A REMARK ON ZHONG-YANG'S EIGENVALUE ESTIMATE

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Geometric inequalities are beautiful and powerful, especially when they are optimal. A classic example is the isoperimetric inequality: let D be a bounded domain in $\mathbb{R}^2$, then its area A and perimeter L satisfy the inequality

$$4\pi A \leq L^2.$$

Moreover we know exactly when the equality holds: if and only if D is a disc. For any optimal geometric inequality it is important to have a complete understanding of the equality case. Sometimes this can be easily achieved by checking the proof of the inequality. Take as an example the following elegant theorem due to Lichnerowicz [L]: let (M,g) be a compact Riemannian manifold of dimension n with $\mathrm{Ric} \geq n-1$, then $\lambda_1 \geq n$, where λ_1 is the first eigenvalue of the Laplacian operator on functions. It was proved by Obata [O] several years later that equality holds iff (M,g) is isometric to the standard sphere S^n. This is not difficult to prove by tracing back each inequality in Lichnerowicz's argument. In other cases characterizing the equality case may not be so easy. Take the Myers theorem proved in 1941: let (M,g) be a compact Riemannian manifold of dimension n with $\mathrm{Ric} \geq n-1$, then its diameter $d \leq \pi$ (see e.g. [P]). To understand the equality case it is far from enough to simply analyze the proof of the inequality as doing so only gives some information along a geodesic. Some new idea is required. It was only proved in 1975 by Cheng [C] that (M,g) is isometric to the standard sphere S^n if $d = \pi$. The proof is not easy. There is an elementary proof [Sh] using the Bishop-Gromov volume comparison theorem.

For a compact Riemannian manifold (M,g) with nonnegative Ricci curvature, Li-Yau [LY, Li] proved the beautiful inequality $\lambda_1 \geq \frac{\pi^2}{2d^2}$. By sharpening Li-Yau's method, Zhong-Yang [ZY] improved the inequality to $\lambda_1 \geq \frac{\pi^2}{d^2}$, which is optimal as equality holds on S^1. It is a natural question if S^1 is the only case for equality. To answer this question it is not enough to go through Zhong-Yang's proof of the inequality. This is raised as an open problem by Sakai in [S].

The main purpose of this short note is to derive the following

Theorem 1. *Let (M,g) be a smooth compact Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and $\lambda_1 = \frac{\pi^2}{d^2}$, then it is isometric to the circle of radius $\frac{d}{\pi}$.*

The solution depends on a different approach to Li-Yau's gradient estimate of the first eigenfunction. Li-Yau derived their gradient estimate by applying the maximum principle to a judiciously chosen auxiliary function. The proof focuses on a point where this auxiliary function attains its maximum. We derive a differential inequality on the dense open set which consists of all regular points of the eigenfunction and then apply the strong maximum principle.

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Lemma 1. *Let u be a nonzero smooth function on a Riemannian manifold such that $-\Delta u=\lambda u$. Set $\phi=|\nabla u|^2+\lambda u^2$. Then on $\Omega=\{\nabla u\neq 0\}$,*

$$\Delta\phi-\frac{\nabla\left(\phi-2\lambda u^2\right)\cdot\nabla\phi}{2\left|\nabla u\right|^2}\geq 2\mathrm{Ric}(\nabla u,\nabla u).$$

Proof. Under a local orthonormal frame, we have

$$\frac{1}{2}\phi_i=\sum_j u_j u_{ji}+\lambda u u_i.$$

Hence

$$\left|\frac{1}{2}\nabla\phi-\lambda u\nabla u\right|^2=\sum_i\left(\sum_j u_j u_{ji}\right)^2\leq\left|D^2u\right|^2\left|\nabla u\right|^2.$$

This implies

$$\frac{1}{4}\left|\nabla\phi\right|^2-\lambda u\nabla u\cdot\nabla\phi\leq\left|\nabla u\right|^2\left(\left|D^2u\right|^2-\lambda^2u^2\right).$$

Therefore On Ω,

$$\left|D^2u\right|^2-\lambda^2u^2\geq\frac{\left|\nabla\phi\right|^2-4\lambda u\nabla u\cdot\nabla\phi}{4\left|\nabla u\right|^2}=\frac{\nabla\left(\phi-2\lambda u^2\right)\cdot\nabla\phi}{4\left|\nabla u\right|^2}.$$

Then by the Bochner formula

$$\begin{aligned}\frac{1}{2}\Delta\phi &= \left|D^2u\right|^2+\nabla u\cdot\nabla\Delta u+\mathrm{Ric}(\nabla u,\nabla u)+\lambda\left|\nabla u\right|^2+\lambda u\Delta u\\ &= \left|D^2u\right|^2-\lambda^2u^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\\ &\geq \frac{\nabla\left(\phi-2\lambda u^2\right)\cdot\nabla\phi}{4\left|\nabla u\right|^2}+\mathrm{Ric}\left(\nabla u,\nabla u\right).\end{aligned}$$

□

We now recall the work of Li-Yau and Zhong-Yang following the presentation in the book [SY, section 4 of chapter 3]. Let (M,g) be an n-dimensional compact Riemannian manifold with $\mathrm{Ric}(g)\geq 0$. Suppose that u is a first eigenfunction and the first eigenvalue is λ_1. We can assume

$$1=\max u>\min u=-k,\quad 0<k\leq 1.$$

Let

$$\widetilde{u}=\frac{u-\frac{1-k}{2}}{\frac{1+k}{2}},$$

then

$$\begin{cases}-\Delta\widetilde{u}=\lambda_1(\widetilde{u}+a),a=\frac{1-k}{1+k}\in[0,1),\\ \max\widetilde{u}=1,\min\widetilde{u}=-1.\end{cases}$$

For small $\epsilon>0$, let $v_\varepsilon=\frac{\widetilde{u}}{1+\epsilon}$. Then

$$\begin{cases}-\Delta v_\varepsilon=\lambda_1(v_\varepsilon+a_\varepsilon),a_\varepsilon=\frac{a}{1+\varepsilon},\\ \max v_\varepsilon=\frac{1}{1+\varepsilon},\min v_\varepsilon=-\frac{1}{1+\varepsilon}.\end{cases}$$

Li and Yau proved the following gradient estimate

$$\frac{\left|\nabla v_\varepsilon\right|^2}{1-v_\varepsilon^2}\leq\lambda_1(1+a_\varepsilon).\tag{0.1}$$

Zhong and Yang established a more precise estimate than (0.1). Set $v_\varepsilon = \sin\theta_\varepsilon$. The function θ_ε has its range in $[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta]$, where δ is specified by

$$\sin(\frac{\pi}{2}-\delta) = \frac{1}{1+\varepsilon}.$$

Define $\psi : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$\begin{cases} \psi(\theta) = \frac{\frac{4}{\pi}(\theta+\cos\theta\sin\theta)-2\sin\theta}{\cos^2\theta}, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \psi(\frac{\pi}{2}) = 1, \psi(-\frac{\pi}{2}) = -1. \end{cases}$$

It is obvious that ψ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\psi(-\theta) = -\psi(\theta)$. Moreover $\psi'(\theta) \geq 0$ and $|\psi(\theta)| \leq 1$. The main result of Zhong and Yang is the following estimate which improves (0.1),

$$|\nabla\theta_\varepsilon|^2 \leq \lambda_1 \left(1 + a_\varepsilon \psi(\theta_\varepsilon)\right). \tag{0.2}$$

From this result one can deduce $\lambda_1 \geq \frac{\pi^2}{d^2}$ as follows. Let p_0 and p_1 be two points such that $\theta_\varepsilon(p_0) = -\frac{\pi}{2}+\delta$ and $\theta_\varepsilon(p_1) = \frac{\pi}{2}-\delta$. Let γ be the shortest geodesic from p_0 to p_1, then

$$\begin{aligned}
\lambda_1^{1/2} d &\geq \lambda_1^{1/2} L(\gamma) \\
&\geq \int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} \frac{d\theta}{\sqrt{1+a_\varepsilon\psi(\theta)}} \\
&= \int_0^{\frac{\pi}{2}-\delta} \left(\frac{1}{\sqrt{1+a_\varepsilon\psi(\theta)}} + \frac{1}{\sqrt{1-a_\varepsilon\psi(\theta)}} \right) d\theta \\
&= 2\int_0^{\frac{\pi}{2}-\delta} \left(1 + \sum_{j=1}^{\infty} \frac{(4j)!}{2^{4j}\,(2j)!^2} a_\varepsilon^{2j} \psi(\theta)^{2j} \right) d\theta \\
&\geq \pi - 2\delta + \frac{3}{4} a_\varepsilon^2 \int_0^{\frac{\pi}{2}-\delta} \psi(\theta)^2 \, d\theta.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, hence $\delta \to 0$ too, we get

$$\lambda_1^{1/2} d \geq \pi + \frac{3(1-k)^2}{4(1+k)^2} \int_0^{\frac{\pi}{2}} \psi(\theta)^2 d\theta.$$

Therefore $\lambda_1 \geq \frac{\pi^2}{d^2}$ and the inequality is strict unless $k=1$.

We now prove Theorem 1. Suppose $\lambda_1 = \frac{\pi^2}{d^2}$. First we have $\min u = -1$. By scaling the metric we assume that $d = \pi$, then $\lambda_1 = 1$. Let $\phi = |\nabla u|^2 + u^2$. By Lemma 1 we have on $\Omega = \{\nabla u \neq 0\}$,

$$\Delta\phi - \frac{\nabla\left(\phi - 2u^2\right)\cdot\nabla\phi}{2\,|\nabla u|^2} \geq 0. \tag{0.3}$$

By the maximum principle

$$\phi = |\nabla u|^2 + u^2 \leq \max_{\{\nabla u = 0\}} \left(|\nabla u|^2 + u^2 \right) = 1.$$

Take two points p_0 and p_1 such that $u(p_0) = -1$ and $u(p_1) = 1$. Let $\gamma : [0, l] \to M$ be a (unit speed) minimizing geodesic from p_0 to p_1. Denote $f(t) = u(\gamma(t))$, then

$$|f'(t)| = |\nabla u(\gamma(t)) \cdot \gamma'(t)| \leq |\nabla u(\gamma(t))| \leq \sqrt{1 - f(t)^2}.$$

Hence for any $\varepsilon > 0$,

$$\pi \geq l \geq \int_{\{0\leq t\leq l, f'(t)>0\}} dt \geq \int_0^l \frac{f'(t)}{\sqrt{1-f(t)^2+\varepsilon}} dt = \int_{-1}^1 \frac{1}{\sqrt{1-x^2+\varepsilon}} dx. \tag{0.4}$$

Let $\varepsilon \to 0^+$

$$\pi \geq l \geq \int_{\{0\leq t\leq l, f'(t)>0\}} dt \geq \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \pi.$$

It follows that $l = \pi$ and $f'(t) > 0$ for a.e. $t \in (0, \pi)$. Hence f is strictly increasing on $[0, \pi]$. From (0.4) we see

$$\int_0^\pi \frac{f'(t)}{\sqrt{1-f(t)^2}} dt = \pi.$$

It follows that $f'(t) = \sqrt{1-f(t)^2}$ for $t \in (0, \pi)$. By continuity we have

$$f'(t) = \sqrt{1-f(t)^2} \quad \text{for all } t \in [0, \pi].$$

Hence $f^2 + f'^2 = 1$. Differentiating with respect to t, we get $ff' + f'f'' = 0$. Since $f'(t) > 0$ for $0 < t < \pi$, we see $f'' + f = 0$. As $f(0) = -1$ and $f'(0) = 0$, we see

$$f(t) = u(\gamma(t)) = -\cos t \quad \text{for } t \in [0, \pi].$$

It follows that $(D^2u)(\gamma'(0), \gamma'(0)) = 1$. Since $\Delta u(p_0) = 1$ and $(D^2u)_{p_0} \geq 0$, we must have

$$(D^2u)_{p_0} = \lambda_{\gamma'(0)} \otimes \lambda_{\gamma'(0)},$$

here for any tangent vector X, λ_X is the dual cotangent vector given by $\lambda_X(Y) = \langle X, Y\rangle$ for any tangent vector Y. To continue, we make the following observation:

Lemma 2. *The set $\{u = \pm 1\}$ has at most four points.*

Proof. Let p be any point with $u(p) = 1$. For a minimizing geodesic $\gamma_p : [0, l_p] \to M$ from p_0 to p, the same argument as before shows $l_p = \pi$ and

$$(D^2u)_{p_0} = \lambda_{\gamma_p'(0)} \otimes \lambda_{\gamma_p'(0)}.$$

It follows that $\gamma_p'(0) = \pm\gamma'(0)$. Hence $p = \exp_{p_0}(\pi\gamma_p'(0))$ has at most two choices. The same argument works for $\{u = -1\}$. □

To finish the proof of Theorem 1, we only need to show the dimension of M must be 1. Indeed, if this is not the case, then $\dim M \geq 2$. Let $M^* = M \backslash \{u = \pm 1\}$, then M^* is still connected. We want to show $|\nabla u|^2 + u^2 = 1$ on M^*. Indeed, let

$$E = \left\{p \in M^* : |\nabla u(p)|^2 + u(p)^2 = 1\right\}.$$

Clearly E is closed. On the other hand, if $p \in E \subset \Omega$, by (0.3) and the strong maximum principle, $|\nabla u|^2 + u^2 \equiv 1$ near p. Hence E must be either the empty set or M^*. Since for any $t \in (0, \pi)$,

$$|\nabla u(\gamma(t))|^2 + (u(\gamma(t)))^2 \geq \sin^2 t + \cos^2 t = 1,$$

we see E is nonempty and therefore $E = M^*$.

Define $X = \frac{\nabla u}{|\nabla u|}$ on M^*. Since $|\nabla u|^2 + u^2 \equiv 1$, by differentiation we have

$$D^2u(X, X) = -u. \tag{0.5}$$

The proof of Lemma 1 shows that

$$\left|D^2u\right|^2 = u^2. \tag{0.6}$$

By (0.5) and (0.6),

$$D^2u = -u\lambda_X \otimes \lambda_X.$$

A simple computation then shows that $D_X X = 0$. In particular all integral curves of X are geodesics. Denote $\Sigma = \{u = 0\}$. Since $|\nabla u| = 1$ on Σ we see Σ is a hypersurface (which may have more than one components). For any $p \in \Sigma$, let α_p be the maximal integral curve of $-X$ with $\alpha_p(0) = p$. Then α_p is a unit speed geodesic. Let $f_p(t) = u(\alpha_p(t))$. We have $f_p(0) = 0$ and $f_p'(t) = -\sqrt{1 - f_p(t)^2}$. It follows that $f_p(t) = -\sin t$ for $t \in \left[0, \frac{\pi}{2}\right)$. On the other hand, as a geodesic on M, α_p is defined on $[0, \infty)$. We have $u(\alpha_p(t)) = -\sin t$ for $t \in \left[0, \frac{\pi}{2}\right]$. In particular $u\left(\alpha_p\left(\frac{\pi}{2}\right)\right) = -1$. The same argument as before shows

$$\left(D^2u\right)_{\alpha_p\left(\frac{\pi}{2}\right)} = \lambda_{\alpha_p'\left(\frac{\pi}{2}\right)} \otimes \lambda_{\alpha_p'\left(\frac{\pi}{2}\right)}.$$

Note that $p = \exp_{\alpha_p\left(\frac{\pi}{2}\right)}\left(-\frac{\pi}{2}\alpha_p'\left(\frac{\pi}{2}\right)\right)$. Since there are at most two points in the set $\{u = -1\}$, we may find q with $u(q) = -1$ and infinitely many $p \in \Sigma$ such that $\alpha_p\left(\frac{\pi}{2}\right) = q$. This clearly leads to a contradiction since $\alpha_p'\left(\frac{\pi}{2}\right)$ has at most two choices. Therefore M must be of one dimension and the main theorem follows.

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