We have the Codazzi equation
$$\nabla_X h(Y,Z) = \nabla_Y h(X,Z).$$
   - Show that $\mathrm{div} h = dH$.
   - Establish the following formula
   $$\frac{1}{2}\Delta |h|^2 = |\nabla h|^2 + \left\langle h, D^2 H\right\rangle + \mathcal{K},$$
   where
   $$\mathcal{K} = -R_{ikjl} h_{ij} h_{kl} + R_{ij} h_{ik} h_{kj}$$
   in terms of an orthonormal frame on M.
   - Deduce the following conclusion (Liebmann's theorem): If M has constant mean curvature H and nonnegative second fundamental form, then it is a round sphere. (Hint: to analyse the curvature term you can choose the frame to diagonalize h. See Page 231 in Petersen's Riemannian Geometry, 2nd edition.)

Date: Due on Nov. 22.