BASIC FACTS ON CONNECTION AND CURVATURE

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1. General theory

Let M^n be a smooth manifold and $\pi : E \to M$ a rank r vector bundle over M.

Definition 1. *A connection is a linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ s.t. for $f \in C^\infty(M), \sigma \in \Gamma(E)$*

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

If X is a vector field on M, we let $\nabla_X \sigma = X \lrcorner \nabla\sigma$.

First note that a connection is local: if $\sigma \in \Gamma(E)$ vanishes on an open set U, then $\nabla\sigma$ also vanishes on U. Indeed, for any open V with $\overline{V} \subset U$ compact we can find $\chi \in C_c^\infty(U)$ s.t. $\chi \equiv 1$ on $\overline{V}$. Then we have $\sigma = (1-\chi)\sigma$ and

$$\nabla\sigma = -d\chi \otimes \sigma + (1-\chi)\nabla\sigma.$$

It follows that $\nabla\sigma = 0$ on V. Therefore, if two sections $\sigma_1, \sigma_2 \in \Gamma(E)$ satisfy $\sigma_1 = \sigma_2$ on U, then $\nabla\sigma_1 = \nabla\sigma_2$ on U. Consequently, for any local section $\sigma \in \Gamma(U, E)$ we can define $\nabla\sigma \in \Gamma(U, T^*M \otimes E)$ by extension.

Given a local frame $\{e_i\}$ on some open set U, we can write

$$\nabla e_i = \omega_i^j \otimes e_j,$$

with $\omega_i^j \in \mathcal{A}^1(U)$. Thus we get a matrix of 1-forms

$$\omega = \begin{bmatrix} \omega_1^1 & \cdots & \omega_r^1 \\ \vdots & & \vdots \\ \omega_1^r & \cdots & \omega_r^r \end{bmatrix}.$$

A local section σ on U can then be written as $\sigma = \sum_i f^i e_i$ with $f^i \in C^\infty(U)$. We have

$$\nabla\sigma = \sum_i \left(df^i + \omega_j^i f^j\right) \otimes e_i.$$

If we write $f = (f_1, \cdots, f_r)^t$ as a column vector, then the connection is locally given by the map

$$f \to df + \omega f.$$

If we use another local frame $\{\widetilde{e}_i\}$ on V, the connection is then determined by another matrix of 1-forms $\widetilde{\omega}$ on V: $\nabla\widetilde{e}_i = \widetilde{\omega}_i^j \otimes \widetilde{e}_j$. On $U \cap V$

$$\widetilde{e}_i = a_i^j e_j,$$

with $A=\left[a_i^j\right]\in C^\infty\left(U\cap V, GL\left(r,\mathbb{R}\right)\right)$. Then

$$\nabla\widetilde{e}_i=\left(da_i^j+a_i^k\omega_k^j\right)\otimes e_j.$$

On the other hand, $\nabla\widetilde{e}_i=\widetilde{\omega}_i^j\otimes\widetilde{e}_j=\widetilde{\omega}_i^k a_k^j\otimes e_j$. Therefore

$$\widetilde{\omega}_i^k a_k^j=da_i^j+a_i^k\omega_k^j,$$

i.e. $A\widetilde{\omega}=dA+\omega A$. Or, equivalently

$$\widetilde{\omega}=A^{-1}dA+A^{-1}\omega A.$$

Let $\mathcal{A}^k\left(E\right)=\Gamma\left(\Lambda^k\left(M\right)\otimes E\right)$ be the space of E-valued differential k-forms. We can extend the connection to $\nabla:\mathcal{A}^k\left(E\right)\to\mathcal{A}^{k+1}\left(E\right)$ as follows: given $\sigma\in\mathcal{A}^k\left(U,E\right)$ we can write

$$\sigma=\sum_i\theta^i\otimes e_i,$$

where $\theta^i\in\mathcal{A}^k\left(U\right)$. Then we define

$$\begin{aligned}\nabla\sigma&=\sum_i d\theta^i\otimes e_i+\left(-1\right)^k\theta^i\wedge\nabla e_i\\&=\sum_i\left(d\theta^i+\omega_k^i\wedge\theta^k\right)\otimes e_i.\end{aligned}$$

It is easy to verify this is well defined.

Proposition 1. *Connections always exist. In fact, the set of all connections on a vector bundle $\pi:E\to M$ is an affine space over $\mathcal{A}^1\left(\mathrm{End}E\right)$.*

Proof. Use a partition of unity. $\square$

Given $\sigma\in\mathcal{A}^k\left(U,E\right)$ we calculate

$$\begin{aligned}\nabla^2\sigma&=\sum_i\left[d\left(d\theta^i+\omega_l^i\wedge\theta^l\right)+\omega_k^i\wedge\left(d\theta^k+\omega_l^k\wedge\theta^l\right)\right]\otimes e_i\\&=\sum_i\left[d\omega_l^i+\omega_k^i\wedge\omega_l^k\right]\wedge\theta^l\otimes e_i\\&=\sum_i\Omega_j^i\wedge\theta^j\otimes e_i,\end{aligned}$$

with $\Omega_j^i=d\omega_j^i+\omega_k^i\wedge\omega_j^k$. The matrix of 2-forms $\Omega=\left[\Omega_i^j\right]$ is called the curvature of the connection. We have

$$\Omega=d\omega+\omega\wedge\omega.$$

If we use another frame, then

$$\begin{aligned}\widetilde{\Omega}&=d\widetilde{\omega}+\widetilde{\omega}\wedge\widetilde{\omega}\\&=d\left(A^{-1}dA+A^{-1}\omega A\right)\\&+\left(A^{-1}dA+A^{-1}\omega A\right)\wedge\left(A^{-1}dA+A^{-1}\omega A\right)\\&=A^{-1}\left(d\omega+\omega\wedge\omega\right)A\\&=A^{-1}\Omega A.\end{aligned}$$

Therefore, the curvature $\Omega\in\mathcal{A}^2\left(\mathrm{End}E\right)$.

There is a natural induced connection on E^*. Let $\{e_i^*\}$ be the dual frame, i.e. $e_i^*(e_j) = \delta_{ij}$. Then

$$0 = \nabla e_i^*(e_j) + e_i^*(\nabla e_j).$$

This implies $\nabla e_i^* = -\omega_j^i \otimes e_j^*$. If we denote the dual frame by $\{e^i\}$ instead, then

$$\nabla e^i = -\omega_j^i \otimes e^j.$$

Therefore

$$\nabla^2 e^i = -\Omega_j^i \otimes e^j.$$

Definition 2. *Let X, Y be two vector fields on M. The curvature operator $R(X,Y) : \Gamma(E) \to \Gamma(E)$ is defined as*

$$R(X,Y) = -\nabla_X \nabla_Y + \nabla_Y \nabla_X + \nabla_{[X,Y]}.$$

Theorem 1. *Locally, for any section $\sigma = \sum_i f^i e_i$*

$$R(X,Y)\sigma = -\sum_{i,j} \Omega_j^i(X,Y) f^j e_i.$$

Proof. First

$$\nabla_X \sigma = \sum_i \left(Xf^i + \omega_j^i(X) f^j\right) e_i.$$

Hence

$$\begin{aligned}
\nabla_Y \nabla_X \sigma &= \sum_i \left(YXf^i + Y\omega_j^i(X) f^j + \omega_j^i(X) Yf^j\right) e_i \\
&\quad + \sum_i \left(Xf^k + \omega_j^k(X) f^j\right) \omega_k^i(Y) e_i \\
&= \sum_i \left(YXf^i + Y\omega_j^i(X) f^j + +\omega_k^i(Y)\omega_j^k(X) f^j\right) e_i \\
&\quad + \sum_i \left(\omega_k^i(X) Yf^k + \omega_k^i(Y) Xf^k\right) e_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma \\
&= \sum_i \left(d\omega_j^i(X,Y) + \omega_k^i(X)\omega_j^k(Y) - \omega_k^i(Y)\omega_j^k(X)\right) f^j e_i \\
&= \sum_i \Omega_j^i(X,Y) f^j e_i.
\end{aligned}$$

□

Proposition 2. *(2nd Bianchi identity) The curvature matrix satisfies the following identity*

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Proof. We calculate

$$\begin{aligned}
d\Omega &= d\omega \wedge \omega - \omega \wedge d\omega \\
&= (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) \\
&= \Omega \wedge \omega - \omega \wedge \Omega.
\end{aligned}$$

□

Proposition 3. *Suppose (E,∇) is flat in the sense that the curvature is zero. Then $\forall p\in M$ there exists a neighborhood U and local frame $\{e_i\}$ on U s.t. $\nabla e_i = 0$.*

Proof. Pick a arbitrary local frame $\{s_i\}$ and let ω be the corresponding connection matrix. We want to find a new local frame $\{e_i\}$

$$e_i = a_i^j s_j,$$

s.t. $\nabla e_i = 0$. In other words, we want to find smooth $A = \left[a_j^i\right] \in GL\left(r,\mathbb{R}\right)$ s.t.

$$dA + \omega A = 0.$$

By the Frobenius theorem, there is a local solution with $A = I$ at p if the curvature is zero. To apply Frobenius, consider the manifold $U\times\mathbb{R}^{n^2}$ with y_j^i the standard coordinates on $\mathbb{R}^{n^2}$. We have n^2 1-forms

$$\theta_j^i = dy_j^i + \omega_k^i y_j^k$$

which are linearly independent. We compute

$$\begin{aligned} d\theta_j^i &= d\omega_k^i y_j^k - \omega_k^i \wedge dy_j^k \\ &= \Omega_k^i y_j^k - \omega_k^i \wedge \theta_j^k. \end{aligned}$$

If $\Omega = 0$, the distribution $\mathcal{D} = \ker\{\theta_{ij}\}$ is involutive. The integral manifold passing through (p, I) is locally the graph of a function $y_j^i = a_j^i\left(x\right)$. □

2. The Levi-Civita connection

A connection on TM is called an affine connection. For an affine connection ∇ we define its torsion by

$$\tau\left(X,Y\right) = \nabla_X Y - \nabla_Y X - \left[X,Y\right].$$

Theorem 2. *Let (M^n, g) be a Riemannian manifold. Then there exists a unique affine connection ∇, called the Levi-Civita connection, that satisfies the following two conditions*

(1) *∇ is compatible with the metric, i.e.*

$$X\left\langle Y,Z\right\rangle = \left\langle \nabla_X Y, Z\right\rangle + \left\langle Y, \nabla_X Z\right\rangle.$$

(2) *∇ is torsion-free, i.e.*

$$\nabla_X Y - \nabla_Y X = \left[X,Y\right].$$

Let $\{e_i\}$ be a frame and $\left\{\theta^i\right\}$ the dual frame. Writing

$$g_{ij} = \left\langle e_i, e_j\right\rangle, \nabla e_i = \omega_i^j \otimes e_j,$$

we have $dg_{ij} = \omega_i^k g_{kj} + g_{ik}\omega_j^k$, i.e

$$dG = \omega^t G + G\omega.$$

It is more convenient to work with a local orthonormal frame.

Proposition 4. *Let $\{e_i\}$ be a local orthonormal frame. The connection matrix ω for the Levi-Civita connection is uniquely determined by the following two identities*

$$\omega^t = -\omega,$$

$$d\theta^i = \theta^j \wedge \omega_j^i.$$

Proof. The 2nd identity follows by direct calculation. Uniqueness follows from Cartan's lemma. □

The curvature $\Omega = d\omega + \omega \wedge \omega$ is a skew-symmetric matix of 2-forms. We have

$$\Omega^i_j = \frac{1}{2} R_{klij} \theta^k \wedge \theta^l.$$

Proposition 5. *Suppose (M^n, g) is flat in the sense that the curvature vanishes identically. Then for any $p \in M$ there is a local chart (U, x) about p s.t.*

$$g = dx_i \otimes dx_i$$

on U.

Proof. By Proposition 3 there exists a local frame $\{e_i\}$ s.t. $\nabla e_i = 0$ and $\langle e_i, e_j \rangle = \delta_{ij}$ at p. As $dg_{ij} = 0$, $\{e_i\}$ is orthonormal. Since $d\theta^i = 0$, there exist local smooth functions f_i s.t. $\theta^i = df_i$. By the implicit function theorem $(f_1, \cdots, f_n)$ is a local coordinate system near p and

$$g = \theta^i \otimes \theta^i = df_i \otimes df_i.$$

□