Notes on Perelman's paper on the entropy formula for the Ricci flow and its geometric applications

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1 Introduction

These are notes on Perelman's paper 'The entropy formula for the Ricci flow and its geometric applications'. The goal of this notes is to give more details to some theorems and arguments that appear in Perelman's paper. So far most of the sections 1-11 of [12] is covered in our notes.

2 The functional $\mathcal{F}$ and its monotonicity

On a closed manifold M consider the following functional in a metric g and a smooth function f on M

$$\mathcal{F}(g,f) = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV. \tag{1}$$

We compute its variation

$$\delta\mathcal{F}(v,h) = \int_M \left[-\langle v, \mathrm{Ric}\,\rangle + \delta^2 v - \Delta \mathrm{tr}\, v + 2\nabla f\cdot\nabla h - \langle v, df\otimes df\rangle + \left(\frac{\mathrm{tr}\, v}{2} - h\right)\left(R + |\nabla f|^2\right)\right] e^{-f} dV.$$

Integrating by parts gives

$$\begin{aligned}\int_M \delta^2 v e^{-f} dV &= \int_M \langle de^{-f}, \delta v\rangle dV \\ &= \int_M \langle D^2 e^{-f}, v\rangle dV \\ &= \int_M \langle -D^2 f + df\otimes df, v\rangle e^f dV.\end{aligned}$$

Therefore

$$\delta\mathcal{F}(v,h) = \int_M \left[-\langle v, \mathrm{Ric}\, + D^2 f\rangle + \left(\frac{\mathrm{tr}\, v}{2} - h\right)\left(2\Delta f - |\nabla f|^2 + R\right)\right] e^{-f} dV. \tag{2}$$

If we fix a measure $dm = e^{-f}dV$ we get a functional $\mathcal{F}^m(g)$ as f is determined by g. Its variation follows from (2) by taking $h = \mathrm{tr}\, v/2$

$$\delta\mathcal{F}^m(v) = \int_M -\langle v, \mathrm{Ric}\, + D^2 f\rangle dm. \tag{3}$$

This leads to the consideration of the gradient flow

$$\frac{\partial g}{\partial t} = -2\left(\text{Ric } + D^2 f\right). \tag{4}$$

As $\frac{\partial f}{\partial t} = \text{tr } \frac{\partial g}{\partial t}/2$, f evolves according to the following backward heat equation

$$\frac{\partial f}{\partial t} = -R - \Delta f. \tag{5}$$

Therefore if g and f evolve according to (4) and (5), we have

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |\text{Ric } + D^2 f|^2 e^{-f} dV. \tag{6}$$

This flow (4) is equivalent to the Ricci flow. To see this let ϕ_t be the flow generated by the time-dependent vector field ∇f. Let $\overline{g}(t) = \phi_t^* g(t)$ and $\overline{f} = f \circ \phi_t$. Then

$$\begin{aligned}
\frac{\partial \overline{g}}{\partial t} &= \phi_t^* \left(\frac{\partial g}{\partial t} + \mathcal{L}_{\nabla f} g\right) \\
&= \phi_t^* \left(-2\text{Ric}\,(g) - 2D^2 f + 2D^2 f\right) \\
&= -2\text{Ric}\,(\overline{g}),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \overline{f}}{\partial t} &= \frac{\partial f}{\partial t} \circ \phi_t + \langle \nabla f, \dot{\phi_t} \rangle \circ \phi_t \\
&= -R \circ \phi_t - \Delta f \circ \phi_t + |\nabla f|^2 \circ \phi_t \\
&= -\overline{R} - \overline{\Delta f} + |d\overline{f}|^2_{\overline{g}},
\end{aligned}$$

where $\overline{R}$ and $\overline{\Delta}$ are the scalar curvature and Laplacian of $\overline{g}$, respectively. On the other hand $\mathcal{F}$ is obviously invariant under diffeomorphisms, so $\mathcal{F}(g, f) = \mathcal{F}(\overline{g}, \overline{f})$.

In summary, we have proved

Theorem 2.1. *If g and f evolve according to the flow*

$$\frac{\partial g}{\partial t} = -2\text{Ric} \tag{7}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 \tag{8}$$

we have the identity

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |\text{Ric } + D^2 f|^2 e^{-f} dV. \tag{9}$$

In particular $\mathcal{F}(g(t), f(t))$ is nondecreasing and the monotonicity is strict unless Ric + $D^2 f = 0$.

Remark 1. *Under the flow (7) and (8), the integral $\int_M e^{-f} dV$ stays constant.*

Define

$$\lambda(g) = \inf \{\mathcal{F}(g,f) | \int_M e^{-f} dV = 1\} \tag{10}$$

Let $u = e^{-f/2}$, then we get the equivalent definition

$$\lambda(g) = \inf \{ \int_M \left(4|\nabla u|^2 + Ru^2\right) dV | \int_M u^2 dV = 1\}. \tag{11}$$

i. e. $\lambda(g) = \lambda_1(-4\Delta + R)$, the first eigenvalue of the operator $-4\Delta + R$.

It is obvious that there is a unique minimizer f in (10) which satisfies the equation

$$2\Delta f - |\nabla f|^2 + R = \lambda. \tag{12}$$

Proposition 2.1. *$\lambda(g)$ has the following properties:*

1. *$\lambda(\phi^* g) = \lambda(g)$ for any diffeomorphism ϕ,*
2. *$\lambda(g(t))$ is non-decreasing under the Ricci flow. Moreover the monotonicity is strict unless* Ric $+ D^2 f = 0$ *for f which achieves $\lambda(g)$.*

Proof. The first property is obvious. We prove the second one. For any t_0, let f_0 be a minimizer in the definition of $\lambda(g(t_0))$. We then solve (8) backward with initial value f_0 at t_0. Then by Theorem (2.1), $\mathcal{F}(g(t), f(t))$ is nondecreasing. Hence we have for $t < t_0$

$$\lambda(g(t)) \leq \mathcal{F}(g(t), f(t)) \leq \mathcal{F}(g(t_0), f(t_0)) = \lambda(g(t_0)).$$

□

Let $\overline{\lambda}(g) = \lambda(g)V(g)^{2/n}$. It has the important property that $\overline{\lambda}(g) = \overline{\lambda}(\alpha\phi^* g)$ for any constant $\alpha > 0$ and any diffeomorphism ϕ.

Proposition 2.2. *$\overline{\lambda}(g)$ is nondecreasing along the Ricci flow whenever it is nonpositive. Moreover the monotonicity is strict unless we are on a gradient soliton.*

Proof. We compute the derivative (understood in the barrier sense)

$$\frac{d\overline{\lambda}(t)}{dt} \geq 2V^{2/n} \int_M |\text{Ric} + D^2 f|^2 dV - \frac{2}{n} V^{(2-n)/n} \lambda \int_M R dV,$$

where f is the minimizer for $\mathcal{F}$.

$$\frac{\int_M RdV}{V} = \mathcal{F}(\log V) \geq \lambda = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV = \int_M \left(R + \Delta f\right) e^{-f} dV.$$

Therefore, if $\lambda \leq 0$, we have

$$\begin{aligned}
\frac{d\overline{\lambda}(t)}{dt} &\geq 2V^{2/n} \left[\int_M |\text{Ric} + D^2 f|^2 e^{-f} dV - \frac{1}{n} \left(\int_M (R + \Delta f) e^{-f} dV \right)^2 \right] \\
&= 2V^{2/n} \int_M |\text{Ric} + D^2 f - \frac{1}{n}(R + \Delta f) g|^2 dV \\
&\quad + \frac{2}{n} V^{2/n} \left(\int_M (R + \Delta f)^2 e^{-f} dV - \left(\int_M (R + \Delta f) e^{-f} dV \right)^2 \right) \\
&\geq 0.
\end{aligned}$$

□

Definition 2.1. *A solution $g(t)$ to the Ricci flow is called a breather if for some $t_1 < t_2$ we have $g(t_2) = \alpha\phi^* g(t_1)$ for some constant α and diffeomorphism ϕ. The cases $\alpha = 1, \alpha > 1$ and $\alpha < 1$ correspond to steady, expanding and shrinking breathers, respectively.*

As an application, we can rule out the existence of steady and expanding breathers other than gradient solitons. The steady case is easy. If $g(t_2) = \phi^* g(t_1)$ then $\lambda(g(t_2)) = \lambda(g(t_1))$. By the monotonicity of λ, we have $\mathrm{Ric} + D^2 f = 0$. Thus a steady breather is necessarily a steady soliton.

For an expanding breather with $g(t_2) = \alpha\phi^* g(t_1)$, where $\alpha > 1$, we have $\overline{\lambda}(g(t_2)) = \overline{\lambda}(g(t_1))$. Since $\alpha > 1$, $V(t_2) > V(t_1)$. Hence for some $t_0 \in (t_1, t_2), V'(t_0) = -\int_M R dV > 0$. Then $\overline{\lambda}(g(t_0) \leq V^{(2-n)/n} \int_M R dV < 0$. By the monotonicity for $\overline{\lambda}$ (Proposition 2.2), it easily follows that $\overline{\lambda}(g(t)) = \textit{constant}$ on $[t_1, t_2]$. Then we must have

$$\mathrm{Ric} + D^2 f - \frac{1}{n}(R + \Delta f)g = 0 \tag{13}$$

$$R + \Delta f = C. \tag{14}$$

Combined with (12), these equations imply that f is a constant and g is Einstein.

The case of shrinking breathers will be addressed in the next section.

3 The functional $\mathcal{W}$

To generalize $\mathcal{F}$ consider the functional

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau\left(|\nabla f|^2 + R\right) + f - n\right] (4\pi\tau)^{-n/2} e^{-f} dV \tag{15}$$

$$= (4\pi\tau)^{-n/2} \tau\mathcal{F}(g, f) + (4\pi\tau)^{-n/2} \int_M (f - n) e^{-f} dV, \tag{16}$$

where $\tau > 0$ is the scale parameter. It is easy to see that for any positive number C and any diffeomorphism ϕ

$$\mathcal{W}(C\phi^* g, \phi^* f, C\tau) = \mathcal{W}(g, f, \tau). \tag{17}$$

We first fix τ. Using the variation formula for $\mathcal{F}$, it is easy to get the variation of $\mathcal{W}$

$$\delta\mathcal{W}(v, h) =$$

$$\int_M \left[-\tau\langle v, \mathrm{Ric} + D^2 f - \frac{1}{2\tau} g\rangle + \left(\frac{\mathrm{tr}\, v}{2} - h\right)\left(\tau\left(2\Delta f - |\nabla f|^2 + R\right) + f - n - 1\right)\right] (4\pi\tau)^{-n/2} e^{-f} dV. \tag{18}$$

If we fix the measure $dm = e^{-f} dV$ we get a functional in g whose L^2-gradient is $\mathrm{Ric} + D^2 f - \frac{1}{2\tau} g$. This leads to the following evolution equations

$$\frac{\partial g}{\partial t} = -2\left(\mathrm{Ric} + D^2 f - \frac{1}{2\tau} g\right) \tag{19}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + \frac{n}{2\tau} \tag{20}$$

let ϕ_t be the flow generated by the time-dependent vector field ∇f. Let $\overline{g}(t) = \phi_t^* g(t)$ and $\overline{f} = f \circ \phi_t$. Then by a computation we did before

$$\begin{aligned}
\frac{\partial \overline{g}}{\partial t} &= -2\mathrm{Ric}\,(\overline{g}) + \frac{1}{\tau}\overline{g}, \\
\frac{\partial \overline{f}}{\partial t} &= -\overline{R} - \overline{\Delta f} + |d\overline{f}|^2_{\overline{g}} + \frac{n}{2\tau}.
\end{aligned}$$

Let $\tilde{g}(s) = C(s)\overline{g}(t(s))$ and $\tilde{f}(s) = \overline{f}(t(s))$.

$$\begin{aligned}
\frac{\partial \tilde{g}}{\partial s} &= C'(s)\overline{g} + C(s)t'(s)\frac{\partial \overline{g}}{\partial t} \\
&= C'(s)\overline{g} + C(s)t'(s)\left(-2\mathrm{Ric}\,(\overline{g}) + \frac{1}{\tau}\overline{g}\right) \\
&= -2C(s)t'(s)\mathrm{Ric}\,(\tilde{g}) + \left(C'(s) + C(s)t'(s)/\tau\right)\overline{g}.
\end{aligned}$$

We want $C(s)t'(s) = 1$ and $C'(s) = -1/\tau$. The solutions are $C(s) = 1 - s/\tau$ and $t(s) = -\tau \log(1 - s/\tau)$. With such choice we have

$$\begin{aligned}
\frac{\partial \tilde{g}}{\partial s} &= -2\mathrm{Ric}\,(\tilde{g}) \\
\frac{\partial \tilde{f}}{\partial s} &= -\tilde{R} - \tilde{\Delta}\tilde{f} + |d\tilde{f}|^2_{\tilde{g}} + \frac{n}{2\tau C(s)}.
\end{aligned}$$

Denote $\tilde{\tau} = \tau C(s) = \tau - s$. By (17) we have

$$\mathcal{W}(g(t), f(t), \tau) = \mathcal{W}(\tilde{g}(s), \tilde{f}(s), \tilde{\tau}(s)). \tag{21}$$

In summary we have proved

Theorem 3.1. *If g, f and τ evolve according to the flow*

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric} \tag{22}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 + \frac{n}{2\tau} \tag{23}$$

$$\dot{\tau} = -1 \tag{24}$$

we have the identity

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_M |\mathrm{Ric} + D^2 f - \frac{1}{2\tau}g|^2 \,(4\pi\tau)^{-n/2}\, e^{-f} dV. \tag{25}$$

In particular $\mathcal{W}(g(t), f(t), \tau(t))$ is nondecreasing and the monotonicity is strict unless $\mathrm{Ric} + D^2 f - \frac{1}{2\tau}g = 0$.

We next consider the following minimization problem

$$\mu(g, \tau) = \inf\{\mathcal{W}(g, f, \tau) | \int_M (4\pi\tau)^{-n/2}\, e^{-f} dV = 1\}. \tag{26}$$

To show the existence of a minimizer, we let $u = e^{-f/2}$ and minimize

$$\mathcal{W}(u) = \int_M \left[\tau \left(4|\nabla u|^2 + Ru^2\right) - u^2 \log u^2 - nu^2\right] (4\pi\tau)^{-n/2}\, dV \tag{27}$$

under the constraint

$$\int_M u^2 \,(4\pi\tau)^{-n/2}\, dV = 1. \tag{28}$$

We show that this problem has a **positive** minimizer.

Since log is concave and $u^2\,(4\pi\tau)^{-n/2}\, dV$ is a probability measure, we have by Jensen's inequality and Sobelev inequality

$$\begin{aligned}\int_M u^2 \log u^2 \,(4\pi\tau)^{-n/2}\, dV &= \frac{n-2}{2}\int_M u^2 \log u^{4/(n-2)}\,(4\pi\tau)^{-n/2}\, dV\\ &\le \frac{n-2}{2}\log \int_M u^{2n/(n-2)}\,(4\pi\tau)^{-n/2}\, dV\\ &\le \frac{n-2}{2}\log\left(C\int_M \left(|\nabla u|^2 + u^2\right) dV\right)^{(n-2)/n} (4\pi\tau)^{-n/2}\\ &= \frac{n}{2}\log C\int_M \tau\left(|\nabla u|^2+u^2\right)(4\pi\tau)^{-n/2}\, dV.\end{aligned}$$

This inequality shows that if $\{u_i\}$ is a minimizing sequence for $\mathcal{W}$, then

$$\tau\int_M |\nabla u_i|^2\,(4\pi\tau)^{-n/2}\, dV \le C. \tag{29}$$

Being bounded in $H^1(M)$, it has a subsequence which converges to some $\bar{u}$ weakly in $H^1(M)$ and strongly in $L^2(M)$. It then follows that $\bar{u}$ satisfies the constraint and achieves the infimum of $\mathcal{W}$. We can also assume that $\bar{u} \ge 0$ (otherwise replace it by its absolute value). It satisfies the Euler-Lagrange equation

$$\tau\left(-4\Delta u + Ru\right) - 2u\log u - nu = \mu(g,\tau)u. \tag{30}$$

Elliptic L^p theory implies $\bar{u} \in C^{1,\alpha}$. To get further regularity and to return to $f = -2\log u$ it is crucial to prove $\bar{u}$ is positive. This is done in Rothaus [13]. So we can conclude that $\bar{u}$ is positive and smooth and achieves $\inf \mathcal{W}$ among all functions u satisfying the constraint (28). Therefore $\mu(g,\tau)$ is well-defined and is achieved by a smooth minimizer $\bar{f} = -2\log\bar{u}$. By Theorem 3.1 we have

Proposition 3.1. *$\mu(g(t), \tau - t)$ is nondecreasing along the Ricci flow $g(t)$.*

The proof uses the same argument in the proof of Proposition 2.1.

Proposition 3.2. *$\mu(g,\tau)$ is negative for small $\tau > 0$ and tends to zero as $\tau \to 0$.*

Proof. Assume $\bar{\tau} > 0$ is so small that the Ricci flow for g exists on $[0,\bar{\tau}]$. Let $\tau_0 = \bar{\tau} - \epsilon$ with $\epsilon > 0$ small. Pick $p \in M$. We use normal coordinates about p on $(M, g(\tau_0))$ to define

$$f_1 = \begin{cases} \frac{|x|^2}{4\epsilon} & d(x,x_0) < \rho_0,\\ \frac{\rho_0^2}{4\epsilon} & \text{elsewhere}\end{cases} \tag{31}$$

where $\rho_0 > 0$ is smaller than the injectivity radius. Note $dV = \left(1 + O(|x|^2\right)$ near p. We compute

$$\int_M (4\pi\epsilon)^{-n/2} e^{-f_1} dV = \int_{|x|\le\rho_0} (4\pi\epsilon)^{-n/2} e^{-|x|^2/4\epsilon} \left(1 + O(|x|^2)\right) dx + O(\epsilon^{-n/2} e^{-\rho_0^2/4\epsilon})$$
$$= \int_{|y|\le\rho_0/\sqrt{\epsilon}} (4\pi)^{-n/2} e^{-|y|^2/4} \left(1 + O(\epsilon|y|^2)\right) dy + O(\epsilon^{-n/2} e^{-\rho_0^2/4\epsilon})$$

The second term goes to zero as $\epsilon \to 0$ while the first term converges to

$$\int_{\mathbb{R}^n} (4\pi)^{-n/2} e^{-|y|^2/4} dy = 1.$$

If we write the integral as e^{-C}, then $C \to 0$ as $\epsilon \to 0$. And $f = f_1 + C$ then satisfies the constraint.

We solve the equation (23) backward with initial value f at τ_0.

$$\begin{aligned}
&\mathcal{W}(g(\tau_0), f(\tau_0), \bar{\tau} - \tau_0) \\
&= \int_{|x|\le\rho_0} \left[\epsilon\left(\frac{|x|^2}{4\epsilon^2} + R\right) + \frac{|x|^2}{4\epsilon} + C - n\right] (4\pi\epsilon)^{-n/2} e^{-|x|^2/4\epsilon - C} \left(1 + O(|x|^2)\right) dx \\
&\quad \int_{M - B(p,\rho_0)} \left(\frac{\rho_o^2}{4\epsilon} + \epsilon R + C - n\right) (4\pi\epsilon)^{-n/2} e^{-\rho_0^2/4\epsilon - C} \\
&= I + II,
\end{aligned}$$

where $I = e^{-C} \int_{|x|\le\rho_0} \left(\frac{|x|^2}{2\epsilon} - n\right) (4\pi\epsilon)^{-n/2} e^{-|x|^2/4\epsilon} \left(1 + O(|x|^2)\right) dx$ and II contains all the remaining terms. It is obvious that $II \to 0$ as $\epsilon \to 0$ while

$$\begin{aligned}
I &= e^{-C} \int_{|y|\le\rho_0/\sqrt{\epsilon}} \left(\frac{|y|^2}{2} - n\right) (2\pi)^{-n/2} e^{-|y|^2/4} \left(1 + O(\epsilon|y|^2)\right) dy \\
&\to \int_{\mathbb{R}^n} \left(\frac{|y|^2}{2} - n\right) (2\pi)^{-n/2} e^{-|y|^2/4} dy = 0
\end{aligned}$$

Therefore $\mathcal{W}(g(\tau_0), f(\tau_0), \bar{\tau} - \tau_0) \to 0$ as $\tau_0 \to \bar{\tau}$. By the monotonicity (Theorem 3.1), $\mu(g(t), \bar{\tau} - t) \le \mathcal{W}(g(t), f(t), \bar{\tau} - t) \le \mathcal{W}(g(\tau_0), f(\tau_0), \bar{\tau} - \tau_0)$ for any $t \le \tau_0$. Let $\tau_0 \to \bar{\tau}$, we get $\mu(g(t), \bar{\tau} - t) \le 0$ for any $t \le \bar{\tau}$.

We next prove by contradiction that $\lim_{\tau\to 0} \mu(g, \tau) = 0$. Suppose there is a sequence $\tau_k \to 0$ such that $\mu_k = \mu(g, \tau_k) \le -\epsilon < 0$. Let $u_k > 0$ be a minimizer for $\mu(g, \tau_k)$ and it satisfies

$$\tau_k \left(-4\Delta u_k + R u_k\right) - 2u_k \log u_k - n u_k = \mu_k u_k. \tag{32}$$

By the maximum principle, $\max u_k \ge C > 0$. Suppose u_k achieves it maximum at p_k. We choose normal coordinates $\{x\}$ about p_k and let $\rho_0 > 0$ be smaller than the injectivity radius. Note in normal coordinates $g_{ij}(x) = \delta_{ij} + O(|x|^2)$. Define

$$\begin{aligned}
a_{ij}(y) &= g_{ij}(\sqrt{2\tau_k} y) \\
\bar{u}_k(y) &= u_k(\sqrt{2\tau_k} y)
\end{aligned}$$

with $y \in B(0, \rho_0/\sqrt{2\tau_k}) \subset \mathbb{R}^n$. By (32), $\bar{u}_k$ satisfies the following equation on $B(0, \rho_0/\sqrt{2\tau_k})$.

$$-\frac{2}{\sqrt{A}} \frac{\partial}{\partial y^i} \left(\sqrt{A} a^{ij} \frac{\partial \bar{u}_k}{\partial y^j} \right) - 2\bar{u}_k \log \bar{u}_k = (\mu_k - R\tau_k + n)\bar{u}_k, \tag{33}$$

where $\left(a^{ij}\right) = \left(a_{ij}\right)^{-1}$ and $A = \det\ (a_{ij})$. We need some estimate on the sequence $\bar{u_k}$.

$$\begin{aligned} 1 &= \int_M u_k^2 \left(4\pi\tau_k\right)^{-n/2} dV \\ &\geq \int_{|x| \leq \rho_0} u_k^2 \left(4\pi\tau_k\right)^{-n/2} \sqrt{\det\ (g_{ij}(x))} dx \\ &= \int_{|y| \leq \rho_0/\sqrt{2\tau_k}} \bar{u}_k^2 \left(2\pi\right)^{-n/2} \sqrt{\det\ (a_{ij}(y))} dy \\ &\geq C \int_{B(0,\rho_0/\sqrt{2\tau_k})} \bar{u}_k^2 dy. \end{aligned}$$

This shows that the sequence $\bar{u_k}$ is bounded in L^2. By the proof of the existence of a minimizer and (29), there is a constant B such that

$$\tau_k \int_M |\nabla u_k|^2 \left(4\pi\tau_k\right)^{-n/2} dV \leq B. \tag{34}$$

Hence

$$\begin{aligned} B \geq& \tau_k \int_{B(p_k,\rho_0)} |\nabla u_k|^2 \left(4\pi\tau_k\right)^{-n/2} dV \\ =& \int_{|y| \leq \rho_0/\sqrt{2\tau_k}} a^{ij} \frac{\partial \bar{u}_k}{\partial y^i} \frac{\partial \bar{u}_k}{\partial y^j} \left(2\pi\right)^{-n/2} \sqrt{\det\ (a_{ij}(y))} dy \\ \geq& C_1 \int_{B(0,\rho_0/\sqrt{2\tau_k})} |\nabla \bar{u}_k|^2 dy. \end{aligned}$$

Therefore $\{\bar{u}_k\}$ is bounded in $H^1(\mathbb{R}^n)$. By elliptic L^p theory applied to (33), $\{\bar{u}_k\}$ is bounded in $C^{1,\alpha}_{\text{loc}}$. Therefore we can take a convergent subsequence $\bar{u}_k \to \bar{u}$. The limit $\bar{u}$ satisfies the equation

$$-2\Delta\bar{u} - 2\bar{u}\log\bar{u} = (\mu + n)\bar{u}, \tag{35}$$

where $\mu = \lim_{k\to\infty} \mu_k$ is negative by our assumption. As $\bar{u}_k(0) \geq C > 0$, we have $\bar{u}(0) > 0$ and hence it is not identically zero. Moreover $\bar{u} \in H^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \bar{u}^2 \left(2\pi\right)^{-n/2} dy \leq 1.$$

From this and (35) one can easily show that

$$\int_{\mathbb{R}^n} \left(2|\nabla\bar{u}|^2 - 2\bar{u}^2 \log\bar{u} - n\bar{u}^2\right) \left(2\pi\right)^{-n/2} = \mu \int_{\mathbb{R}^n} \bar{u}^2 < 0. \tag{36}$$

Let $\bar{u} = \phi e^{-|x|^2/4}$ and $dm = e^{-|x|^2/2} \left(2\pi\right)^{-n/2} dx$ (the Gaussian measure on $\mathbb{R}^n$). Then we have

$$\int_{\mathbb{R}^n} \phi^2 dm \leq 1 \tag{37}$$

and, by a little computation from (36)

$$\int_{\mathbb{R}^n} \left(|\nabla\phi|^2 - \phi^2 \log\phi\right) dm < 0. \tag{38}$$

This is impossible in view of the Logarithmic Sobolev inequality (see Gross[4])

$$\int_{\mathbb{R}^n} \phi^2 \log\phi dm \le \int_{\mathbb{R}^n} |\nabla\phi|^2 dm + \int_{\mathbb{R}^n} \phi^2 dm \log\left(\int_{\mathbb{R}^n} \phi^2 dm\right)^{1/2}. \tag{39}$$

□

If $\lambda(g) = \lambda_1(-4\Delta + R) > 0$, then $\lim_{\tau\to\infty} \mu(g,\tau) = +\infty$ and therefore

$$\nu(g) = \inf{}_{\tau>0}\mu(g,\tau). \tag{40}$$

Moreover by Proposition 3.2 the inf is achieved by some $\bar\tau > 0$. A crucial property of $\nu(g)$, which follows from (17), is that it is invariant under scaling and diffeomorphism. If $g(t)$ is a solution to the Ricci flow such that $\lambda(g(t)) > 0$ then $\nu(g(t))$ is nondecreasing along the flow. This just follows from the corresponding monotonicity of μ.

We now can rule out the existence of shrinking breathers other than gradient solitons. Suppose $g(t)$ is a solution to the Ricci flow and there exist $t_1 < t_2$ such that $g(t_2) = \alpha\phi^* g(t_1)$ for some $\alpha > 1$ and some diffeomorphism ϕ. If $\lambda_1(-4\Delta + R) \le 0$ for some $t_0 \in [t_1, t_2]$, then we can employ the monotonicity of $\overline{\lambda}$ in the same way as we handled the expanding breathers to show that $g(t)$ must be a gradient soliton. So we assume $\lambda(g(t)) > 0$ for all $t \in [t_1, t_2]$. Then we have $\nu(g(t))$ which is nondecreasing on $[t_1, t_2]$. On the other hand, we have $\nu(g(t_2)) = \nu(g(t_1))$ because ν is invariant under scaling and diffeomorphism. Therefore $\nu(g(t))$ is constant and $g(t)$ is a gradient soliton.

4 non-collapsing theorem I

Definition 4.1. *Let $g(t)$ be a solution to the Ricci flow on $[0,T)$. We say $g(t)$ is locally collapsing at T if $\exists t_k \to T$ and $B_k = B(p_k, r_k)$ at t_k such that r_k^2/t_k is bounded, $|\mathrm{Rm}|(g(t_k) \le Cr_k^{-2}$ in B_k and $r_k^{-n}\mathrm{Vol}\, B_k \to 0$.*

Theorem 4.1. *If M is closed and $T < \infty$, then $g(t)$ is not locally collapsing at T.*

Proof. Suppose it is locally collapsing at T, then we have a sequence $t_k \to T$ and and B_k as described in the definition.

We again work with $u = e^{-f/2}$. It is proved previously that $\mu(g,\tau)$ is the infimum of

$$\mathcal{W}(u) = \int_M \left[\tau\left(4|\nabla u|^2 + Ru^2\right) - u^2\log u^2 - nu^2\right](4\pi\tau)^{-n/2} dV \tag{41}$$

under the constraint

$$\int_M u^2 (4\pi\tau)^{-n/2} dV = 1. \tag{42}$$

Let $\tau = r_k^2$. Define

$$u_k = e^{C_k}\phi(r_k^{-1}d(x,p_k)) \tag{43}$$

at t_k, where ϕ is a smooth function on $\mathbb{R}$, equal 1 on $[0,1/2]$, decreasing on $[1/2,1]$ and equal 0 on $[1,\infty)$. C_k is a constant to make u satisfy the constraint (42) i. e.

$$\begin{aligned}(4\pi)^{n/2} =& e^{2C_k} r_k^{-n}\int_{B(p_k,r_k)} \phi(r_k^{-1}d(x,p_k))^2 dV\\ \le& e^{2C_k} r_k^{-n}\mathrm{Vol}\, B_k.\end{aligned}$$

Since $r_k^{-n}\mathrm{Vol}\, B_k \to 0$, this shows that $C_k \to +\infty$. We compute

$$\begin{aligned}\mathcal{W}(u_k) =& (4\pi)^{-n/2}\, r_k^{-n} e^{2C_k}\int_{B(p_k,r_k)} \left(4|\phi'(r_k^{-1}d(x,p_k))|^2 - 2\phi^2\log\phi\right) dV\\ &+ r_k^2\int_{B(p_k,r_k)} Ru^2\,(4\pi)^{-n/2}\, r_k^{-n} dV - n - 2C_k\\ \le& (4\pi)^{-n/2}\, r_k^{-n} e^{2C_k}\int_{B(p_k,r_k)} \left(4|\phi'|^2 - 2\phi^2\log\phi\right) dV\\ &+ r_k^2 \max{}_{B_k} R - n - 2C_k.\end{aligned}$$

Let $V(r) = \mathrm{Vol}\, B(p_k,r)$. As $\mathrm{Ric} \ge -(n-1)C^2 r_k^{-2}$ in B_k, we compare with the space $\mathbb{H}_{-C^2 r_k^{-2}}$ of constant sectional curvature. Let $\bar V(r)$ be the corresponding volume in $\mathbb{H}_{-Cr_k^{-1}}$. It is easy to see that $\bar V(r_k)/\bar V(r_k/2)$ is bounded above by a constant C'. By Bishop comparison theorem, $V(r_k)/V(r_k/2) \le \bar V(r_k)/\bar V(r_k/2) \le C'$. Hence $V(r_k) - V(r_k/2) \le C'V(r_k/2)$. Therefore

$$\begin{aligned}\int_{B(p_k,r_k)} \left(4|\phi'|^2 - 2\phi^2\log\phi\right) dV \le& C\left(V(V(r_k) - V(r_k/2)\right)\\ \le& CV(r_k/2)\\ \le& C\int_{B_k}\phi^2 dV.\end{aligned}$$

Plugging into the previous estimate for $\mathcal{W}$ and using the constraint (42), we get

$$\mathcal{W}(u_k) \le C'' - 2C_k. \tag{44}$$

Since $C_k \to +\infty$ and $\mu(g(t_k), r_k^2) \le \mathcal{W}(g(t_k), u_k, r_k^2)$, we conclude that $\mu(g(t_k), r_k^2) \to -\infty$. By the monotonicity $\mu(g(0), t_k + r_k^2) \le \mu(g(t_k), r_k^2)$ and hence $\mu(g(0), t_k + r_k^2) \to -\infty$. This is impossible for $t_k + r_k^2$ is bounded. □

Remark 2. *In the proof we only use a lower bound for* Ric *and an upper bound for R in B_k.*

Corollary 4.1. *Let $g(t), t \in [0,T)$ be a solution to the Ricci flow on a closed manifold $M, T < \infty$. Assume that for some sequence $t_k \to T, p_k \in M$ and some constant C we have $Q_k = |Rm|(p_k,t_k) \to \infty$ and $|Rm|(x,t) \le CQ_k$, whenever $t < t_k$. Then (a subsequence of) the scalings of $g(t_k)$ at p_k with factors Q_k converges to a complete ancient solution to the Ricci flow, which is κ-noncollapsed on all scales for some $\kappa > 0$*

Proof. Let $\tilde{g}_k(s) = Q_k g(t_k + s/Q_k)$. This is a solution to the Ricci flow on $(-t_k Q_k, 0]$ with bounded curvature.

Let $r_k = 1/\sqrt{Q_k}$, then $|Rm|(g(t_k)) \le C r_k^{-2}$ and r_k^2/t_k is obviously bounded. By the non-collapsing theorem, there exists $\kappa > 0$ such that (passing to a subsequence)

$$\mathrm{Vol}_{\tilde{g}_k(0)}(B(p_k, 1)) = \frac{\mathrm{Vol}_{g(t_k)}(B(p_k, r_k))}{r_k^n} \ge \kappa.$$

As $|Rm|(\tilde{g}_k(0)) \le C$, we conclude that $\mathrm{inj}_{p_k} \tilde{g}_k(0)$ has a positive lower bound by Cheeger's estimate on injectivity radius. The convergence then follows from Hamilton's compactness theorem.

□

5 Arguments for section 6

We can interpret the formal expressions arising in the study of a Ricci flow as the natural geometrric quantities for a certain Riemannian manifold of potentially infinite dimension. Bishop-Gromov comparison theorem can be interpreted as another monotonicity formula for the Ricci flow.

Consider the Ricci flow

$$\frac{d}{dt} g_{ij} = -2R_{ij}$$

and let

$$P_{ijk} = D_i R_{jk} - D_j R_{ik}$$

$$M_{ij} = \Delta R_{ij} - \frac{1}{2} D_i D_j R + 2R_{ijkl} R_{kl} - R_{ik} R_{jk}$$

Harnack quadratic for $g(t)$ on M, introduced by Richard Hamilton is

$$Z = (M_{ab} + \frac{1}{2t} R_{ab}) W_a W_b + 2P_{abc} U_{ab} W_c + R_{abcd} U_{ab} U_{cd}$$

where U_{ab} is any 2 form and W_a is any 1 form. It is found by the fact that it vanishes identically on a homothetically expanding soliton $D_a V_b + D_b V_a = 2R_{ab} + \frac{1}{t} g_{ab}$, where V_a is a vector field.

Look at the manifold $\tilde{M} = M \times S^N \times R^+$ with the following metric:

$$\begin{aligned}
\tilde{g}_{ij} &= g_{ij} \\
\tilde{g}_{\alpha\beta} &= \tau g_{\alpha\beta} \\
\tilde{g}_{00} &= \frac{N}{2\tau} + R \\
\tilde{g}_{i\alpha} &= \tilde{g}_{i0} = \tilde{g}_{\alpha 0} = 0
\end{aligned}$$

where i, j are coordinate indices on M, α, β are coordinate indices on S^N and τ is a coordinate indice on R^+. We will consider backward Ricci flow $(g_{ij})_\tau = 2R_{ij}$.The metric $G_{\alpha\beta}$ on S^N is a metric with constant sectional curvature $\frac{1}{2N}$.

Theorem 5.1. *Components of the curvature tensor of this metric coincide (mod N^{-1}) with the components of the matrix Harnack expression.*

Proof. Fix a point $(p, g, t) \in M \times S^N \times R^+$. Choose normal coordinates around $p \in M$ such that $\Gamma^k_{ij}(p) = 0 \;\; \forall i \;\; j \;\; k$. The list of the Christoffel symbols of our metric $\tilde{g}$ is:

$$\begin{aligned}
\tilde{\Gamma}^k_{\alpha\beta} &= 0 \\
\tilde{\Gamma}^k_{i\beta} &= 0 \\
\tilde{\Gamma}^k_{i\tau} &= g^{ks} R_{is} \\
\tilde{\Gamma}^\tau_{\alpha\beta} &= -\frac{1}{2} g^{\tau\tau} g_{\alpha\beta} \\
\tilde{\Gamma}^k_{\alpha\tau} &= 0 \\
\tilde{\Gamma}^\tau_{ij} &= -g^{\tau\tau} R_{ij} \\
\tilde{\Gamma}^\tau_{i\beta} &= 0 \\
\tilde{\Gamma}^\tau_{\tau\tau} &= \frac{1}{2} g^{\tau\tau}(-\frac{N}{2\tau^2} + R_\tau) \\
\tilde{\Gamma}^\tau_{\alpha\tau} &= 0 \\
\tilde{\Gamma}^l_{\tau\tau} &= -\frac{1}{2} g^{ls} \frac{\partial R}{\partial x_s} \\
\tilde{\Gamma}^\tau_{i\tau} &= \frac{1}{2} g^{\tau\tau} \frac{\partial R}{\partial x_i} \\
\tilde{\Gamma}^\gamma_{\tau\tau} &= 0 \\
\tilde{\Gamma}^\gamma_{\alpha\tau} &= \frac{1}{2\tau} g^{\gamma\delta} g_{\alpha\delta}
\end{aligned}$$

Compute $\tilde{\mathrm{R}}_{\mathrm{ijkl}}$:

$$\begin{aligned}
\tilde{R}_{ijkl} &= \tilde{g}_{ks} \tilde{R}^s_{ijl} \\
\tilde{R}^s_{ijl} &= \frac{\partial}{\partial x_i} \Gamma^s_{jl} - \frac{\partial}{\partial x_j} \Gamma^s_{il} + \tilde{\Gamma}^s_{ip} \tilde{\Gamma}^p_{jl} - \tilde{\Gamma}^s_{jp} \tilde{\Gamma}^p_{il} \\
\tilde{R}^s_{ijl} &= R^s_{ijl} + \tilde{\Gamma}^s_{i\tau} \tilde{\Gamma}^\tau_{jl} - \tilde{\Gamma}^s_{j\tau} \tilde{\Gamma}^\tau_{il} \\
\tilde{R}_{ijkl} &= R_{ijkl} - g_{ks} g^{sr} R_{ir} g^{\tau\tau} R_{jl} + g_{ks} g^{sr} R_{jr} g^{\tau\tau} R_{il} \\
\tilde{R}_{ijkl} &= R_{ijkl} + O(\frac{1}{N})
\end{aligned}$$

since $g^{\tau\tau}$ is of order $\frac{1}{N}$ and we see that $\tilde{R}_{ijkl}$ correspond to the third coefficient in the Harnack quadratic.

Compute $\tilde{\mathrm{R}}_{\mathrm{i}\tau\mathrm{j}\tau}$:

$$\begin{aligned}
\tilde{R}_{i\tau j\tau} &= \tilde{g}_{js}\tilde{R}^s_{i\tau\tau} \\
\tilde{R}^s_{i\tau\tau} &= \frac{\partial}{\partial x_i}\tilde{\Gamma}^s_{\tau\tau} - \frac{\partial}{\partial \tau}\tilde{\Gamma}^s_{i\tau} + \tilde{\Gamma}^s_{i\tau}\tilde{\Gamma}^\tau_{\tau\tau} - \tilde{\Gamma}^s_{\tau l}\tilde{\Gamma}^l_{i\tau} - \tilde{\Gamma}^s_{\tau\tau}\tilde{\Gamma}^\tau_{i\tau} \\
\tilde{R}^s_{i\tau\tau} &= -\frac{1}{2}\frac{\partial}{\partial x_i}g^{sr}\frac{\partial}{\partial x_r}R - \frac{1}{2}g^{sr}\frac{\partial^2}{\partial x_i \partial x_r}R - g^{sr}(R_{ir})_\tau + 2g^{sp}g^{rq}R_{pq}R_{ir} + \\
+ \quad & g^{sl}R_{il}\frac{1}{2}g^{\tau\tau}(-\frac{N}{2\tau^2} + R_\tau) - g^{sr}R_{lr}g^{lp}R_{ip} + \frac{1}{2}g^{sr}\frac{\partial}{\partial x_r}R\frac{g^{\tau\tau}}{2}\frac{\partial}{\partial x_i}R \\
\tilde{R}_{i\tau j\tau} &= -\frac{1}{2}g^{js}\frac{\partial}{\partial x_i}g^{sr}\frac{\partial}{\partial x_r}R - \frac{1}{2}g^{js}g^{sr}\frac{\partial^2}{\partial x_i \partial x_r}R - \frac{\partial}{\partial \tau}R_{ij} + 2g^{rq}R_{jq}R_{ir} - \frac{1}{2\tau}R_{ij} - g^{lp}R_{lj}R_{ip} + O(\frac{1}{N})
\end{aligned}$$

We have the evolution equation for R_{ij}:

$$\frac{d}{d\tau}R_{ij} = -\Delta R_{ij} - 2R_{ipjq}R_{pq} + 2R_{pi}R_{pj}$$

Using that we get the following expression for $\tilde{R}_{i\tau j\tau}$:

$$\tilde{R}_{i\tau j\tau} = (\Delta R_{ij} - \frac{1}{2}D_iD_jR + 2R_{ipjq}R_{pq} - R_{pi}R_{pj}) - \frac{1}{2\tau}R_{ij} + O(\frac{1}{N})$$

Finally we get:

$$\tilde{R}_{i\tau j\tau} = M_{ij} - \frac{1}{2\tau}R_{ij} + O(\frac{1}{N})$$

.

which correspond to the first coefficient in Harnack quadratic Z.

Compute $\tilde{\mathrm{R}}_{\mathrm{ij}\tau\mathrm{k}}$:

$$\begin{aligned}
\tilde{R}_{ij\tau k} &= \tilde{g}_{\tau\tau}\tilde{R}^\tau_{ijk} \\
\tilde{R}^\tau_{ijk} &= \frac{\partial}{\partial x_i}\tilde{\Gamma}^\tau_{jk} - \frac{\partial}{\partial x_j}\tilde{\Gamma}^\tau_{ik} + \tilde{\Gamma}^\tau_{i\tau}\tilde{\Gamma}^\tau_{jk} - \tilde{\Gamma}^\tau_{j\tau}\tilde{\Gamma}^\tau_{ik} \\
\tilde{R}_{ij\tau k} &= \frac{\partial}{\partial x_j}R_{ik} - \frac{\partial}{\partial x_i}R_{jk} + O(\frac{1}{N})
\end{aligned}$$

Finally we get:

$$\tilde{R}_{ij\tau k} = -P_{ijk} + O(\frac{1}{N})$$

$$\tilde{R}_{ijk\tau} = P_{ijk} + O(\frac{1}{N})$$

which corresponds to the second coefficient in Harnack quadratic Z.

□

Theorem 5.2. *All components of the Ricci tensor of $\tilde{g}$ are 0 (mod N^{-1}).*

Proof. The proof of the theorem is just routine computation, using the formulas for the curvature and computed Christoffel symbols. For the convenience of a reader we will compute $\tilde{R}_{ij}$.

$$\begin{aligned}\tilde{R}_{ij} &= \tilde{g}^{ab}\tilde{R}_{iajb} = g^{kl}\tilde{R}_{ijkl} + g^{\alpha\beta}\tilde{R}_{i\alpha j\beta} + \tilde{g}^{\tau\tau}\tilde{R}_{i\tau j\tau} \\ \tilde{R}_{ij} &= g^{kl}R_{ijkl} + g^{\alpha\beta}\tilde{R}_{i\alpha j\beta} + O(\frac{1}{N})\end{aligned}$$

Easy computation will give us

$$\tilde{R}_{i\alpha j\beta} = -\frac{1}{2}g_{js}g^{sr}R_{ir}g_{\alpha\beta}g^{\tau\tau}$$

From here we immediatelly see that $\tilde{R}_{i\alpha j\beta} = O(\frac{1}{N})$ and

$$\begin{aligned}\tilde{g}^{\alpha\beta}\tilde{R}_{i\alpha j\beta} &= -\frac{1}{2}\tilde{g}^{\alpha\beta}g_{js}g^{sr}R_{ir}g_{\alpha\beta}g^{\tau\tau} \\ &= -\frac{1}{2\tau}g^{\alpha\beta}g_{\alpha\beta}R_{ij}g^{\tau\tau} = -\frac{N}{2\tau}R_{ij}g^{\tau\tau} \\ &= -R_{ij} + O(\frac{1}{N})\end{aligned}$$

Finally, we get:

$$\tilde{R}_{ij} = O(\frac{1}{N})$$

□

One more thing that we can notice is that the heat equation and the conjugate heat equation(conjugate heat operator is $\Box^* = -\frac{\partial}{\partial t} - \Delta + R$) can be interpreted via Laplace equations on $\tilde{M}$.

Theorem 5.3.

(a) If u satisfies the heat equation on M then $\tilde{u}$ satisfies $\tilde{\Delta}\tilde{u} = 0$ (mod N^{-1}), where $\tilde{u}$ is the extension of u to $\tilde{M}$, constant along the S^N fibers.

(b) If u satisfies the conjugate heat equation on M then $\tilde{u}^* = \tau^{-\frac{N-1}{2}}\tilde{u}$ satisfies $\tilde{\Delta}\tilde{u}^* = 0$ (mod N^{-1}).

Proof. We will prove only (a), since the proof of (b) is very similar to the proof of (a).

$$\tilde{\Delta}\tilde{u} = \frac{1}{\sqrt{det\tilde{g}}}\frac{\partial}{\partial x_a}(\tilde{g}^{ab}\frac{\partial}{\partial x_b}\tilde{u}\sqrt{det\tilde{g}})$$

$$\sqrt{\det\tilde{g}} = \sqrt{\det g}\sqrt{\frac{N}{2\tau} + R}\sqrt{\det g_{S^N}}\,\tau^{\frac{N}{2}}$$

Since $\tilde{u}$ is constant along the S^N fibers,we have

$$
\begin{aligned}
\tilde{\Delta}\tilde{u} &= \frac{1}{\sqrt{\det\tilde{g}}}(\frac{\partial}{\partial x_k}(\tilde{g}^{kl}\sqrt{\det\tilde{g}}\frac{\partial}{\partial x_l}\tilde{u}) + \frac{\partial}{\partial\tau}(\tilde{g}^{\tau\tau}\sqrt{\det\tilde{g}}\frac{\partial}{\partial\tau}\tilde{u})) \\
&= \frac{1}{\sqrt{\det\tilde{g}}}[\frac{\partial}{\partial x_k}(g^{kl}\sqrt{\det g_M\frac{\partial}{\partial x_l}u})\tau^{\frac{N}{2}}\sqrt{\frac{N}{2\tau}+R}\sqrt{\det g_S^N} + \frac{\partial}{\partial\tau}\tilde{u}\tilde{g}^{\tau\tau}\frac{N}{2}\tau^{\frac{N}{2}-1}\sqrt{\det g_M\det g_S(\frac{N}{2\tau}+R)} + \\
&+ g^{kl}\sqrt{\det g_M}\tau^{\frac{N}{2}}\sqrt{\det g_S}\frac{\partial}{\partial x_l}\tilde{u}\frac{\frac{\partial}{\partial x_k}R}{2\sqrt{\frac{N}{2\tau}+R}} + \tilde{g}^{\tau\tau}\sqrt{\det\tilde{g}}\frac{\partial^2}{\partial\tau^2}\tilde{u} + \\
&+ \frac{\partial}{\partial\tau}\tilde{u}\sqrt{\det\tilde{g}}(\frac{-\frac{N}{2\tau^2}+R_\tau}{)}(\tilde{g}^{\tau\tau})^2 + \frac{\partial}{\partial\tau}\tilde{u}\tilde{g}^{\tau\tau}\tau^{\frac{N}{2}}\sqrt{\det g_S(\frac{N}{2\tau}+R)}\frac{\partial}{\partial\tau}\sqrt{\det g_M} + \\
&+ \frac{\partial}{\partial\tau}\tilde{g}^{\tau\tau}\tau^{\frac{N}{2}}\sqrt{\det g_S\det g_M}\frac{(-\frac{N}{2\tau^2}+R_\tau)}{2\sqrt{\frac{N}{2\tau}+R}}]
\end{aligned}
$$

Analyzing term by term we will see that all exepct the first two are mod $O(\frac{1}{N})$, i.e.

$$\tilde{\Delta}\tilde{u} = \Delta u + \frac{\partial}{\partial\tau}\tilde{u}(\tilde{g}^{\tau\tau}\frac{N}{2\tau}) + O(\frac{1}{N})$$

$$\tilde{\Delta}\tilde{u} = \Delta u + \frac{\partial}{\partial\tau}u + O(N^{-1})$$

So if u is a solution of the heat equation, $\tilde{\Delta}\tilde{u} = 0$ (mod N^{-1}).

□

6 Interpretation of Bishop-Gromov relative comparison principle

This section is just supposed to be a motivation for another monotonicity formula of reduced volumes of a Ricci flow and proofs here will not be rigorous. We just want to give an intuitive picture of what will follow later in a more rigorous setting.

Consider a metric ball in $(\tilde{M}, \tilde{g})$ centered at p, where $\tau = 0$. The shortest geodesic between p and arbitrary q is always orthogonal to S^N fibre, since at $\tau = 0$ the metric of the sphere S^N degenerates and it shrinks to a point. Let $\gamma(\tau) = (\gamma_M(\tau), *, \tau)$ be a shortest geodesic between $p = (p, s, 0)$ and $q = (q, s_1, \tau(q))$, where s and s_1 are points on the sphere. Length of $\gamma(\tau)$ is:

$$l(\gamma(\tau)) = \int_0^{\tau(q)} \sqrt{|\dot{\gamma}(\tau)|^2 + \frac{N}{2\tau} + Rd\tau}$$

where $|\dot{\gamma_M}(\tau)|$ is the length of the given vector in the metric g on M. The above length we can write in the form

$$
\begin{aligned}
l(\gamma(\tau)) &= \int_0^{\tau(q)} (1 + \frac{R\tau}{N} + \frac{|\tau\dot{\gamma}(\tau)|^2}{N})\sqrt{\frac{N}{2\tau}}d\tau + O(N^{-\frac{3}{2}}) \\
l(\gamma(\tau)) &= \sqrt{2N\tau(q)} + \frac{1}{\sqrt{2N}}\int_0^{\tau(q)} \sqrt{\tau}(R + |\dot{\gamma}(\tau)|^2)d\tau + O(N^{-\frac{3}{2}})
\end{aligned}
$$

The shortest geodesic should minimize

$$
\mathcal{L}(\gamma) = \int_0^{\tau(q)} \sqrt{\tau}(R + |\dot{\gamma}_M(\tau)|^2)d\tau
$$

Let $L(q_M)$ be the corresponding minimum.

Claim 3. *The metric sphere* $S(\sqrt{2N\tau(q)}) \subset \tilde{M}$*, centred at* p*, in metric* $\tilde{g}$ *is* $O(N^{-1})$ *close to a hypersurface* $\tau = \tau(q)$*.*

Proof. Let $x \in S(\sqrt{2N\tau(q)})$. The distance between x and p is:

$$
\sqrt{2N\tau(q)} = d(x,p) = \sqrt{2N\tau(x)} + \frac{1}{\sqrt{2N}}L(x) + O(N^{-\frac{3}{2}})
$$

From the above equation we get

$$
\sqrt{2N\tau(x)} - \sqrt{2N\tau(q)} + \frac{1}{\sqrt{2N}}L(x) + O(N^{-\frac{3}{2}}) = 0
$$

This we can write as

$$
\sqrt{\tau(x)} - \sqrt{\tau(q)} = -\frac{1}{2N}L(x) + O(N^{-2}) = O(N^{-1})
$$

and then our claim follows immediatelly. □

Call the hypersurface from the previous claim H.Let $I = \frac{\mathrm{Vol}(S(\sqrt{2N\tau(q)}))}{\tilde{\mathrm{Vol}}(S(\sqrt{2N\tau(q)}))}$, where $\tilde{\mathrm{Vol}}$ is the volume in a simply connected manifild with constant sectional curvature 0, mod N^{-1}. Let

$$
J(\tau) = \int_M \tau(q)^{\frac{N}{2}} e^{-\frac{1}{\sqrt{2\tau(q)}}L(x)} dV_M
$$

Claim 4. $J(\tau)$ *is decreasing in* τ *modulo* N^{-1}*.*

Proof.

$$
\begin{aligned}
\mathrm{Vol}(S(\sqrt{2N\tau(q)})) &= \int_H dV_H + O(N^{-1}) = \int_{M\times S^N} dV_g dV_h \\
&= \int_M \tau(x)^{\frac{N}{2}} vol_h(S^N) dV_M + O(N^{-1}) \\
&= \mathrm{Vol}_h(S^N)\int_M (\sqrt{\tau(q)} - \frac{1}{2N}L(x) + O(N^{-2}))^N dV_M
\end{aligned}
$$

where h is a metric on S^N.

$$\tilde{\text{Vol}}(S(\sqrt{2N\tau(q)})) = (2N\tau(q))^{\frac{N+n}{2}}\text{Vol}(S^N)$$

where $\text{Vol}(S^N)$ is a volume of a standard sphere.

$$\text{Vol}_h(S^N) = (2N\tau)^{\frac{N}{2}}\text{Vol}(S^N)$$

$$I(\tau(q)) = C(N\tau(q))^(-\frac{n}{2})\int_M e^{-\frac{1}{\sqrt{2\tau(q)}}L(x)}dV_M + O(N^{-1})$$

We have seen in the previous section that $(\tilde{M},\tilde{g})$ is Ricci flat modulo N^{-1} so we can apply Bishop-Gromov volume comparison principle to $\tilde{M}$ to conclude that $I(\tau)$ and therefore $J(\tau)$ (because of the previous equality) is decreasing in τ, modulo N^{-1}. □

7 Arguments for section 7

Let M be a manifold and $g(\tau)$ a solution to a Ricci flow $(g_{ij}(\tau))_\tau = 2R_{ij}$. The assumption is that $g_{ij}(\tau)$ is complete and have uniformly bounded curvature. Define the $\mathcal{L}$ - length for each curve $\gamma(\tau)$ for $0 < \tau_1 \le \tau \le \tau_2$:

$$\mathcal{L} = \int_{\tau_1}^{\tau_2}\sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2)d\tau$$

Let $X(\tau) = \dot{\gamma}(\tau)$, $Y(\tau)$ be any vector field along $\gamma(\tau)$. We know that $[X,Y] = 0$.

Claim 5. *$\mathcal{L}$ - geodesic satisfies:*

$$\nabla_X X - \frac{1}{2}\nabla R + \frac{1}{2\tau}X + 2\text{Ric}(X,\cdot) = 0$$

Proof. The first variation formula is:

$$\begin{aligned}
\delta_Y(\mathcal{L}) &= \int_{\tau_1}^{\tau_2}\sqrt{\tau}(\langle Y,\nabla R\rangle) + 2\langle\nabla_Y X, X\rangle)d\tau \\
&= \int_{\tau_1}^{\tau_2}\sqrt{\tau}(\langle Y,\nabla R\rangle) + 2\langle\nabla_X Y, X\rangle)d\tau \\
&= \int_{\tau_1}^{\tau_2}\sqrt{\tau}(\langle Y,\nabla R\rangle + 2\frac{d}{d\tau}\langle Y,X\rangle - 2\langle Y,\nabla_X X\rangle - 2\text{Ric}(Y,X))d\tau \\
&= 2\sqrt{\tau}\langle Y,X\rangle|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\sqrt{\tau}(\langle Y,\nabla R\rangle - 2\langle Y,\nabla_X X\rangle - 2\text{Ric}(Y,X) - \frac{1}{\tau}\langle X,Y\rangle)d\tau
\end{aligned}$$

Now we immediatelly get the claim. □

For any p and q and $\tau_2 > \tau_1 > 0$, there is always an $\mathcal{L}$ - shortest geodesic $\gamma(\tau)$ for $\tau \in [\tau_1,\tau_2]$ such that $\gamma(\tau_1) = p$ and $\gamma(\tau_2) = q$.

Claim 6. *It can be extended to $\tau_1 = 0$.*

Proof. Multiply the previous equation for $\mathcal{L}$ geodesic by $\sqrt{\tau}$. We get:

$$\nabla_X(\sqrt{\tau}X) - \frac{\sqrt{\tau}}{2}\nabla R + 2\sqrt{\tau}\mathrm{Ric}(X,\cdot) = 0$$

From here we get that $\int_{\tau_1}^{\tau_2} |\nabla(X\sqrt{\tau})|^2 d\tau \leq C(\tau_2 - \tau_1)$. Therefore, $\sqrt{\tau}X$ is continuous and has a limit when $\tau \to 0$, and we can extend the definition of $\mathcal{L}$ - length to $\tau = 0$.

□

From now on, we fix p and $\tau_1 = 0$.Denote by $L(q,\bar{\tau})$ the $\mathcal{L}$ - length of the $\mathcal{L}$ -shortest curve $\gamma(\tau)$ for $0 \leq \tau \leq \bar{\tau}$, connecting p and q.

From the first variation formula we get that $\nabla L(q,\bar{\tau}) = 2\sqrt{\bar{\tau}}X$ so, $|\nabla L|^2 = 4\bar{\tau}|X|^2 = -4\bar{\tau}R + 4\bar{\tau}(R + |X|^2)$.

$$\frac{d}{d\tau}L(\gamma(\tau),\tau)|_{\tau=\bar{\tau}} = \langle \nabla L, X\rangle + L_\tau(\gamma(\bar{\tau}),\bar{\tau})$$

$$\frac{d}{d\tau}L(\gamma(\tau),\tau)|_{\tau=\bar{\tau}} = \frac{d}{d\tau}\int_0^\tau \sqrt{\tau'}(R + |X|^2)d\tau'|_{\tau=\bar{\tau}} = \sqrt{\bar{\tau}}(R(\bar{\tau}) + |X|^2)$$

From the above equations we get

$$L_\tau(q,\bar{\tau}) = \sqrt{\bar{\tau}}(R + |X|^2) - \langle X, \nabla L\rangle = 2\sqrt{\bar{\tau}}R - \sqrt{\bar{\tau}}(R + |X|^2)$$

Now compute:

$$\begin{aligned}
\frac{d}{d\tau}(R(\gamma(\tau),\tau) + |X|^2) &= R_\tau + \langle \nabla R, X\rangle + 2\langle \nabla_X X, X\rangle + 2\mathrm{Ric}(X,X) \\
&= R_\tau + 2\langle \nabla R, X\rangle - \frac{1}{\tau}|X|^2 - 2\mathrm{Ric}(X,X) \\
&= -H(X) - \frac{1}{\tau}(R + |X|^2)
\end{aligned}$$

where $H(X) = -R_\tau - \frac{1}{\tau}R - 2\langle \nabla R, X\rangle + 2\mathrm{Ric}(X,X)$.

$$\begin{aligned}
\frac{d}{d\tau}(\tau^{\frac{3}{2}}(R + |X|^2))|_{\tau=\bar{\tau}} &= \frac{1}{2}\sqrt{\bar{\tau}}(R + |X|^2) - \bar{\tau}^{\frac{3}{2}}H \\
&= \frac{1}{2}\frac{d}{d\tau}(L(\gamma(\tau),\tau))|_{\tau=\bar{\tau}} - \bar{\tau}^{\frac{3}{2}}H
\end{aligned}$$

Denote by $K(\gamma,\bar{\tau}) = \int_0^{\bar{\tau}} \tau^{\frac{3}{2}}H d\tau$. Then we have:

$$\bar{\tau}^{\frac{3}{2}}(R + |X|^2) = \frac{1}{2}L(q,\bar{\tau}) - K$$

Now we get:

$$\begin{aligned}
L_\tau(q,\bar{\tau}) &= 2\sqrt{\bar{\tau}}R + \frac{1}{\bar{\tau}}K - \frac{1}{2\bar{\tau}}L \\
|\nabla L|^2 &= -4\bar{\tau}R - \frac{4}{\sqrt{\bar{\tau}}}K + \frac{2}{\sqrt{\bar{\tau}}}L
\end{aligned}$$

$$2\sqrt{\bar{\tau}}L_\tau + |\nabla L|^2 = -\frac{2}{\sqrt{\bar{\tau}}}K + \frac{1}{\sqrt{\bar{\tau}}}L$$

Next we will compute the second variation:

$$\delta_Y^2(\mathcal{L}) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R \cdot Y \cdot Y + 2\langle \nabla_Y \nabla_Y X, X\rangle + 2|\nabla_Y X|^2)d\tau$$

$$\begin{aligned}
\frac{d}{d\tau}\langle \nabla_Y Y, X\rangle &= \frac{d}{d\tau}(g_{ij}(Y^k \frac{\partial}{\partial x_k}Y^i + \Gamma_{kl}^i Y^k Y^l)X^j) \\
&= 2R_{ij}Y^k \frac{\partial Y^i}{\partial x_k}X^j + g_{ij}\frac{\partial \Gamma_{kl}^i}{\partial \tau}Y^k Y^l X^j + \langle \nabla_Y Y, \nabla_X X\rangle + \langle \nabla_X \nabla_Y Y, X\rangle \\
&= 2R_{ij}Y^k \frac{\partial Y^i}{\partial x_k}X^j + (R_{kj,l} + R_{lj,k} - R_{kl,j})Y^k Y^l X^j + \langle \nabla_Y Y, \nabla_X X\rangle + \langle \nabla_X \nabla_Y Y, X\rangle \\
&= 2R_{ij}Y^k \frac{\partial Y^i}{\partial x_k}X^j + 2\frac{\partial R_{kj}}{\partial x_l}Y^k Y^l X^j - \frac{\partial R_{kl}}{\partial x_j}Y^k Y^l X^j + \langle \nabla_Y Y, \nabla_X X\rangle + \langle \nabla_X \nabla_Y Y, X\rangle \\
&= 2\frac{\partial}{\partial x_k}(R_{ij}Y^i X^j)Y^k - 2R_{ij}Y^k Y^i \frac{\partial X^j}{\partial x_k} - X^j \frac{\partial}{\partial x_j}(R_{kl}Y^k Y^l) + \\
&+ 2R_{kl}X^j \frac{\partial Y^k}{\partial x_j}Y^l + \\
&+ \langle \nabla_Y Y, \nabla_X X\rangle + \langle \nabla_X \nabla_Y Y, X\rangle \\
&= \langle \nabla_Y Y, \nabla_X X\rangle + \langle \nabla_X \nabla_Y Y, X\rangle + 2Y \cdot \mathrm{Ric}(X, Y) - X \cdot \mathrm{Ric}(Y, Y)
\end{aligned}$$

If $Y(0) = 0$ the formula for the second variation becomes:

$$\begin{aligned}
\delta_Y^2(\mathcal{L}) &= \int_0^{\bar{\tau}} \sqrt{\bar{\tau}}(Y \cdot Y \cdot R + 2\frac{d}{d\tau}\langle \nabla_Y Y, X\rangle + 2\langle R(Y, X)Y, X\rangle - 2\langle \nabla_Y Y, \nabla_X X\rangle - 4Y \cdot \mathrm{Ric}(X, Y) + \\
&+ 2X \cdot \mathrm{Ric}(Y, Y) + 2|\nabla_Y X|^2)d\tau \\
&= \int_0^{\bar{\tau}} \sqrt{\bar{\tau}}(Y \cdot Y \cdot R + 2\frac{d}{d\tau}\langle \nabla_Y Y, X\rangle + 2\langle R(Y, X)Y, X\rangle - 2\langle \nabla_Y Y, \frac{1}{2}\nabla R - \frac{1}{2\tau}X - 2\mathrm{Ric}(X, \cdot)\rangle - \\
&- 4Y \cdot \mathrm{Ric}(X, Y) + 2X \cdot \mathrm{Ric}(Y, Y) + 2|\nabla_Y X|^2 d\tau \\
&= \int_0^{\bar{\tau}} \sqrt{\bar{\tau}}(\nabla_Y \nabla_Y R + 2\frac{d}{d\tau}\langle \nabla_Y Y, X\rangle + 2\langle R(Y, X)Y, X\rangle - 4\nabla_Y \mathrm{Ric}(X, Y) - 4\mathrm{Ric}(\nabla_Y X, Y) - \\
&- 4\mathrm{Ric}(X, \nabla_Y Y) + 2\nabla_X \mathrm{Ric}(Y, Y) + 4\mathrm{Ric}(\nabla_Y X, Y) + 2|\nabla_Y X|^2 + \frac{1}{\tau}\langle \nabla_Y Y, X\rangle + 4\mathrm{Ric}(X, \nabla_Y Y))d\tau \\
&= \int_0^{\bar{\tau}} \sqrt{\bar{\tau}}[2\frac{d}{d\tau}(\sqrt{\tau}\langle \nabla_Y Y, X\rangle) + \sqrt{\tau}(\nabla_Y \nabla_Y R + \\
&+ 2\langle R(Y, X)Y, X\rangle - 4\nabla_Y \mathrm{Ric}(X, Y) + 2\nabla_X \mathrm{Ric}(Y, Y) + 2|\nabla_Y X|^2)]d\tau \\
&= 2\sqrt{\bar{\tau}}\langle \nabla_Y Y, X\rangle(\bar{\tau}) + \int_0^{\bar{\tau}} \sqrt{\tau}(\nabla_Y \nabla_Y R + 2\langle R(Y, X)Y, X\rangle - 4\nabla_Y \mathrm{Ric}(X, Y) + \\
&+ 2\nabla_X \mathrm{Ric}(Y, Y) + 2|\nabla_Y X|^2)d\tau
\end{aligned}$$

Fix Y at $\tau = \bar{\tau}$, assuming $|Y(\bar{\tau})| = 1$ and construct Y by solving the following ODE on $[0, \bar{\tau}]$:

$$\nabla_X Y = -\mathrm{Ric}(Y,\cdot) + \frac{1}{2\tau}Y$$

Then:

$$\frac{d}{d\tau}\langle Y,Y\rangle = 2\mathrm{Ric}(Y,Y) + 2\langle \nabla_X Y, Y\rangle = \frac{1}{\tau}\langle Y,Y\rangle$$

It follows that $|Y(\tau)|^2 = \frac{\tau}{\bar\tau}$ and $Y(0) = 0$.

Claim 7.

$$\mathrm{Hess}_L(Y,Y) \leq \frac{1}{\sqrt{\bar\tau}} - 2\sqrt{\bar\tau}\mathrm{Ric}(Y,Y) - \int_0^{\bar\tau} \sqrt{\tau} H(X,Y)d\tau$$

where $H(X,Y) = -\nabla_Y\nabla_Y R - 2\langle R(Y,X)Y,X\rangle - 4(\nabla_X \mathrm{Ric}(Y,Y) - \nabla_Y \mathrm{Ric}(Y,X)) - 2\mathrm{Ric}_\tau(Y,Y) + 2|\mathrm{Ric}(Y,\cdot)|^2 - \frac{1}{\tau}\mathrm{Ric}(Y,Y)$.

Proof. Remember that $L(q,\bar\tau) = 2\sqrt{\bar\tau}X$. Then $\langle \nabla_Y Y, \nabla L\rangle = 2\sqrt{\bar\tau}\langle \nabla_Y Y, X\rangle$. We also have that $\nabla_X Y = -\mathrm{Ric}(Y,\cdot) + \frac{1}{2\tau}Y$ and $|Y(\tau)|^2 = \frac{\tau}{\bar\tau}$.

$$\begin{aligned}
\mathrm{Hess}_L(Y,Y) &\leq Y\cdot Y(L)(\bar\tau) - \nabla_Y Y(L)(\bar\tau)\\
&\leq \delta_Y^2 \mathcal{L} - \nabla_Y Y(\mathcal{L})\\
&\leq \int_0^{\bar\tau} \sqrt{\tau}(\nabla_Y\nabla_Y R + 2\langle R(Y,X)Y,X\rangle - 4\nabla_Y \mathrm{Ric}(X,Y) + 2\nabla_X \mathrm{Ric}(Y,Y) +\\
&+ 2|\mathrm{Ric}(Y,\cdot)|^2 + \frac{1}{2\tau\bar\tau} - \frac{2}{\tau}\mathrm{Ric}(Y,Y)
\end{aligned}$$

In the equations above we have the inequality sign, for the following reason: if $L(q,\bar\tau)$ is not smooth at $(q,\bar\tau)$, we can take a barrier function $\tilde{L}(x,\tau) = L(x,\tau-\epsilon)+\delta$ where for each ϵ small, we choose δ such that $L(q,\bar\tau) = \tilde{L}(q,\bar\tau)$. It is easy to show that $\mathrm{Hess}L(q,\bar\tau) \leq \mathrm{Hess}\tilde{L}(q,\bar\tau)$.

To simply further, we compute

$$\begin{aligned}
\frac{d}{d\tau}\mathrm{Ric}(Y(\tau),Y(\tau)) &= \mathrm{Ric}_\tau(Y,Y) + 2\mathrm{Ric}(\nabla_X Y, Y) + \nabla_X \mathrm{Ric}(Y,Y)\\
&= \mathrm{Ric}_\tau(Y,Y) + \nabla_X \mathrm{Ric}(Y,Y) + \frac{1}{\tau}\mathrm{Ric}(Y,Y) - 2|\mathrm{Ric}(Y,\cdot)|^2
\end{aligned}$$

Finally we have:

$$\mathrm{Hess}_L(Y,Y) \leq \frac{1}{\sqrt{\bar\tau}} - 2\sqrt{\bar\tau}\mathrm{Ric}(Y,Y) - \int_0^{\bar\tau} \sqrt{\tau} H(X,Y)d\tau$$

□

Lemma 7.1.

$$\Delta L|_{\bar\tau} \leq \frac{n}{\sqrt{\bar\tau}} - 2\sqrt{\bar\tau}R - \frac{1}{\bar\tau}K$$

Proof. At a given time $\bar{\tau}$ choose an orthonormal basis $\{Y_\alpha\}$ such that

$$\langle Y_\alpha, Y_\beta\rangle = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

Solve for vector fields $Y_\alpha(\tau)$ denoted by Y_α again, such that

$$\nabla_X Y_\alpha = -\mathrm{Ric}(Y_\alpha, \cdot) + \frac{1}{2\tau} Y_\alpha$$

$\langle Y_\alpha, Y_\beta\rangle(\tau) = \frac{\tau}{\bar{\tau}}\delta_{\alpha\beta}$ since:

$$\frac{d}{d\tau}\langle Y_\alpha, Y_\beta\rangle = \langle \nabla_X Y_\alpha, Y_\beta\rangle + \langle \nabla_X Y_\beta, Y_\alpha\rangle + 2\mathrm{Ric}(Y_\alpha, Y_\beta) = \frac{1}{\tau}\langle Y_\alpha, Y_\beta\rangle$$

It follows that $\{\sqrt{\frac{\bar{\tau}}{\tau}} Y_\alpha(\tau)\}$ form an orthonormal basis at τ. Consider $H(X,Y)$.

$$\begin{aligned}
\sum_\alpha H(X, Y_\alpha)(\tau) &= -\frac{\tau}{\bar{\tau}}\Delta R + 2\mathrm{Ric}(X,X) - 4(R_{ij,k}Y_\alpha^i Y_\alpha^j X^k - R_{ij,k}Y_\alpha^i X^j Y_\alpha^k) - 2\mathrm{Ric}_\tau(Y_\alpha, Y_\alpha) + \\
&+ \frac{2\tau}{\bar{\tau}}|\mathrm{Ric}|^2 - \frac{1}{\bar{\tau}}R \\
&= \frac{\tau}{\bar{\tau}}(-\Delta R + 2\mathrm{Ric}(X,X) - 2\frac{\bar{\tau}}{\tau}\mathrm{Ric}_\tau(Y_\alpha, Y_\alpha) + 2|\mathrm{Ric}|^2 - \frac{1}{\tau}R) - \\
&- 4((R_{ij}Y_\alpha^i Y_\alpha^j)_k X^k - 2R_{ij}Y_{\alpha,k}^i Y_\alpha^j X^k - R_{ij,k}Y_\alpha^i X^j Y_\alpha^k) \\
&= \frac{\tau}{\bar{\tau}}(-\Delta R + 2\mathrm{Ric}(X,X) - X(R) + 2|\mathrm{Ric}|^2 - \frac{1}{\tau}R) - 2\frac{\tau}{\bar{\tau}}g^{ij}\mathrm{Ric}_{\tau ij} - 8|\mathrm{Ric}|^2\frac{\tau}{\bar{\tau}} \\
&= \frac{\tau}{\bar{\tau}}(-\Delta R + 2\mathrm{Ric}(X,X) - 2X(R) + 2|\mathrm{Ric}|^2 - \frac{1}{\tau}R - 8|\mathrm{Ric}|^2 - 2R_\tau + 4|\mathrm{Ric}|^2) \\
&= \frac{\tau}{\bar{\tau}}(-\Delta R + 2\mathrm{Ric}(X,X) - 2X(R) - 2|\mathrm{Ric}|^2\frac{1}{\tau}R - 2R_\tau) \\
&= \frac{\tau}{\bar{\tau}}(-R_\tau + 2\mathrm{Ric}(X,X) - \frac{1}{\tau}R - 2X(R)) \\
&= \frac{\tau}{\bar{\tau}}H(X)
\end{aligned}$$

Writing down the inequalities that have been proved above

$$\mathrm{Hess}_L(Y,Y) \leq \frac{1}{\tau} - 2\sqrt{\tau}\mathrm{Ric}(Y,Y) - \int_0^\tau \sqrt{\tilde{\tau}} H(X,Y) d\tilde{\tau}$$

for every Y_α and summing them over α, using that $\sum_\alpha H(X, Y_\alpha) = \frac{\tau}{\bar{\tau}}H(X)$ we get that

$$\Delta L \leq \frac{n}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}}R - \frac{1}{\bar{\tau}}K$$

□

Definition 7.1. *A field $Y(\tau)$ along $\mathcal{L}$ - geodesic $\gamma(\tau)$ is called $\mathcal{L}$-Jacobi field if it is a derivative of a variation of γ along $\mathcal{L}$-geodesics.*

If Y is a $\mathcal{L}$ - Jacobi field with $|Y(\bar{\tau})| = 1$, we have:

Let $\gamma(q,\tau,s)$ be a variation of $\mathcal{L}$ - geodesic γ. Then $Y = \frac{\partial \gamma}{\partial s}$. $L(q,\tau)$ is $\mathcal{L}$ - length of $\gamma(q,\tau,s)$. Then:

$$\langle \nabla L, Y \rangle = 2\sqrt{\tau}\langle X, Y \rangle$$

$$\frac{\partial}{\partial s}\langle \nabla L, Y \rangle = 2\sqrt{\tau}\langle \nabla_Y X, Y \rangle + 2\sqrt{\tau}\langle \nabla_Y Y, X \rangle$$

$$2\sqrt{\bar{\tau}}\langle \nabla_Y X, Y \rangle = Y \cdot Y \cdot L - \nabla_Y Y \cdot L = \mathrm{Hess}_L(Y,Y)$$

$$\begin{aligned}
\frac{d}{d\tau}|Y|^2 &= 2\mathrm{Ric}(Y,Y) + 2\langle \nabla_X Y, Y \rangle \\
&= 2\mathrm{Ric}(Y,Y) + 2\langle \nabla_Y X, Y \rangle \\
&= 2\mathrm{Ric}(Y,Y) + \frac{1}{\sqrt{\bar{\tau}}}\mathrm{Hess}_L(Y,Y) \\
&\leq \frac{1}{\bar{\tau}} - \frac{1}{\sqrt{\bar{\tau}}}\int_0^{\bar{\tau}} \tau^{\frac{1}{2}} H(X,\tilde{Y})d\tau
\end{aligned}$$

where $\tilde{Y}$ is obtained by solving $\nabla_X Y = -\mathrm{Ric}(Y,\cdot) + \frac{1}{2\tau}Y$ with initial data $\tilde{Y}(\bar{\tau}) = Y(\bar{\tau})$. If the last inequality is the equality, then $\tilde{Y}$ has to be a $\mathcal{L}$ - Jacobi field. Then $\tilde{Y} = Y$ and $|Y|^2 = |\tilde{Y}|^2 = \frac{\tau}{\bar{\tau}}$. It implies that:

$$\frac{d}{d\tau}|Y|^2 = \frac{1}{\bar{\tau}} = 2\mathrm{Ric}(Y,Y) + \frac{1}{\sqrt{\bar{\tau}}}\mathrm{Hess}_L(Y,Y)$$

Definition 7.2. *The $\mathcal{L}$ - exponential map $\mathcal{L}\exp : T_pM \times R^+ \to M$ is defined as follows: $\forall\ \ X \in T_pM$ let:*

$$\mathcal{L}\exp_X(\bar{\tau}) = \gamma(\bar{\tau})$$

where $\gamma(\tau)$ is the $\mathcal{L}$ - geodesic, starting at p and having X as the limit of $\sqrt{\tau}\dot{\gamma}(\tau$ as $\tau \to 0$.

Denote by $\mathcal{J}(\tau)$ the Jacobian of $\mathcal{L}\exp(\tau) : T_pM \to M$. As usual $Y := d\mathcal{L}{-}\exp(\bar{\tau})(Y_0)$ is a $\mathcal{L}$ - Jacobi field. Using the estimates on Jacobi fields and choosing the coordinates such that $Y_i := d\mathcal{L} - \exp(\bar{\tau})(\frac{\partial}{\partial x_i})$ are orthogonal to each other at $\gamma(\bar{\tau})$, we get:

$$\begin{aligned}
\frac{d}{d\tau}\ln \mathcal{J}(\tau)|_{\tau=\bar{\tau}} &= \frac{1}{2}\frac{d}{d\tau}\sum |Y_i|^2 \\
&\leq \frac{n}{2\bar{\tau}} - \frac{1}{2\sqrt{\bar{\tau}}}\sum_i \int_0^{\bar{\tau}} \sqrt{\tau}H(X,\tilde{Y}_i)d\tau \\
&= \frac{n}{2\bar{\tau}} - \frac{1}{2}\bar{\tau}^{-\frac{3}{2}}K
\end{aligned}$$

with equality iff $2\mathrm{Ric} + \frac{1}{2\sqrt{\bar{\tau}}}\mathrm{Hess}_L = \frac{1}{\bar{\tau}}g$.

Definition 7.3. *$l(q,\tau)=\frac{1}{2\tau}L(q,\tau)$ is called the reduced distance. $\tilde{V}(\tau)=\int_M \tau^{-\frac{n}{2}}e^{-l}dV_M$ is the reduced volume.*

$$\frac{d}{d\tau}l(\tau)=-\frac{1}{2\bar{\tau}}l+\frac{1}{2}(R+|X|^2)=-\frac{1}{2}\bar{\tau}^{-\frac{3}{2}}K$$

So $\tau^{-\frac{n}{2}}\exp(-l(\tau))\mathcal{J}(\tau)$ is nonincreasing in τ and monotonicity is strict unless we are on a gradient shrinking soliton.

Finally, we have that $\tilde{V}(\tau)=\int_{U\subset T_pM}\tau^{-\frac{n}{2}}\exp(-l(\tau))\mathcal{J}(\tau)dX$. Let $\tau(x)=\sup\{\tau|\gamma(\tau)$ is a minimizing geodesic $\}$. Then $U=U_\tau=\{x\in M|\tau\leq\tau(x)\}$. This family U_τ is decreasing in τ. Therefore, we have the theorem:

Theorem 7.1. *The reduced volume $\tilde{V}(\tau)=\int_M \tau^{-\frac{n}{2}}\exp(-l(q,\tau))dq$ is nonincreasing along the backward Ricci flow.*

We have showed that:

$$L_\tau=2\sqrt{\tau}R-\frac{1}{2\tau}L+\frac{1}{\tau}K$$
$$|\nabla L|^2=-4\tau R+\frac{2}{\sqrt{\tau}}L-\frac{4}{\sqrt{\tau}}K$$

In terms of l these two equations can be written as

$$l_\tau=R-\frac{l}{\tau}+\frac{K}{2\tau^{3/2}}$$
$$|\nabla l|^2=-R+\frac{l}{\tau}-\frac{K}{\tau^{3/2}}$$

Therefore

$$l_\tau+|\nabla l|^2=-\frac{K}{2\tau^{3/2}}$$

By Lemma (7.1) we have

$$\Delta l\leq -R+\frac{n}{2\tau}-\frac{K}{\tau^{3/2}}.$$

From the above two inequalities we have :

$$l_\tau-\Delta l+|\nabla l|^2-R+\frac{n}{2\tau}\geq 0$$

Let $\phi=\tau^{-\frac{n}{2}}\exp(-l)$. Then

$$\begin{aligned}\phi_\tau &= \left(-\frac{n}{2\tau}-l_\tau\right)\phi\\ &\leq (-\Delta l+|\nabla l|^2-R)\phi\\ &= \Delta\phi-R\phi\end{aligned}$$

i.e. we have proved the following inequality

$$\phi_\tau-\Delta\phi+R\phi\leq 0. \tag{45}$$

Finally:

$$\frac{d}{d\tau}\tilde{V}(\tau) = \int_M (\phi_\tau + R\phi)dq \leq \int_M \Delta\phi dq = 0$$

It follows again that $\tilde{V}(\tau)$ is nonincreasing and equality holds iff g is a Ricci soliton.

7.1 The applications of reduced volume monotonicity formula

In this section we will prove the weakened version of no local collapsing theorem that is the application of the comparison inequalities proved in the previous section.

First we will get the upper bound on the minimum of $l(\cdot,\tau)$ for every τ.

Lemma 7.2. $\min_M l(\cdot,\tau) \leq \frac{n}{2}$ *for every* τ.

Proof. First recall a few inequalities that have appeared in our discussion in the previous section.

1. $L_\tau = 2\sqrt{\tau}R - \frac{1}{2\tau}L + \frac{1}{\tau}K$
2. $\Delta L \leq \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau}R - \frac{1}{\tau}K$
3. $|\nabla L|^2 = -4\tau R + \frac{2}{\sqrt{\tau}}L - \frac{4}{\sqrt{\tau}}K$

1 + 2 + 3 gives us

$$l_\tau - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} \geq 0$$

2 + 3 gives us

$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\tau} \leq 0$$

Let $\bar{L} = 2\sqrt{\tau}L$. 1 + 2 gives us

$$\bar{L}_\tau + \Delta\bar{L} \leq 2n$$

$$(\bar{L} - 2n\tau)_\tau + \Delta(\bar{L} - 2n\tau) \leq 0$$

i.e. $\min(\bar{L} - 2n\tau)$ is nonincreasing and therefore $\min_{q\in M} l(\cdot,\tau) \leq \frac{n}{2}$, what we wanted to prove.

□

Lemma 7.3. *If we have a Ricci flow* $(g_{ij})_\tau = 2R_{ij}$, *then* $R(\cdot,\tau) \geq -\frac{n}{2(\tau_0-\tau)}$ *whenever the flow exists for* $\tau \in [0,\tau_0]$.

Proof. The evolution equation for a scalar curvature is $R_\tau = -\Delta R - 2|\mathrm{Ric}|^2$. Look at the corresponding ODE, $R_\tau = -2|\mathrm{Ric}|^2$. Since $R = tr(\mathrm{Ric})$, $|\mathrm{Ric}|^2 \geq \frac{1}{n}R^2$ and therefore, $-2|\mathrm{Ric}|^2 \leq -\frac{2|R|^2}{n}$, i.e. $R_\tau \leq -\frac{2|R|^2}{n}$. By solving this equation we get that the set $R(\cdot,\tau) \geq -\frac{n}{2(\tau_0-\tau)}$ is preserved by the ODE and therefore it is preserved by the corresponding PDE (the evolution equation for R).

□

Lemma 7.4. *If the metrics $g_{ij}(\tau)$ have nonnegative curvature operator and if the flow exists for $\tau \in [0,\tau_0]$, then*

$$|\nabla l|^2 + R \leq \frac{Cl}{\tau}$$

for some constant C, whenever τ is bounded away from τ_0, say $\tau \leq (1-c)\tau_0$, where $c > 0$.

Proof.

$$\begin{aligned} H(X,Y) &= -\nabla_Y\nabla_Y R - 2\langle R(Y,X)Y,X\rangle - 4(\nabla_X \mathrm{Ric}(Y,Y) - \nabla_Y \mathrm{Ric}(Y,X)) - 2\mathrm{Ric}_\tau(Y,Y) + \\ &+ 2|\mathrm{Ric}(Y,\cdot)|^2 + \frac{1}{\tau_0-\tau}\mathrm{Ric}(Y,Y) - (\frac{1}{\tau_0-\tau}\mathrm{Ric}(Y,Y) + \frac{1}{\tau}\mathrm{Ric}(Y,Y)) \\ &\geq -(\frac{1}{\tau} + \frac{1}{\tau_0-\tau})\mathrm{Ric}(Y,Y) \\ &\geq -R(\frac{1}{\tau} + \frac{1}{\tau_0-\tau})|Y|^2 \end{aligned} \tag{46}$$

We know that if we choose an orthonormal basis $\{Y_\alpha\}$ at time $\bar{\tau}$ and if we solve equations $\nabla_X Y = -\mathrm{Ric}(Y,\cdot) + \frac{1}{2\tau}Y$ for $Y_\alpha(\tau)$ then $\{\sqrt{\frac{\bar{\tau}}{\tau}}Y_\alpha(\tau)\}$ will be the orthonormal basis at τ. We also know that $\sum_\alpha H(X,Y_\alpha)(\tau) = \frac{\tau}{\bar{\tau}}H(X)$ and therefore if we write inequlities 46 for all Y_α and sum them up over α we will get

$$H(X) \geq -R(\frac{1}{\tau} + \frac{1}{\tau_0-\tau})$$

From $|\nabla L|^2 = -4\tau R + \frac{2}{\sqrt{\tau}}L - \frac{4}{\sqrt{\tau}}K$ we get that

$$|\nabla l|^2 + R = \frac{l}{\tau} - \frac{1}{\tau\sqrt{\tau}}K$$

$$\begin{aligned} -\tau^{-\frac{3}{2}}\int_0^\tau \tau'^{\frac{3}{2}}H(X)d\tau' &\leq 2\tau^{-1}(\frac{1}{2\sqrt{\tau}}\int_0^\tau R\sqrt{\tau'}(1+\frac{\tau'}{c\tau_0})d\tau') \\ &\leq C\tau^{-1}(\frac{1}{2\sqrt{\tau}}\int_0^\tau \sqrt{\tau'}(R+|\dot{\gamma}|^2)d\tau') = \frac{C}{\tau}l \end{aligned}$$

Finally, we have that

$$|\nabla l|^2 + R \leq \frac{Cl}{\tau}$$

for some constant C.

□

Now we will state the important application of the monotonicity formula, so called noncollapsing theorem.

Theorem 7.2. *Let* $|Rm|(x,t) \leq r_k^{-2}$ *for* $x \in B_{t_k}(x_k, r_k) = B_k$ *and* $t_k - r_k^2 \leq t \leq t_k$*. Then* $r_k^{-n}\mathrm{Vol}(B_k) \geq \kappa > 0$ *for some* $\kappa > 0$*.*

We will prove it by contradiction. Let $\tau_k(t) = t_k - t$, $p = p_k$ and $\epsilon_k = r_k^{-1}\mathrm{Vol}(B_k)^{\frac{1}{n}}$. In a discussion that follows, we will have a sequence of reduced distances l_k and a sequence of reduced volumes $\tilde{V}_k$, defined with respect to points p_k. We will refer to them as $\tilde{V}$ and l, omitting the subscript k. We will apply the same to other quantities that appear in study of reduced volume (as $\mathcal{L}$ - length, $\mathcal{L} - \exp$ map, etc). First we will prove the following lemma:

Proposition 7.1. $\tilde{V}_k(\epsilon_k r_k^2) < 3\epsilon_k^{\frac{n}{2}}$

Proof. Using the $\mathcal{L}$ - exponential map, we can integrate over T_pM rather than M. For any $X \in T_pM$ we find a $\mathcal{L}$ - geodesic $\gamma(\tau)$, starting at p, with $\lim_{\tau\to 0}\sqrt{\tau}\dot{\gamma}(\tau) = X$.

From the fact that $\gamma(\tau)$ is an $\mathcal{L}$ - geodesic we have that:

$$\nabla_{\dot{\gamma}(\tau)}\dot{\gamma}(\tau) - \frac{1}{2}\nabla R + \frac{1}{2\tau}\dot{\gamma} + 2\mathrm{Ric}(\dot{\gamma}(\tau), \cdot) = 0$$

$$\frac{d}{d\tau}(\sqrt{\tau}\dot{\gamma}(\tau)) - \frac{1}{2}\sqrt{\tau}\nabla R + 2\sqrt{\tau}\mathrm{Ric}(\dot{\gamma}(\tau), \cdot) = 0$$

Using this we can get the following equation:

$$\begin{aligned} \frac{d}{d\tau}\langle\sqrt{\tau}\dot{\gamma}(\tau), \sqrt{\tau}\dot{\gamma}(\tau)\rangle &= 2\langle\nabla_{\dot{\gamma}(\tau)}\sqrt{\tau}\dot{\gamma}(\tau), \sqrt{\tau}\dot{\gamma}(\tau)\rangle + 2\mathrm{Ric}(\sqrt{\tau}\dot{\gamma}(\tau), \sqrt{\tau}\dot{\gamma}(\tau)) \qquad (47) \\ &= \langle\sqrt{\tau}\nabla R, \sqrt{\tau}\dot{\gamma}(\tau)\rangle - 2\mathrm{Ric}(\sqrt{\tau}\dot{\gamma}, \sqrt{\tau}\dot{\gamma}) \end{aligned}$$

In the lemma that follows we will see more precisely how $\sqrt{\tau}\gamma'(\tau)$ behaves when $\tau \leq \epsilon_k r_k^2$, for k big enough.

Lemma 7.5. *With the above notation we have:*

$$||\sqrt{\tau}\gamma'(\tau)| - |X|| \leq C\epsilon_k(|X| + 1)$$

for ϵ_k *sufficiently small and* $\tau \leq \epsilon_k r_k^2$*, if* $\gamma(\tilde{\tau}) \in B_k$ *for* $\tilde{\tau} < \tau$*.*

Proof. Let $\tilde{\gamma}(t) = \gamma(t^2)$,where $\tau = t^2$.Then $\tilde{\gamma}'(t) = 2\sqrt{\tau}\gamma'(\tau)$

$$||\sqrt{\tau}\gamma'(\tau)| - |X|| = \frac{1}{2}||\tilde{\gamma}'(t)| - |\tilde{\gamma}'(0)||$$
$$= \frac{1}{2}\int_0^{\sqrt{\tau}} \frac{d}{ds}|\tilde{\gamma}'(s)|ds = \int_0^{\tau} \frac{d}{d\tilde{\tau}}|\sqrt{\tilde{\tau}}\gamma'(\tilde{\tau})|d\tilde{\tau}$$

From the equation 47 we have

$$\frac{d}{d\tau}|\tau\gamma'(\tau)| = 2\langle\sqrt{\tau}\nabla R, \frac{\sqrt{\tau}\gamma'(\tau)}{|\sqrt{\tau}\gamma'(\tau)|}\rangle - 2\mathrm{Ric}(\sqrt{\tau}\gamma'(\tau), \frac{\sqrt{\tau}\gamma'(\tau)}{|\sqrt{\tau}\gamma'(\tau)|})$$

Since $R \leq \frac{1}{r_k^2}$, we have also bound on $|\nabla R|$ of order $\frac{1}{r_k^2}$ and we get

$$||\sqrt{\tau}\gamma'(\tau)| - |X|| \leq \frac{C}{r_k^2}\int_0^{\tau}(\sqrt{\tau} + |\sqrt{\tau}\gamma'(\tau)|) \leq C\epsilon_k(|X| + 1)$$

□

Lemma 7.6. *If $\gamma(\tilde{\tau}) \in B_k$ for $\tilde{\tau} < \tau$ and $|X| \leq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}$, then $\gamma(\tau) \in B_k = B(p_k, r_k)$, where $\tau \leq \epsilon_k r_k^2$.*

Proof. By lemma 7.5 we have that for small ϵ_k, $|\sqrt{\tau}\gamma'(\tau)| \leq \frac{1}{2}\epsilon_k^{-\frac{1}{2}}$. Therefore:

$$|\gamma(\tau) - p_k| \leq \int_0^{\tau}|\gamma'(\tilde{\tau})|d\tilde{\tau} \leq \frac{1}{2}\epsilon_k^{-\frac{1}{2}}\int_0^{\tau}\frac{d\tilde{\tau}}{\sqrt{\tilde{\tau}}} = \epsilon_k^{-\frac{1}{2}}\sqrt{\tau} \leq r_k$$

So $\gamma(\tau) \in B_k$.

□

Lemmas 7.5 and 7.6 give us that $\gamma(\tau) \in B_k$ for all $\tau \leq \epsilon_k r_k^2$. We will prove one more lemma before we finish the proof of proposition 7.1.

Lemma 7.7. *$\tau^{-\frac{n}{2}}\mathcal{J}(\tau) \to 1$ as $\tau \to 0$, where $\mathcal{J}$ is the Jacobian of $\mathcal{L}$ - exponential map $\mathcal{L}\exp : T_pM \times R^+ \to M$.*

Proof.

$$\mathcal{L}(\gamma) = \int_0^{\tau}\sqrt{\tilde{\tau}}(R + |\gamma'|^2)d\tilde{\tau}$$

Let $\tilde{\gamma}(t) = \gamma(\tilde{\tau}) = \gamma(\tilde{\tau}^2)$, where $\tilde{\tau} = t^2$. This gives us that $\tilde{\gamma}'(t) = 2\sqrt{\tilde{\tau}}\gamma'(\tilde{\tau})$.

If we change the variable in the formula for the $\mathcal{L}$ - length of curve γ we get

$$\mathcal{L}(\gamma) = \int_0^{\sqrt{\tau}}(2t^2 R + \frac{|\tilde{\gamma}(t)|^2}{2})dt$$

$$\mathcal{L}(\gamma) = \frac{1}{2}E(\tilde{\gamma}) + o(\sqrt{\tau})$$

since $\int_0^{\sqrt{\tau}} t^2 R dt \leq \frac{1}{r_k^2}\int_0^{\sqrt{\tau}} t^2 dt = \frac{1}{3r_k^2}\sqrt{\tau}\tau \leq \frac{\epsilon_k}{3}\sqrt{\tau}$, where $E(\tilde{\gamma}) = \int_0^{\tau}|\tilde{\gamma}'|^2 ds$. Therefore, we have that $\mathcal{L}_X\exp(\tau) = \exp\frac{\sqrt{\tau}X}{2} + o(\sqrt{\tau})$.

$$\mathcal{J}(\mathcal{L}\exp(\tau)) = \mathcal{J}_0(\tau)\tau^{\frac{n}{2}} + o(\tau^{\frac{n}{2}})$$

$$\mathcal{J}(\mathcal{L}\exp(\tau))\tau^{-\frac{n}{2}} = \mathcal{J}_0(\tau) + o(1)$$

where $\mathcal{J}_0$ is a usual Jacobian of the exponential map and $o(1)$ depends on ϵ_k. When $\epsilon_k \to 0$, $\tau \to 0$ as well and therefore $\mathcal{J}(\mathcal{L}\exp(\tau))\tau^{-\frac{n}{2}} \to 1$ as $\tau \to 0$.

□

Another way to see that $\tau^{-\frac{n}{2}}\mathcal{J}(\tau)$ has a limit and is bounded as $\tau \to 0$ is explained in the following remark. We will consider only set $\{|X| \geq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}\} \subset T_pM$, since we will use the estimate on $\tau^{-\frac{n}{2}}\mathcal{J}$ only when we integrate over that set, for small values of τ.

Remark 8.

$$\frac{d}{d\tau}\ln\mathcal{J}(\tau) \leq \frac{n}{2\tau} - \frac{1}{2}\tau^{-\frac{3}{2}}K$$

where $K = \int_0^\tau \tau'^{\frac{3}{2}}H(X)d\tau'$. *The above inequality is equvivalent to*

$$\frac{d}{d\tau}\tau^{-\frac{n}{2}}\mathcal{J}(\tau) \leq -\frac{1}{2}\tau^{-\frac{3}{2}}K$$

Choose $\tilde{\tau} = (\frac{r_k}{4|X|})^2$, *where* $|\sqrt{\tau}\gamma'(\tau)| \sim |X|$ *as* $\tau \to 0$ *(if* $\gamma(\tau) \in B_k$*). This is justified below, when we discuss the case* $|X| \geq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}$. *Let* $\tau \leq \tilde{\tau}$.

Since we have our curvature bound, we can estimate $H(X)$ *as follows*

$$|H(X)| \leq Cr_k^{-2}(1 + \frac{1}{\tau'} + 2|X| + |X|^2)$$

at time $t = \tau'$. *We can write this as* $|H(X)| \leq Cr_k^{-2}(1 + \frac{1}{\tau'} + 3\frac{r_k^2}{4\tilde{\tau}})$. *Therefore,* $K(X) \leq Cr_k^{-2}((1 + \frac{3r_k^2}{4\tilde{\tau}})\tau^{\frac{5}{2}} + \frac{2}{3}\tau^{\frac{3}{2}})$ *which gives us*

$$\frac{d}{d\tau}\ln(\tau^{-\frac{n}{2}}\mathcal{J}(\tau)) \leq Cr_k^{-2}(1 + \frac{3r_k^2}{4\tilde{\tau}}\tau + \frac{2}{3}\tau^{\frac{3}{2}})$$

Let τ_1 *and* τ_2 *be* $\leq \tilde{\tau} \leq \epsilon_k r_k^2$. *Then:*

$$|\tau_1^{\frac{n}{2}}\mathcal{J}(\tau_1) - \tau_2^{\frac{n}{2}}\mathcal{J}(\tau_2)| \leq |\tau_1 - \tau_2|\frac{C}{r_k^2}(\tilde{\tau} + 3r_k^2 + \frac{2}{3})$$

$$|\tau_1^{\frac{n}{2}}\mathcal{J}(\tau_1) - \tau_2^{\frac{n}{2}}\mathcal{J}(\tau_2)| \leq C\epsilon_k(\tilde{\tau} + 3r_k^2 + \frac{2}{3}) \leq \tilde{C}\epsilon_k$$

It follows that there exists a limit $\lim_{\tau\to 0}\tau^{-\frac{n}{2}}\mathcal{J}(\tau)$ *and therefore* $\tau^{-\frac{n}{2}}\mathcal{J}(\tau) \leq C$ *for small* τ.

To finish the proof of the proposition 7.1 we will consider two cases.

1. $|X| \leq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}$

   By lemma 7.6 we have that in this case $q = \mathcal{L}\exp_X(\epsilon_k r_k^2) \in B_k = B(p_k, r_k)$. We want to estimate:

$$I = \int_{|X|\leq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}} (\epsilon_k r_k^2)^{-\frac{n}{2}} \exp(-\frac{1}{\sqrt{\epsilon_k} r_k} L(q, \epsilon_k r_k^2)\mathcal{J}(\epsilon_k r_k^2))dq \leq \int_{B_k} (\epsilon_k r_k^2)^{-\frac{n}{2}} e^{-l} dq$$

   where we have used lemma 7.6.

   $L(q, \epsilon_k r_k^2)$ we can estimate as follows:

$$L(q, \epsilon_k r_k^2) = \int_0^{\epsilon_k r_k^2} \sqrt{\tilde{\tau}}(R + |\gamma'|^2)d\tilde{\tau} \geq -C r_k^{-2}(\epsilon_k r_k^2)^{\frac{3}{2}} = -C\epsilon_k^{\frac{3}{2}} r_k$$

   Therefore we have:

$$I \leq 2\epsilon_k^{-\frac{n}{2}} r_k^{-n} e^{C\epsilon_k} \mathrm{Vol}(B_k) \leq 2\epsilon_k^{\frac{n}{2}}$$

2. $|X| \geq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}$

   We will try to estimate:

$$I'(\bar{\tau}) = \int_{|X|\geq \frac{1}{4}\epsilon_k^{-\frac{1}{2}}} (\epsilon_k r_k^2)^{-\frac{n}{2}} \exp(-l)\mathcal{J}(\bar{\tau})dX$$

   where $\bar{\tau} = \epsilon_k r_k^2$. We have that $|\sqrt{\tau}\gamma'(\tau)| \sim |X|$ if $\gamma(\tau) \in B_k$ for $\tau \leq \bar{\tau}$. Choose $\tilde{\tau} = (\frac{r_k}{2|X|})^2$. This is justified, since $(\frac{r_k}{2|X|})^2 \leq r_k^2\epsilon_k = \bar{\tau}$.For this choice of $\tilde{\tau}$, for every $\tau \leq \tilde{\tau}$: if for all $\tau' < \tau$, $\gamma(\tau') \in B_k$, then $\gamma(\tau) \in B_k$, since

$$d(\gamma(\tau), p_k) \leq \int_0^{\tau} |\gamma'| \sim 2|X|\sqrt{\tau} \leq 2|X|\sqrt{\tilde{\tau}} \leq r_k$$

   By monotonicity, we know that $\bar{\tau}^{-\frac{n}{2}} \exp(-l)\mathcal{J}(\bar{\tau}) \leq \tilde{\tau}^{-\frac{n}{2}} \exp(-l)\mathcal{J}(\tilde{\tau})$ for $\tilde{\tau} \leq \bar{\tau}$ and therefore it is enough to estimate $I'(\tilde{\tau})$. We have proved that for small τ, $\mathcal{J}\tau^{-\frac{n}{2}} \leq C$, where C is a universal constant that does not depend on k.

$$\begin{aligned} l(q, \tilde{\tau}) &= \frac{1}{2\sqrt{\tilde{\tau}}} \int_0^{\tilde{\tau}} \sqrt{\tau}(R + |\gamma'|^2)d\tau \geq \frac{1}{2\sqrt{\tilde{\tau}}}(-\frac{2}{3r_k^2}\tilde{\tau}^{\frac{3}{2}} + \int_0^{\tilde{\tau}} \frac{(\sqrt{\tau}|\gamma'|)^2}{\sqrt{\tau}} d\tau) \\ &\sim -\frac{1}{3r_k^2}\tilde{\tau} + |X|^2 \geq -C\epsilon_k + \frac{|X|^2}{2} \end{aligned}$$

   Now,

$$I'(\tilde{\tau}) \leq Ce^{C\epsilon_k} \int_{|X|\geq \frac{1}{4}\epsilon^{-\frac{1}{2}}} e^{-\frac{|X|^2}{2}} d|X|dS^{n-1} \leq 2e^{-\frac{1}{8}\epsilon_k^{-1}}$$

Anyhow, we get that $\tilde{V}_k(\epsilon_k r_k^2) \leq \epsilon_k^{\frac{n}{2}}$ for k big enough.

□

Now we can give the proof of theorem 7.2 using the proposition 7.1 as follows.

Proof. On one hand we have that $\tilde{V}_k(t_k) \leq \tilde{V}_k(\epsilon_k r_k^2)$, by monotonicity of $\tilde{V}$. By proposition 7.1 we get that

$$\tilde{V}_k(t_k) \leq 3\epsilon_k^{\frac{n}{2}} \tag{48}$$

On the other hand we want to show the following claim.

Claim 9. *There exists some $C > 0$ such that $L(q, t_k) \leq C$ on M for all k.*

Proof. By lemma 7.2 we have that the minimum of $l(\cdot, \tau)$ does not exceed $\frac{n}{2}$ for each $\tau > 0$. Our sequence $t_k \to T$ as $k \to \infty$ and choose q_k such that the minimum of $l(\cdot, t_k - \frac{T}{2})$ is attained at q_k. We want to obtain the upper bound on $l_k(\cdot, t_k)$ by considering only curves γ with $\gamma(t_k - \frac{T}{2}) = q_k$. All geometric quantities in g_{ij} are uniformly bounded when $t \in [0, \frac{T}{2}]$ and the claim follows.

□

Since $t_k \sim T$ when k is big enough, by the above claim we get that $\tilde{V}_k(t_k) \geq C > 0$. This contradicts 48.

□

8 Arguments for section 8

Definition 8.1. *A solution to $(g_{ij})_t = -2R_{ij}$ is said to be κ - collapsed at (x_0, t_0) on the scale $r > 0$ if $|Rm| \leq \frac{1}{r^2}$ for all (x, t) satisfying $d_{t_0}(x, x_0) \leq r$, $t_0 - r^2 < t \leq t_0$ and $\mathrm{Vol}_{t_0} B(x_0, r) \leq \kappa r^n$.*

The main theorem that we will prove in theis section is:

Theorem 8.1. *For any $A > 0$ there exists $\kappa > 0$ with the following property: if $g(t)$ is a solution for $0 \leq t \leq r_0^2$, such that $|\mathrm{Rm}|(x, t) \leq r_0^{-2}$ for $d_0(x, x_0) < r_0$ and $\mathrm{Vol} B(x_0, r_0) \geq A^{-1} r_0^n$ at time zero, then $g(t)$ can not be κ - collapsed on the scales $r < r_0$ at a point (x, r_0^2) with $\mathrm{dist}_{r_0^2}(x, x_0) \leq A r_0$.*

Lemma 8.1. *Suppose we have a solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$.*

1. *Suppose $\mathrm{Ric}(x, t_0) \leq (n-1)K$ when $\mathrm{dist}_{t_0}(x, x_0) < r_0$. Then the distance function $d(x, t) = \mathrm{dist}_t(x, x_0)$ satisfies at $t = t_0$, outside $B(x_0, r_0)$, the differential inequality:*

$$d_t - \Delta d \geq -(n-1)(\frac{2}{3} K r_0 + r_0^{-1})$$

where $d_t = \frac{d}{dt} \mathrm{dist}_t$.

2. Suppose $\mathrm{Ric}(x, t_0) \leq (n-1)K$ *when* $\mathrm{dist}_{t_0}(x, x_0) < r_0$ *or* $\mathrm{dist}_{t_0}(x, x_1) < r_0$. *Then:*

$$\frac{d}{dt}\mathrm{dist}_t(x_0, x_1) \geq -2(n-1)(\frac{2}{3}Kr_0 + r_0^{-1})$$

at $t = t_0$.

Proof. We will give a proof only of the first statement, since the proof of the second one is similar.

We will assume that x and x_0 are not conjugate in metrics $g(t_0)$, because otherwise the inequality that we want to prove can be understood in a barrier sense. Let $\gamma(s) = \exp_x sX$,for $s \in [0, L]$, where $X = \dot{\gamma}(0)$, be a minimal geodesic between x and x_0, such that $\gamma(0) = x_0$ and $\gamma(L) = x$. Let $(X, e_1, \ldots e_{n-1})$ be the orthonormal basis of $T_{x_0}M$.Let E_i (with $1 \leq i \leq n-1$) be the parallel vector fields along $\gamma(s)$ such that $E_i(0) = e_i$. Let $X_i(s)$ be the Jacobi fields along $\gamma(s)$, such that, $X_i(L) = E_i(L)$ and $X_i(0) = 0$ (they exist, since we assumed x and x_0 are not conjugate points). The formula for the laplacian of the distance function (can be found in the book [8]) is:

$$\Delta \mathrm{dist}(x, x_0) = \sum_{i=1}^{n-1} s''_{X_i}(\gamma)$$

where $s''_{X_i}(\gamma)$ is the second variation along X_i of the length of γ, where $X_i(t)$ are Jacobi fields constructed above.

$$s''_{X_k}(\gamma) = \int_0^L (|X'_k(s)|^2 - R(\dot{\gamma}, X_k, \dot{\gamma}, X_k))ds$$

what is again proved in the book [8]. The RHS of the equality above is usually denoted by $I(X_k, X_k)$.

Define the vector fields Y_k as follows:

$$Y_k = \begin{cases} \frac{s}{r_0}E_k(s) & \text{if } s \in [0, r_0] \\ E_k(s) & \text{if } s \in [r_0, L] \end{cases}$$

We can notice that the vector fields Y_k have the same values at the ends of γ as a Jacobi field X_k and it is the known fact that $I(X_k, X_k) \leq I(Y_k, Y_k)$. Now we can compute:

$$\begin{aligned}
\Delta \mathrm{dist}(x, x_0) &\leq \sum_{i=1}^{n-1} I(Y_i, Y_i) \\
&\leq \sum_{i=1}^{n-1}(\int_0^{r_0} \frac{1}{r_0^2} - \frac{s^2}{r_0^2}R(X, E_k, X, E_k) + \int_{r_0}^{\mathrm{dist}(x,x_0)} (-R(X, E_k, X, E_K))) \\
&= \int_\gamma (-\mathrm{Ric}(X, X)) + \int_0^{r_0} \mathrm{Ric}(X, X)(1 - \frac{s^2}{r_0^2}) + \frac{n-1}{r_0} \\
&\leq d_t + (n-1)(\frac{2}{3}Kr_0 + r_0^{-1})
\end{aligned}$$

where we have used the fact that:

$$d_t(x,x_0) = \frac{d}{dt}\mathrm{dist}(x,x_0)|_{t=t_0} = \frac{d}{dt}\int_0^L |\dot\gamma_t(s)|_{g(t)}|_{t=t_0} = -\int_0^L \mathrm{Ric}(X,X)ds$$

where X is the unit tangent vector to the path γ_t in metric $g(t)$.

□

We will now prove theorem 8.1. By scaling we may assume that $r_0 = 1$.We will prove the theorem by contradiction.

Assume that there exists $A > 0$ such that for every $\epsilon_k \to 0$ as $k \to \infty$, there exists $p_k \in M$ and $1 > r_k > 0$, such that $\mathrm{dist}_1(p_k, x_0) \le A$ and g_{ij} is ϵ_k - collapsed at $(p_k, 1)$ on the scale less than r_k. That means:

- $|\mathrm{Rm}|(x,t) \le r_k^{-2}$ for all (x,t) satisfying $\mathrm{dist}_t(x,p_k) \le r_k$ and $1 - r_k^2 \le t \le 1$
- $\mathrm{Vol}B(p_k, r_k^2) \le \epsilon_k r_k^n$ at time $t = 1$

Let $\tau(t) = 1 - t$. Arguing as in the section on the applications of a comparison geometry approach to the Ricci flow we can get that $\tilde V_k(\epsilon_k r_k^2)$ must be very small as $k \to \infty$.$\tilde V_k$ is the reduced volume defined with respect to point x_k By the monotonicity of the reduced volume, we get that $\tilde V_k(1) \le \tilde V_k(\epsilon_k r_k^2) \to 0$ as $k \to \infty$. We have that:

$$\tilde V_k(1) = \int_M \exp(-l_k)dq \ge \int_{B_{t=0}(x_0,1)} \exp(-l_k)dq$$

where l_k is the reduced distance defined with respect to point p_k. From the above equality we can see that it would be enough to show that $l_k(x,1) \le C$ for $x \in B_0(x_0,1)$, since then we would have:

$$\tilde V_k(1) \ge e^{-C}\mathrm{Vol}_0 B_0(x_0,1) \ge e^{-C}A = \tilde C$$

and we would get a contradiction. So, our goal is to find an upper uniform bound on $l_k(x,1) \le C$ for $x \in B_0(x_0,1)$.

Claim 10. *There exists a uniform constant C, such that $\min_{B_{\frac{1}{2}}(x_0,\frac{1}{10})} l_k(x,\frac{1}{2}) \le C$ for all k.*

Proof. Define $h_k(y,t) = \phi(d(y,t) - A(2t-1))(\bar L_k(y, 1-t) + 2n + 1)$ where $d(y,t) = \mathrm{dist}_t(y,x_0)$ and ϕ is a function of one variable, equal to 1 on $(-\infty\frac{1}{20})$ and rapidly increasing to infinity on $(\frac{1}{20}, \frac{1}{10})$, in such a way that:

$$2\frac{(\phi')^2}{\phi} - \phi'' \ge (2A + 100n)\phi' - C(A)\phi \tag{49}$$

for some constant $C(A) < \infty$. Fom the lower bound on the scalar curvature proved in the previous section we have that for $t \geq \frac{1}{2}$, $R(\cdot, 1-t) \geq -\frac{n}{2\tau}$. $\bar{L}(\tau) = 2\sqrt{\tau}L$, so for $\tau = 1 - t \leq \frac{1}{2}$ we have that

$$\bar{L}(\tau) = 2\sqrt{\tau}\int_0^\tau \sqrt{u}(R + |\dot{\gamma}|^2)du \geq -\frac{n}{\sqrt{\tau}}\int_0^\tau \sqrt{u}du = -\frac{2}{3}n\tau \geq -\frac{1}{3}n \geq -2n$$

We get that $\bar{L}_k \geq -2n$ for $t \geq \frac{1}{2}$ and therefore $\bar{L}_k + 2n + 1 \geq 1$. Keeping this in mind, since $\phi \geq 1$ tending to infinity when the argument is close to $\frac{1}{10}$, we have the following:

$$\min_{y \in M} h_k(y, \frac{1}{2}) \geq \min_{B_{\frac{1}{2}}(x_0, \frac{1}{10})} \bar{L}_k(y, \frac{1}{2})$$

since $\min h_k(y, \frac{1}{2})$ is achieved for some y satisfying $d(y, \frac{1}{2}) < \frac{1}{10}$. We also have that $\min_M h_k(y, 1) \leq h_k(p_k, 1) = 2n + 1$, since $\bar{L}(p_k, 0) = 0$. Now we compute:

$$\Box(h) = (\bar{L} + 2n + 1)(-\phi'' + (d_t - \Delta d - 2A)\phi') - 2\langle \nabla\phi, \nabla\bar{L}\rangle + (\bar{L}_t - \Delta\bar{L})\phi \qquad (50)$$

$$\nabla h = (\bar{L} + 2n + 1)\nabla\phi + \phi\nabla\bar{L}$$

At a minimum point of h we have $\nabla h = 0$ so the equation 50 becomes:

$$\Box h = (\bar{L} + 2n + 1)(-\phi'' + (d_t - \Delta d - 2A)\phi' + 2\frac{(\phi')^2}{\phi}) + (\bar{L}_t - \Delta\bar{L})\phi$$

Now since $d(y, t) \geq \frac{1}{20}$ whenever $\phi' \neq 0$ and since $\mathrm{Ric} \leq n - 1$ in $B_{\frac{1}{2}}(x_0, \frac{1}{20})$ we can apply the first part of lemma 8.1 to get $d_t - \Delta d \geq -100(n-1)$ on the set where $\phi' \neq 0$.

We also know that $\bar{L}_\tau + \Delta\bar{L} \leq 2n$. Using this and equation 49, we get, at the minimum point of h:

$$\Box h \geq -(\bar{L} + 2n + 1)C(A)\phi - 2n\phi \geq -(2n + C(A))h = -\tilde{C}h$$

Let $m_k(t) = \inf_{y \in M} h_k(y, t)$ for $t \geq \frac{1}{2}$. At the minimum point of h_k, $\Box h_k \leq h_k'(t)$ and therefore we have $m_k'(t) \geq -Cm_k(t)$, i.e. $m_k(\frac{1}{2}) \leq e^{\frac{C}{2}} m_k(1)$. Since $m_k(1) \leq 2n + 1$ we have the claim.

$\Box$

Let q_k be such that $l_k(q_k, \frac{1}{2}) = \inf_{B_{\frac{1}{2}}(x_0, \frac{1}{10})} l_k(x, \frac{1}{2})$. Take $x \in B_0(x_0, 1)$ and connect it to q_k with a geodesic γ, such that $\gamma(\frac{1}{2}) = q_k$. We have actually chosen radius $\frac{1}{10}$ such that for all shortest geodesics $\gamma(s)$ connecting $x \in B_0(x_0, 1)$ and $q \in B_{\frac{1}{2}}(x_0, \frac{1}{10})$, so that $\gamma(\frac{1}{2}) = q$, $(s, \gamma(s))$ remains in the region $\{(x, t) | 0 \leq t \leq 1; x \in B_t(x_0, 1)\}$ where we have uniform bound on the curvature.

Let γ_k be a shortest $\mathcal{L}$ - geodesic connecting x_k and q_k such that $\gamma_k(\frac{1}{2}) = q_k$. Then $L_k(x,1)$, for $x \in B_0(x_0,1)$ can be bounded from above in terms of $L_k(q_k,\frac{1}{2})$ and the uniform bound on the curvature in above mentioned region for (x,t). $l_k(q_k,\frac{1}{2})$ and therefore $L_k(q_k,\frac{1}{2})$ has already been bounded from above by a uniform constant. Now our claim follows.

Remark 11. *To justify the choice of ϕ, we can put $u = \frac{\phi'}{\phi}$. Then the condition for ϕ is equivalent to:*

$$3u^2 - u' - Bu \geq -C(A)$$

for some constants B and $C(A)$. This we can solve for u.

9 Arguments for section 9

Proposition 9.1. *Let $g_{ij}(t)$ be a solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \leq t \leq T$, and let $u = (4\pi(T-t))^{-\frac{n}{2}}e^{-f}$ satisfy the conjugate heat equation $\Box^* u = -u_t - \Delta u + Ru$. Then $v = [(T-t)(2\Delta f - |\nabla f|^2 + R) + f - n]u$ satisfies*

$$\Box^* v = -2(T-t)|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2(T-t)}g_{ij}|^2$$

Proof. Let $v = Hu$. Then

$$\frac{d}{dt}v = \frac{d}{dt}Hu + H\frac{d}{dt}u$$

$$\Delta v = \Delta H u + 2\nabla H\nabla u + H\Delta u$$

Using the equation for u gives us

$$\Box^* v = (-\frac{d}{dt}H - \Delta H)u - 2\nabla H\nabla u$$

$$\begin{aligned}\frac{d}{dt}H &= -(2\Delta f - |\nabla f|^2 + R) + f_t + (T-t)(2\frac{d}{dt}(g^{ij}(\frac{\partial^2 f}{\partial x_i\partial x_j} + \\ &+ \frac{\partial f}{\partial x_k}\Gamma^k_{ij})) - 2R^{ij}\nabla_i f\nabla_j f - 2g^{ij}\frac{\partial f_t}{\partial x_j}\frac{\partial f}{\partial x_i} + R_t)\end{aligned}$$

We have that

$$f_t = -\Delta f + |\nabla f|^2 - R + \frac{n}{2(T-t)}$$

$$R_t = \Delta R + 2|\mathrm{Ric}|^2$$

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\frac{\partial g_l i}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l})$$

We can assume that we chose a normal coordinates around a point at which we are computing $\Box^* v$, so that we have

$$g^{ij}\frac{d}{dt}\Gamma_{ij}^k = g^{kl}g^{ij}(-R_{li,j} - R_{lj,i} + R_{ij,l}) = 0$$

Using this equation in the expression for $\frac{d}{dt}H$ we get

$$\frac{d}{dt}H = -2\Delta f + |\nabla f|^2 - R + f_t + (T-t)(4R^{ij}\nabla_i\nabla_j f + 2\Delta f_t - 2R^{ij}\nabla_i f\nabla_j f - 2\nabla_i f_t \nabla^i f + R_t)$$

$$\begin{aligned}
\Delta H &= \Delta f + (T-t)(2\Delta^2 f - \Delta|\nabla f|^2 + \Delta R) \\
&= \Delta f + (T-t)(2\Delta^2 f - 2|\nabla\nabla f| - 2\nabla_i\nabla_i\nabla_j f\nabla_j f + \Delta R)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}H + \Delta H &= (-\Delta f + |\nabla f|^2 + f_t - R) + (T-t)(4R^{ij}\nabla_i\nabla_j f + 2\Delta f_t + 2\Delta^2 f - \\
&- 2R^{ij}\nabla_i f\nabla_j f - 2|\nabla\nabla f|^2 - 2\nabla_i\nabla_i\nabla_j f\nabla_j f - 2\nabla_i f_t\nabla^i f + 2\Delta R + 2|\mathrm{Ric}|^2 \\
&= (-\Delta f + |\nabla f|^2 + f_t - R) + (T-t)(4R^{ij}\nabla_i\nabla_j f + \\
&+ 2\Delta(|\nabla f|^2 - R) - \Delta(|\nabla f|^2) - 2R^{ij}\nabla_i f\nabla_j f - 2\nabla_i f_t\nabla^i f + 2|\mathrm{Ric}|^2 \\
&= (-\Delta f + |\nabla f|^2 + f_t - R) + (T-t)(4R^{ij}\nabla_i\nabla_j f + \Delta|\nabla f|^2 - 2R^{ij}\nabla_i f\nabla_j f + 2|\mathrm{Ric}|^2 + \\
&+ 2\nabla_i(\Delta f)\nabla^i f - 2\nabla_i|\nabla f|^2\nabla_i f + 2\nabla_i R\nabla^i f)
\end{aligned}$$

$$\begin{aligned}
\frac{2\nabla u\nabla H}{u} &= -2\nabla f\nabla H = -2|\nabla f|^2 - 2(T-t)\nabla_i f\nabla_i(2\Delta f - |\nabla f|^2 + R) = \\
&= -2|\nabla f|^2 - 2(T-t)(2\nabla f\nabla(\Delta f) - \nabla f\nabla|\nabla f|^2 + \nabla f\nabla R)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial t} + \Delta H + \frac{2\nabla u\nabla H}{u} &= -\Delta f - |\nabla f|^2 + f_t - R + (T-t)(4R^{ij}\nabla_i\nabla_j f + \Delta|\nabla f|^2 - \\
&- 2R^{ij}\nabla_i f\nabla_j f + 2|\mathrm{Ric}|^2 - 2\nabla f\nabla(\Delta f))
\end{aligned}$$

$$\begin{aligned}
\Delta|\nabla f|^2 &= 2\nabla_i\nabla_i\nabla_j f\nabla_j f = 2\nabla_i\nabla_i\nabla_j f\nabla_j f + 2|\nabla\nabla f|^2 = \\
&= 2\nabla_i(\Delta f)\nabla_i f + 2R^{ij}\nabla_i f\nabla_j f + 2|\nabla\nabla f|^2
\end{aligned}$$

This gives us

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \Delta H + \frac{2\nabla u \nabla H}{u} &= -2\Delta f - 2R + \frac{n}{2(T-t)} + (T-t)(4R^{ij}\nabla_i\nabla_j f + 2|\mathrm{Ric}|^2 + 2|\nabla\nabla f|^2) \\
&= 2(T-t)[2R^{ij}\nabla_i\nabla_j f + |\mathrm{Ric}|^2 + |\nabla\nabla f|^2 - \frac{\Delta f}{T-t} - \frac{R}{T-t} + \frac{n}{4(T-t)^2}] \\
&= 2T-t)|R_{ij}\nabla_i\nabla_j f - \frac{1}{2(T-t)} g_{ij}|^2
\end{aligned}
$$

This implies $\Box^* v = -2(T-t)|R_{ij}\nabla_i\nabla_j f - \frac{1}{2(T-t)} g_{ij}|^2 u$.

□

Corollary 9.1. *The proposition 9.1 implies the monotonicity formula for our functional $W(g, f, \tau)$.*

Proof. By partial integration we get

$$
\int_M v = \int_M [(T-t)(|\nabla f|^2 + R) + f - n] e^{-f} (4\pi(T-t))^{-\frac{n}{2}} dV = W(g(t), f(t), \tau(t))
$$

$$
\begin{aligned}
\frac{d}{dt} W &= \frac{d}{dt}\int_M v dV_t = \int_M v_t DV_t - \int_M -Rv dV_t \\
&= \int_M (-\Box^* v - \Delta v) dV_t = -\int_M \Box^* v \geq 0
\end{aligned}
$$

by proposition 9.1.

□

Corollary 9.2. *Under the same assumptions, on a closed manifold M, or whenever the application of the maximum principle can be justified,* $\max_M \frac{v}{u}$ *is nondecreasing in t.*

Proof. After choosing the normal coordinates around the maximum point of $\frac{v}{u}$, we have that $\nabla_i v u - \nabla_i u v = 0$ at the maximum point and :

$$
\Delta \frac{v}{u} = \frac{1}{u^4}[(\Delta v u - \Delta u v)u^2 - 2u\nabla_i u(\nabla_i v u - \nabla_i v) \leq 0
$$

This implies $\Delta u v - \Delta v u \geq 0$ at the point where $\frac{v}{u}$ attains maximum.

$$
\frac{d}{dt}\frac{v}{u} = -\frac{\Box^* v}{u} + \frac{\Delta u v - \Delta v u}{u^2}
$$

Since $\Box^* v \leq 0$, we have that $\frac{dm}{dt} \geq 0$, where $m = \max_M \frac{v}{u}$.

□

Corollary 9.3. *Under the same assumptions, if u tends to a δ function as $t \to T$, then $v \leq 0$ for all $t < T$.*

Proof. Let $h \geq 0$ satisfies the heat equation $\Delta h = h_t$. Then

$$\frac{d}{dt}\int_M hu = \int_M \Delta hu - \int_h u_t - \int_M Ruh = 0$$

by partial integration and by the fact that $u_t = -\Delta u + Ru$. Similarly,

$$\frac{d}{dt}\int_M hv = -\int_M \Box^* vh \geq 0$$

Assume for a moment that we have proved $\lim_{t\to T}\int_M hv \leq 0$. Since $\int_M hv$ increases in t, we have that $\int_M hv \leq 0$ for all times $t < T$ and all $h \geq 0$ such that $\Delta h = h_t$.

We want to prove that $v \leq 0$ for all $t < T$. Assume there exists $t_0 < T$, $x_0 \in M$, such that $v(x_0, t_0) > 0$. In that case there would exist a ball $B(x_0, \delta + \eta)$ such that $v(x, t_0) \geq b > 0$. Choose t_0 to be our initial point and choose h_ϵ, where $\epsilon < \eta$, such that

$$\Delta h_\epsilon = (h_\epsilon)_t$$

$$h_\epsilon(x, t_0) = k_\epsilon(x)$$

where

$$k_\epsilon(x) = \begin{cases} \frac{1}{b} & x \in B(x_0, \delta) \\ 0 & \text{outside } B(x_0, \delta + \epsilon) \end{cases}$$

for $t \in [t_0, T]$. We choose k_ϵ such that it is a nonnegative function on M.From what we have assumed above for a moment, that will be proved in the claim 12 below, $k_\epsilon v(x, t_0) \leq 0$. On the other hand $k_\epsilon v(x, t_0)$ is positive on M (strictly positive on $B(x_0, \delta)$) and we get a contradiction. Therefore, $v \leq 0$ for all $t < T$ on M.

To complete the proof of the corollary we have to show the following claim.

Claim 12. $\lim_{t\to T}\int_M hv = 0$.

Proof. Let $I = \lim_{t\to T}\int_M hv$.

$$\int_M hv = \int_M [2\Delta f - |\nabla f|^2 + R)(T - t) + f - n]hu$$

Let $\tau = T - t$. Let $f'(\tau) = f(T - \tau)$, $u'(\tau) = u(T - \tau)$ and $h'(\tau) = h(T - \tau)$. Then

$$I = \lim_{\tau\to 0}\int_M [\tau(2\Delta f' - |\nabla f'|^2) + f' - n]h'u'$$

since $\lim_{\tau\to 0}\int_M R(q, T-\tau)h'u'dq = \lim_{\tau\to 0}\tau R(p, T)h'(p, T) = 0$. Let $\tau_i = sr_i$, where $r_i \to 0$ and $s \in [0, T]$. Let $\bar{g}_i(s) = \frac{1}{r_i}g(T - sr_i)$, $\bar{u}_i(s) = u'(r_i s)$, $\bar{f}_i(s) = f'(r_i s)$ and $\bar{h}_i(s) = h'(r_i s)$. After scaling metric by factor r_i^{-1} we have

$$(\bar{u}_i)_s = \Delta_i \bar{u}_i(s) - R_i(s)\bar{u}_i(s)$$

where $\bar{u}_i(s)$ tend to δ function as $s \to 0$.

$$(\bar{h}_i)_s = -\Delta_i \bar{h}_i(s)$$

Since our function $h(x,t)$ is smooth, we have that $\lim_{i\to\infty} h'(r_i s) = h'(0)$ in C^∞ norm.$|\mathrm{Rm}_i(s)| = r_i|\mathrm{Rm}(g(T - r_i s))| \to 0$ as $i \to \infty$, i.e. the limit metric is a flat metric (we can choose r_i small, such that the injectivity radii of $\bar{g}_i$ tend to infinity). Denote the limit manifold by $(\bar{M}, \bar{p}, \bar{g})$ (we are considering pointed Gromov-Hausdorff convergence). We also know that $\bar{u}_i \to \bar{u}$, where $\bar{u}(s)$ is the fundamental solution of the conjugate heat equation on the limit flat manifold, where our flow is now stationary. That means $\bar{u}$ is the solution of $(\bar{u})_s = \Delta\bar{u} - R\bar{u}$, where the metric is now fixed. Let $F_i = \bar{u}_i\bar{h}_i[(T-s)(2\Delta_i\bar{f}_i - |nabla_i\bar{f}_i|^2) + \bar{f}_i - n]$ and let (U_j, μ_j) be a partition of unity of $\bar{M}$. We will look at $\lim_{s\to 0}\int_{\bar{M}} \nu_j F dV_{\bar{g}}$, where $F = \lim_{i\to\infty} F_i$. Let

$$Q_j(s) = \int_{\bar{M}} \nu_j|(s(2\Delta\bar{f} - |\bar{f}|^2) + \bar{f} - n|\bar{h}\bar{u}dV_{\bar{g}}$$

Since we are in the case when metric is stationary along a Ricci flow, we can approximate $\bar{u}$ in C^3 norm by a heat kernel up to order s^k, where k is an arbitrary integer. In other words (see [7]) we have that there exists N such that if $k_s^N(x,y) = e^{-\frac{\mathrm{dist}(x,y)^2}{4s}}\sum_{i=0}^N t^i\phi_i(x,y)$, where $\phi_i(x,x) = 1$ we have that

$$|k_s^N(x,p) - \bar{u}(x,s)|_{C^3} = O(s^{N-\frac{n}{2}-\frac{3}{2}})$$

Denote by $r(x)$ a distance from $\bar{p}$ to $x \in \bar{M}$. Let $l_s^N = \frac{r^2}{4s} - \ln\sum_{i=0}^N s^i\phi_i$. Then

$$\begin{aligned}
s\int_{\bar{M}} \nu_j\bar{h}\bar{u}|\nabla\bar{f}|^2 &= s\int_{\bar{M}} \nu_j\bar{h}|\nabla l_s^N|^2 + O(s) \\
&= s\int_{\bar{M}} \nu_j(\frac{r^2}{4s} + s\frac{|\sum_{i=1}^N s^i\nabla\phi_i|^2}{(1+\sum_{i=1}^N s^i\phi_i)^2} + \qquad (51)\\
&+ \sum_j \frac{r}{2s}\frac{\partial r}{\partial x_j}\frac{\sum_{i=1}^N s^i\nabla_j\phi_i}{1+\sum_{i=1}^N s^i\phi_i})(4\pi s)^{-\frac{n}{2}}e^{-\frac{r^2}{4s}}\sum_{i=0}^N s^i\phi_i + O(s) \\
&= s\int_{\bar{M}} \nu_j(4\pi s)^{-\frac{n}{2}}e^{-\frac{r^2}{4s}}\bar{h}\frac{r^2}{4s}dV\bar{g} + O(s) \qquad (52)
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
s\int_{\bar{M}} \nu_j\bar{h}\bar{u}\Delta\bar{f} &= \int_{\bar{M}} \nu_j\bar{h}\Delta l_s^N k_s^N + O(s) \\
&= s\int_{\bar{M}} \nu_j(\frac{n}{2s} - \frac{(\sum_{i=1}^N s^i\Delta\phi_i)\sum_{i=0}^N s^i\phi_i - |\nabla\sum_{i=1}^N s^i\phi_i|^2}{(\sum_{i=0}^N s^i\phi_i)^2}(4\pi s)^{-\frac{n}{2}}e^{-\frac{r^2}{4s}}\sum_{i=0}^N s^i\phi_i) \\
&= \int_{\bar{M}} \nu_j n e^{-\frac{r^2}{4s}}dV_{\bar{g}} + O(s)
\end{aligned}$$

and also

$$\int_{\bar{M}} \nu_j \bar{h}\bar{f}\bar{u} = \int_{\bar{M}} \nu_j (4\pi)^{-\frac{n}{2}} \bar{h}\frac{r^2}{4s} e^{-\frac{-r^2}{4s}} + O(s)$$

$$Q_j(s) = \int_{\bar{M}} \nu_j (4\pi s)^{-\frac{n}{2}} e^{-\frac{r^2}{4s}} |n - \frac{r^2}{4s} + \frac{r^2}{4s} - n| dx + O(s) = O(s)$$

Since $\int_{\bar{M}} |F(s)| dV_{\bar{g}} = \sum_j \int_{\bar{M}} \nu_j |F(s)| dV_{\bar{g}}$, by Lebesgue monotone convergence theorem (if we pass to a sequence $s_k \to 0$ as $k \to \infty$, we get that $\int_{\bar{M}} |F(s_k)| dV_{\bar{g}} \to 0$ as $k \to \infty$. This implies $\int_{\bar{M}} F(s_k) \to 0$ as $k \to \infty$.

Now we can conclude that $\int_M hv \to 0$ over some sequence $t_i \to T$ as $i \to \infty$. Since $\int_M hv \nearrow$ in t, there exists $\lim_{t\to T} \int_M hv$ and from the previous discussion we conclude that $\lim_{t\to T} \int_M hv = 0$.

□

□

Corollary 9.4. *Under the assumptions of the previous corollary, for any smooth curve $\gamma(t)$ in M holds:*

$$-\frac{d}{dt} f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(T-t)} f(\gamma(t), t)$$

Proof. Previous corollary gives us that $v \leq 0$, i.e. $(T-t)(2\Delta f - |\nabla f|^2 + R) + f - n \leq 0$. The equation for f is $f_t = -\Delta f + |\nabla f|^2_R + \frac{n}{2(T-t)}$. From these we get

$$f_t + \frac{1}{2}R - \frac{1}{2}|\nabla f|^2 - \frac{f}{2(T-t)} \geq 0 \tag{53}$$

On the other hand $-\frac{d}{dt} f(\gamma(t), t) = -f_t - \langle \nabla f, \dot{\gamma}(t) \rangle \leq -f_t + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\dot{\gamma}|^2$. Summing this inequality with inequality 53 we get

$$-\frac{d}{dt} f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\gamma(t)|^2) - \frac{1}{2(T-t)} f(\gamma(t), t)$$

□

Corollary 9.5. *If under the assumptions of the previous corollary, p is the point where the limit δ function is concentrated, then $f(q, t) \leq l(q, T-t)$, where l is the reduced distance, defined in section 7, using p and $\tau(t) = T - t$.*

Proof. From the previous corollary we have that

$$-\frac{d}{dt}(f(\gamma(t), t)\sqrt{T-t}) \leq \frac{1}{2}\sqrt{T-t}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2)$$

$$\sqrt{T-t}f(q,t) \le \lim_{s\to T}\sqrt{T-s}f(\gamma(s),s) + \frac{1}{2}\int_t^T \sqrt{T-s}(R(\gamma(s),s)+|\dot{\gamma}|^2)ds$$

Take infimum over all curves $\gamma(s)$ for $s \in [t,T]$, such that $\gamma(t) = q$ and $\gamma(T) = p$.

$$\sqrt{T-t}f(q,t) \le \lim_{s\to T}(\sqrt{T-t}f(\gamma(s),s)) + \frac{1}{2}L(q,T-t)$$

Since $u = (4\pi(T-t))^{-\frac{n}{2}}e^{-f}$ and since $u(p,T) = \infty$, we have that

$$\lim_{s\to T}\sqrt{T-s}f(\gamma(s),s) = \lim_{s\to T}(-\sqrt{T-t}\ln u - \frac{n}{2}\sqrt{T-s}\ln(4\pi(T-s))) \le 0$$

Therefore, $f(q,t) \le \frac{1}{2\sqrt{T-t}}L(q,T-t) = l(q,T-t)$, i.e. $f(q,t) \le l(q,T-t)$.

□

10 Arguments for section 10

The main theorem in this section is the following:

Theorem 10.1. *For every $\alpha > 0$ there exists $\delta > 0$, $\epsilon > 0$ with the following property. Suppose we have a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$ for $0 \le t \le (\epsilon r_0)^2$ and assume that at $t = 0$ we have $R(x) \ge -r_0^{-2}$ and $\mathrm{Vol}(\partial\Omega)^n \ge (1-\delta)c_n\mathrm{Vol}(\Omega)^{n-1}$ for any x and $\Omega \subset B(x_0,r_0)$, where c_n is the euclidean isoperimetric constant. Then we have an estimate $|\mathrm{Rm}|(x,t) \le \alpha t^{-1} + (\epsilon r_0)^{-2}$, whenever $0 < t \le (\epsilon r_0)^{-2}$, $d(x,t) = \mathrm{dist}_t(x,x_0) \le \epsilon r_0$.*

This theorem gives us that under the Ricci flow, the almost singular regions (where curvature is large) can not instantly significantly influence the almost euclidean regions.

Proof. The argument is by contradiction.By scaling assume that $r_0 = 1$. We may also assume that α is small, say $\alpha < \frac{1}{100n}$.From now on fix α and denote by M_α the set of pairs (x,t), such that $|\mathrm{Rm}|(x,t) \ge \alpha t^{-1}$.

We will prove our theorem by proving the sequence of claims.

Claim 13. *For any $A > 0$, if $g_{ij}(t)$ solves the Ricci flow equation on $0 \le t \le \epsilon^2$, $A\epsilon < \frac{1}{100n}$ and $|\mathrm{Rm}|(x,t) > \alpha t^{-1} + \epsilon^2$ for some (x,t), satisfying $0 \le t \le \epsilon^2$, $d(x,t) < \epsilon$, then one can find $(\bar{x},\bar{t}) \in M_\alpha$, with $0 < \bar{t} \le \epsilon^2$, $d(\bar{x},\bar{t}) < (2A+1)\epsilon$, such that*

$$|\mathrm{Rm}|(x,t) \le 4|\mathrm{Rm}|(\bar{x},\bar{t}) \tag{54}$$

whenever

$$(x,t) \in M_\alpha \quad 0 < t \le \bar{t} \quad d(x,t) \le d(\bar{x},\bar{t}) + A|\mathrm{Rm}|^{-\frac{1}{2}}(\bar{x},\bar{t}) \tag{55}$$

Proof. We will construct $(\bar{x},\bar{t})$ as a limit of a finite sequnce of points. Take an arbitrary (x_1,t_1), such that $0 < t_1 \le \epsilon^2$ and $d(x_1,t_1) < \epsilon$. Assume we have already constructed (x_k,t_k). If we can not take it for $(\bar{x},\bar{t})$, there exists some (x,t), such that $|\mathrm{Rm}|x,t) \ge 4|\mathrm{Rm}|(x_k,t_k)$ and $(x,t) \in M_\alpha$, $0 < t \le t_k$ and $d(x,t) \le d(x_k,t_k) + A|\mathrm{Rm}|^{-\frac{1}{2}}(x_k,t_k)$. Take (x,t) for (x_{k+1},t_{k+1}). Then:

- $|\mathrm{Rm}|(x_k,t_k) \ge 4^{k-1}|\mathrm{Rm}|(x_1,t_1) \ge 4^{k-1}\epsilon^{-2}$
- $d(x_k,t_k) \le d(x_1,t_1) + \sum_{i=1}^{k} A|\mathrm{Rm}|^{-\frac{1}{2}}(x_i,t_i)$

  i.e.

  $d(x_k,t_k) \le \epsilon + A\epsilon\sum_{i=1}^{k} 2^{i-1} \le (2A+1)\epsilon$

Since the solution is smooth, the sequence is finite and its last element fits.

□

Claim 14. *For $(\bar{x},\bar{t})$ constructed above 54holds whenever*

$$\bar{t} - \frac{1}{2}\alpha Q^{-1} \le t \le \bar{t} \quad \mathrm{dist}_{\bar{t}}(x,\bar{x}) \le \frac{1}{10}AQ^{-\frac{1}{2}} \tag{56}$$

for big enough A.

Proof. We only need to show that if (x,t) satisfies 56 then it must satisfy 54 or 55. Since $(\bar{x},\bar{t}) \in M_\alpha$, we have $Q \ge \alpha\bar{t}^{-1}$, so $\bar{t} - \frac{1}{2}\alpha Q^{-1} \ge \frac{1}{2}\bar{t}$ and therefore $|\mathrm{Rm}|(x,t) \ge 4Q \ge 4\alpha\bar{t}^{-1} \ge \frac{2\alpha}{t}$, i.e. $(x,t) \in M_\alpha$.

We want to apply lemma 8.1 from section 8.Let (x,t) be a point satisfying the relations 56. Choose $r_0 = \frac{1}{10}AQ^{-\frac{1}{2}}$ and fix points x and x_0. Consider two cases:

1. $d_t(x',x_0) \le r_0$

   In this case $d_t(x',x_0) \le d(\bar{x},\bar{t}) + AQ^{-\frac{1}{2}}$ and therefore, by claim 13 we get that $|\mathrm{Rm}|(x,t) \le 4Q$.

2. $d_t(x,x') \le r_0$

   In this case $d_t(x',x_0) \le d_t(x',x) + d_t(x_0,x)$, i.e. $d_t(x',x_0) \le \frac{1}{10}AQ^{-\frac{1}{2}} + d_t(x_0,x)$. We have, by triangle inequality that for every $x \in \bar{B} = \bar{B}_{\bar{t}}(\bar{x}, \frac{1}{10}AQ^{-\frac{1}{2}})$:

   $$d_{\bar{t}}(x,x_0) \le d(\bar{t},\bar{x}) + d_{\bar{t}}(\bar{x},x) \le d(\bar{t},\bar{x}) + \frac{1}{10}AQ^{-\frac{1}{2}} < d(\bar{t},\bar{x}) + \frac{8}{10}AQ^{-\frac{1}{2}}$$

   Choose t_0 such that $d_t(x_0,x) \le d(\bar{t},\bar{x}) + \frac{9}{10}AQ^{-\frac{1}{2}}$ for all $x \in \bar{B}$ and all $\bar{t} \ge t \ge t_0$. This is possible, since $\bar{B}$ is compact, the distance function is a continuous function and since $d_{\bar{t}}(x,x_0) \le d(\bar{t},\bar{x}) + \frac{9}{10}AQ^{-\frac{1}{2}}$ for all $x \in \bar{B}$. This will give us that $d_t(x',x_0) \le d(\bar{t},\bar{x}) + AQ^{-\frac{1}{2}}$, so again by claim 13 we get that $|\mathrm{Rm}|(x',t) \le 4Q$ for all $t \in [t_0,\bar{t}]$.

After applying lemma 8.1 from section 8, we get:

$$d_{t_0}(x,x_0) \le d_{\bar{t}}(x,x_0) + \int_t^{\bar{t}} (\frac{8}{3}\frac{AQ^{\frac{1}{2}}}{10} + \frac{10Q^{\frac{1}{2}}}{A})ds$$

Since $\bar{t} - t \le \frac{1}{2}\alpha Q^{-1}$ we get that

$$d_{t_0}(x,x_0) \le d_{\bar{t}}(x,\bar{x}) + d(\bar{x},\bar{t}) + \frac{1}{2}\alpha(\frac{8}{30}AQ^{-\frac{1}{2}} + 10\frac{Q^{-\frac{1}{2}}}{A})$$

Since α is small we can choose A big enough, such that the last term on the right hand side of the inequality above is $\le \frac{8}{10}AQ^{-\frac{1}{2}}$. Finally we get that:

$$d_{t_0}(x,x_0) < d(\bar{x},\bar{t}) + \frac{9}{10}AQ^{-\frac{1}{2}}$$

i.e. we can continue this inequality beyond t_0. That implies the condition 55 holds for all (x,t) satisfying 56.

□

Continuing the proof of the theorem 10.1 and arguing by contradiction, take sequences $\epsilon_i \to 0$, $\delta_i \to 0$ and solutions $g_i(t)$ violating the statement. Take $A_i = \frac{1}{100n\epsilon_i} \to \infty$. Construct the sequence of $(\bar{x}_i, \bar{t}_i)$ as in the claim 13 and consider solutions $u_i = (4\pi(\bar{t}_i - t))^{-\frac{n}{2}}e^{-f_i}$ of the conjugate heat equation, starting from δ - functions at $(\bar{x}_i, \bar{t}_i)$. Consider the corresponding functions $v_i = [(\bar{t}_i - t)(2\Delta_i f_i - |\nabla_i t_i|^2 + R_i)]u_i$. Prove the following claim.

Claim 15. *As $\epsilon_i\ \delta_i \to 0$, we can find times $\tilde{t}_i \in [\bar{t}_i - \frac{1}{2}\alpha Q_i^{-1}, \bar{t}_i]$, where $Q_i = |\mathrm{Rm}|(\bar{x}_i, \bar{t}_i)$, such that the integral $\int_{B_i} v_i$ stays bounded away from zero, where B_i is the ball at time $\tilde{t}_i$ of radius $\sqrt{\bar{t}_i - \tilde{t}_i}$, centered at $\bar{x}_i$.*

Proof. The argument is by contradiction. We will divide the proof of this claim in several steps.

Step 15.1. *The first thing that we will show is that the statement of the claim 15 is invariant under scaling by a factor Q_i.*

Proof. Let $\tilde{g}_i = Q_i g_i(\bar{t}_i + \frac{t}{Q_i})$. Let $s = \bar{t}_i + \frac{t}{Q_i}$. Let $u_i(s)$ be the solution to a conjugate heat equation for $g_i(s)$. Then:

$$\frac{d}{ds}f_i = -\Delta_i f_i + |\nabla_i f_i|^2 - R_i + \frac{n}{2(\bar{t}_i - s)}$$

Let $\tilde{f}_i(t) = f_i(\bar{t}_i + \frac{t}{Q_i})$.Now it is straightforward to get that:

$$\frac{d}{dt}\tilde{f}_i(t) = -\tilde{\Delta}_i\tilde{f}_i + |\tilde{\nabla}_i\tilde{f}i|^2 - \tilde{R}_i + \frac{n}{(-2t)}$$

Let $\tilde{u}_i = e^{-\tilde{f}_i}(4\pi(-t))^{-\frac{n}{2}}$. Then $(\tilde{u}_i)_t = -\tilde{\Delta}_i\tilde{u}_i + \tilde{R}_i\tilde{u}_i$ and

$$\lim_{t\to 0}\int_M \tilde{u}_i = \lim_{s\to \bar{t}_i}\int (4\pi(\bar{t}_i - s)Q_i)^{-\frac{n}{2}} e^{-f_i} Q_i^{\frac{n}{2}}\, dV_{g_i} = 1$$

We see that $\tilde{u}_i$ is the fundamental solution to a conjugate heat equation, starting at $(\bar{x}_i, 0)$ as a δ function. Therefore, it is enough to prove that $\int_{B_i} \tilde{v}_i \le -\beta$ for all i and some uniform constant β, where $\tilde{v}_i$ are the corresponding functions (defined as v_i above), associated to $\tilde{u}_i$.

□

Our metrics $\tilde{g}_i(t)$ are defined for $t \in [-\frac{\alpha}{2}, 0]$ and $|\tilde{\mathrm{Rm}}_i|(\bar{x}_i, 0) = 1$, $|\tilde{\mathrm{Rm}}|(x,t) \le 4$ for all $t \in [-\frac{\alpha}{2}, 0]$ and $x \in M_i = B_0(\bar{x}_i, \frac{A_i}{10})$.

First we will assume that the injectivity radii of $\tilde{g}_i$ are uniformly bounded from below. Now we have that $(M_i, \tilde{g}_i(t), \bar{x}_i)$ converge to some manifold $(N, g_\infty(t), \bar{x})$ as $i \to \infty$, where $g_\infty(t)$ is a solution of a Ricci flow, by Hamilton's compactness theorem.

Fundamental solutions of the conjugate heat equations associated to $\tilde{g}_i$ converge to such a solution on a limit manifold. Denote by u the fundamental solution of a conjugate heat equation on a limit manifold and by v the corresponding function, as above.

Step 15.2. *If $\int_B v = 0$ at time $t = -\frac{\alpha}{2}$, then $g_\infty(t)$ is a gradient shrinking soliton for $t \in [-\frac{\alpha}{2}, 0]$, where B is a ball in N of radius $\sqrt{\frac{\alpha}{2}}$, centered at $\bar{x}$.*

Proof. We know that $v \le 0$ on B, for all times. If $\int_B v = 0$ at $t = -\frac{\alpha}{2}$ then $v \equiv 0$ on B at $t = -\frac{\alpha}{2}$. We have the equation for v

$$v_t = -\Delta v - \Delta^* v - Rv$$

Since $v \le 0$ we have that $v_t \le 0$ at time $t = -\frac{\alpha}{2}$ and our equation for v becomes $0 \ge v_t = -\Delta v - \Delta^* v$. Since 0 is a maximal value of v, we have that $\Delta v \le 0$ at the points where it equals zero and therefore we get that $-\Delta^* v \le 0$. On the other hand, we know that $-\Delta^* v \ge 0$, so $\Delta^* v \equiv 0$ on N at time $t = -\frac{\alpha}{2}$. That is equivalent to:

$$R_{ij} + \nabla_i \nabla_j f + \frac{2}{\alpha}(g_\infty)_{ij} = 0 \tag{57}$$

i.e. g_∞ is a Ricci soliton at time $t = -\frac{\alpha}{2}$. Since g_∞ is a Ricci flow, g_∞ is a Ricci soliton for all times $t \in [-\frac{\alpha}{2}, 0]$.

□

So far we have proved that if $\int_B v = 0$ at time $-\frac{\alpha}{2}$, $g_\infty(t)$ would be a gradient shrinking soliton for all times $t \in [-\frac{\alpha}{2}, 0]$. In the next step we will see that it is not possible, since $|\mathrm{Rm}|(\bar{x}, 0) = 1$.

Step 15.3. *Since $|\mathrm{Rm}|(\bar{x}, 0) = 1$, $g_\infty(t)$ can not be a Ricci soliton.*

Proof. If $g_\infty(t)$ were a Ricci soliton, from equation 57 we would have that $R + \Delta f - \frac{n}{(-2t)} = 0$ for all $t \in [-\frac{\alpha}{2}, 0]$. $u = (-4\pi t)^{-\frac{n}{2}} e^{-f}$ tends to δ function as $t \to 0$.

$$\int_B Ru + \Delta f u = -\frac{n}{2t} \int_B u \tag{58}$$

We know that $\lim_{t\to 0} \int_B u = 1$ and that $\lim_{t\to 0} \int_B Ru = R(\bar{x}, 0) = 1$. Look at $\int_\Delta f u dV_{g_\infty(ts)}$. Let $\bar{g}_t(s) = \frac{1}{2t} g_\infty(2ts)$. Since the curvature of g_∞ is uniformly bounded, $\{\bar{g}_t(s)\}$ converge to a flat metric when $t \to 0$ and balls B in metric $\bar{g}_t$ converge to a metric ball of infinite radius, i.e.

$$\int_B (-2t)\Delta f u dV_{g_\infty(ts)} = \int_{B_t} \Delta \bar{f}_t(s) \bar{u}_t(s) dV_{\bar{g}_t(s)} \stackrel{t\to 0}{\longrightarrow} \int_{R^n} \Delta \bar{f}(s) \bar{u}(s) dV_{\bar{g}} = I(s)$$

where $\bar{u}_t$ is a delta function in metric $\bar{g}_t(s)$ and $\bar{f}_t(s)$ is a corresponding function f. We know that a limit of delta functions is a delta function in a limit metric. Delta function of our limit metric is denoted by $\bar{u}(s)$. One can easily show that (since our limit is a euclidean metric) $\lim_{s\to 0} I(s) = \int_{R^n} -\frac{n}{2s} e^{-\frac{r^2}{4s}} (4\pi(-s))^{-\frac{n}{2}} dV = -n$. Finally, we get that $\int_B \Delta f u \sim -\frac{n}{2t}$ when $t \to 0$. If we put this asymptotic relation together with other asymptotic relations that we have got above, in equation 58, around $t = 0$, we get a contradiction.

□

Now assume that the injectivity radii of the scaled metrics $\tilde{g}_i$ tend to zero. Denote the injectivity radii by $I_i \to 0$ as $i \to \infty$. In that case we can change the scaling factor and we can consider the sequence of scaled metrics $\bar{g}_i(t) = \frac{Q_i}{I_i} g_i(\bar{t}_i + \frac{tI_i}{Q_i})$. We have that $|\bar{\mathrm{Rm}}_i| \le 4I_i \to 0$ as $i \to \infty$. The metrics $\bar{g}_i(t)$ are defined for $t \in [-\frac{\alpha}{2I_i}, 0]$. Now the sequence of manifolds $\{(B_0(\bar{x}_i, \frac{A_i}{10\sqrt{I_i}}), \bar{g}_i(t), \bar{x}_i)\}$ converge to a flat manifold $(N, g_\infty, \bar{x})$, with finite injectivity radius. Now we can repeat the above argument for the sequence of metrics $\bar{g}_i$ and the metric g_∞, to get a contradiction (we assume that $\int_B v = 0$ at some time $t = T < 0$, where B is the ball as above).

Anyhow, we get that in either of these cases there exists some time $\tilde{t}$ and some constant β, such that $\int_B v \le -\beta$ on the limit manifold N. Therefore, if we go from scalings back to our original sequence of metrics $g_i(s)$ we can conclude that there exist times $\tilde{t}_i \in [\bar{t}_i - \frac{1}{2}\alpha Q^{-1}, \bar{t}_i]$ such that $\int_{B_i} v_i \le -\beta$ at time $\tilde{t}_i$ for all i, where B_i are the balls as in the statement of the claim 15.

□

Our next goal is to construct an appropriate cut-off function. Let $h_i(y,t) = \phi(\frac{\tilde{d}_i(y,t)}{10A_i\epsilon_i})$, where $\tilde{d}_i(y,t) = d_i(y,t) + 200n\sqrt{t}$ and ϕ is a smooth function of one variable, equal to one on $(-\infty, 1]$ and decreasing to zero on $[1,2]$. It is easy to notice that h_i vanishes at $t = 0$ outside $B_i(x_0, 20A_i\epsilon_i)$ and that it is equal to one, near $(\bar{x}, \bar{t})$.

Compute:

$$\Box h_i = \frac{1}{10A_i\epsilon_i}((d_i)_t - \Delta_i d_i + \frac{100n}{\sqrt{t}}\phi') - \frac{1}{(10A_i\epsilon_i)^2}\phi''$$

By reducing ϵ_i we can assume that $|\mathrm{Rm}|_i(x,t) \le \alpha t^{-1} 2\epsilon_i^{-2}$, whenever $0 \le t \le \epsilon_i^2$ and $d_i(x,t) \le \epsilon_i$. For every $t \in (0, \epsilon_i^2]$ choose $r_0(t) = \sqrt{t} \le \epsilon_i$, so that

$$|\mathrm{Ric}| \le (n-1)(\alpha t^{-1} + 2\epsilon_i^{-2})$$

for $(d_i)_t(x, x_0) \le r_0(t)$. We can apply lemma 8.1 from section 8 to our case, to get (where $\phi' \ne 0$):

$$\begin{aligned}(d_i)_t - \Delta_i d_i &\ge -(n-1)(\frac{2}{3}(\frac{\alpha}{t} + \frac{2}{\epsilon^2})\sqrt{t} + \sqrt{t}^{-1}) \ge \\ &\ge -\frac{1}{\sqrt{t}}(n-1)(\frac{2}{3}(\alpha+2)+1) \ge -\frac{100n}{\sqrt{t}}\end{aligned}$$

So, $(d_i)_t - \Delta_i d_i + \frac{100n}{\sqrt{t}} \ge 0$, when $\phi' \ne 0$. We may also choose ϕ so that $\phi'' \ge -10\phi$, $(\phi')^2 \le 10\phi$. $\phi' \le 0$ implies that $\Box h_i \le \frac{10\phi}{(10A_i\epsilon_i)^2} \le \frac{1}{(A_i\epsilon_i)^2}$. Since $(\int_M h_i u_i)_t = \int_M \Box h_i u_i$, we have that $(\int_M h_i u_i)_t \le \frac{1}{(A_i\epsilon_i)^2}$ and therefore integration from 0 to $\bar{t}_i$ will give us:

$$\int_M h_i u_i|_{t=0} \ge \int_M h_i u_i|_{t=\bar{t}_i} - \frac{\bar{t}_i}{(A_i\epsilon_i)^2} \ge 1 - \frac{1}{A_i^2} \tag{59}$$

$$\begin{aligned}(\int_M (-h_i v_i)_t &= \int_M \Box(-h_i)v_i - \int_M (-h_i)\Delta^* v_i \le \int_M \Box(-h_i)v_i \\ &= \frac{1}{10A_i\epsilon_i}\int_M -((d_i)_t - \Delta d_i + \frac{100n}{\sqrt{t}})\phi' v_i + \frac{1}{(10A_i\epsilon_i)^2}\int_M \phi'' v_i \\ &\le \frac{1}{(A_i\epsilon_i)^2}\int_M (-h_i)v_i\end{aligned}$$

Using claim 15 this will give us:

$$\begin{aligned}-\int_M h_i v_i|_{t=0} &\ge \int_M (-h_i v_i)|_{\bar{t}_i} e^{-\frac{\bar{t}_i}{(A_i\epsilon_i)^2}} \\ &\ge \beta(1 - \frac{\bar{t}_i}{(A_i\epsilon_i)^2}) \ge (1 - \frac{1}{A_i^2})\beta\end{aligned}$$

From now on we will work at $t=0$ only. Let $\tilde{u}_i = h_i u_i$ and $\tilde{f}_i = f_i - \ln h_i$. Now we have:

$$\begin{aligned}\beta(1 - \frac{1}{A_i^2}) \le -\int_M h_i v_i &= ((-2\Delta_i f_i + |\nabla_i f_i|^2 - R_i)\bar{t}_i - f_i + n)u_i h_i \\ &= \int_M [-\bar{t}_i|\nabla_i \tilde{f}_i|^2 - \tilde{f}_i + n]\tilde{u}_i + \int_M [\bar{t}_i(\frac{|\nabla_i h_i|^2}{h_i} - R_i h_i) - h_i \ln h_i]u_i\end{aligned}$$

Since $\int_M(-\Delta_i h_i u_i + \frac{|\nabla_i h_i|^2}{h_i}u_i + \nabla_i \tilde{f}_i \nabla_i h_i u_i) = 0$ by partial integration. $R \geq -1$ by assumption of the theorem 10.1 and hence $-\int_M R_i h_i u_i \leq \int_M h_i u_i \leq 1$, for we are at a point $t = 0$. This implies $-\bar{t}_i \int_M R_i h_i u_i \leq \epsilon_i^2$. Also, $-\int_M h_i \ln h_i u_i \leq \int_{B(x_0,20A_i\epsilon_i)\backslash B(x_0,10A_i\epsilon_i)} u_i$. On the other hand, $\int_{B(x_0,10A_i\epsilon_i)} u_i \geq \int_M \bar{h}_i u_i|_{t=0} \geq 1 - \frac{1}{A_i^2}$, where $\bar{h}_i = \phi(\frac{\tilde{d}_i}{5A_i\epsilon_i})$. Finally, $-\int_M h_i \ln h_i u_i \leq \int_M u_i - \int_{B(x_0,10A_i\epsilon_i)} u_i \leq \frac{1}{A_i^2}$. Also, $\int_M \bar{t}_i \frac{|\nabla_i h_i|^2}{h_i} u_i \leq 10\epsilon_i^2$. Therefore, we get:

$$\int_M [-\bar{t}_i|\nabla_i \tilde{f}_i|^2 - \tilde{f}_i + n]\tilde{u}_i \geq \beta(1 - \frac{1}{A_i^2}) - \frac{1}{A_i^2} - 100\epsilon_i^2 = C > 0 \tag{60}$$

for big enough i.

Step 15.4. *The inequality 60 contradicts with the Gaussian logarithmic Sobolev inequality.*

Proof. Scale our metrics $g_i(t)$ by the factor $\frac{1}{2}\bar{t}_i$, i.e. let $\hat{g}_i(s) = \frac{1}{2}\bar{t}_i^{-1} g_i(2\bar{t}_i s)$. Now $\hat{\mathrm{Rm}}_i \leq 2\bar{t}_i(\frac{\alpha}{\bar{t}_i} + \frac{2}{\epsilon_i^2}) = 2(\alpha + 2) = \tilde{C}$. Since $R_i \geq -1$, after scaling it becomes $\hat{R}_i(0) = 2\bar{t}_i R_i(0) \geq -2\bar{t}_i \to 0$ as $i \to \infty$. We know the following for the scaled metrics $\{\hat{g}_i(0)\}$, for $x \in M_i = B_0(x_0, \frac{\sqrt{\bar{t}_i}}{\sqrt{2\bar{t}_i}}) = B_0(x_0, \frac{1}{\sqrt{2}})$, after scaling:

- $|\hat{\mathrm{Rm}}_i| \leq C$
- $\mathrm{Vol}B_0(x_0, \frac{1}{\sqrt{2}}) \geq C(\frac{1}{\sqrt{2}})^n$
- $\mathrm{diam}(M_i) \leq \frac{1}{\sqrt{2}}$

The volume estimate is expressed at time $t = 0$. From the isoperimetric inequality we get the lower bound on the volume, since it is invariant under scaling. The facts listed above will give us the uniform lower bound on the injectivity radii of $\hat{g}_i$ at time $t = 0$. Call these metrics simply $\hat{g}_i$. To everything that we have said by now we can apply Hamilton's compactness theorem to conclude that there exists a subsequence $\hat{g}_i(t)$, such that $\hat{g}_i(t)$ converge to some metrics $\hat{g}(t)$, such that $\hat{R} \geq 0$. From the evolving equation for $\hat{R}$, by maximum principle applied to a heat equation, we get that $\hat{R}(t) \geq 0$ for all times t in $[0, \epsilon_i^2]$. Also, since $\delta_i \to 0$ as $i \to \infty$, we have that for our limit metric $\hat{g}$ the following holds:

$$\mathrm{Vol}(\partial\Omega)^n \geq c_n \mathrm{Vol}(\Omega)^{n-1} \tag{61}$$

where c_n is the euclidean isoperimetric constant.

In general, we have the following asymptotic expansion of the volume $B(x_0, r)$ in an arbitrary manifold (M, g) (as in for example [8]):

$$\mathrm{Vol}(B(x_0, r)) = r^n \mathrm{Vol}(B^e(1))(1 - \frac{R(x_0)}{6(n+2)}r^2 + o(r^2))$$

where $B^e(1)$ is a euclidean ball of radius 1.

Using this asymptotic expansion for our metric $\hat{g}$, together with isoperimetric inequality 61 we get:

$$\mathrm{Vol}B^e(1) \le \frac{\mathrm{Vol}B(x,r)}{r^n} = \mathrm{Vol}B^e(1)(1 - \frac{\hat{R}}{6(n+2)}r^2 + o(r^2))$$

Since we apply this locally, around each point of our limit manifold, we get that $\hat{R} \le 0$. Since $\hat{R} \ge 0$ from efore, we get that $\hat{R} \equiv 0$. From the evolution equation for $\hat{R}$

$$\frac{d}{dt}\hat{R} = \Delta\hat{R} + 2|\hat{\mathrm{Ric}}|^2$$

we get that $\hat{\mathrm{Ric}}(0) \equiv 0$. Bishop volume comparison principle tells us that $\frac{\mathrm{Vol}B(r)}{r^n} \downarrow$. Clearly, $\lim_{r\to 0}\frac{\mathrm{Vol}B(r)}{r^n} = w_n$ where w_n is the volume of the unit ball in R^n. Therefore $\frac{\mathrm{Vol}B(r)}{r^n} \le w_n$. On the other hand, our $\hat{g}$ satisfies the isoperimetric inequality with euclidean constant c_n and therefore, $\frac{\mathrm{Vol}B(r)}{r^n} \ge w_n$. Finally, $\frac{\mathrm{Vol}B(r)}{r^n} = w_n$ for all r. This is possible only if our limit manifold is isometric to R^n.

We have that our limit metric $\hat{g}$ is a flat euclidean metric. After scaling the metrics g_i by factors $\frac{1}{2}\bar{t}_i^{-1}$ and after passing to a limit when $i \to \infty$, the inequality 60 will give us:

$$\int_{R^n}[-\frac{1}{2}|\nabla f|^2 - f + n]u \ge C > 0$$

for some function f, such that $\int_{R^n} u dx = 1$, where $u = (2\pi)^{-\frac{n}{2}}e^{-f}$. This contradicts the Gaussian logarithmic Sobolev inequality due to L. Gross. To pass to its standard form, take $f = \frac{|x|^2}{2} - 2\ln\phi$ and integrate by parts.

□

This finishes the proof of the theorem 10.1.

□

Now we will state a theorem with slightly different conditions from the conditions of theorem 10.1, but which proof is essentially the same as the proof that we have given above.

Theorem 10.2. *There exist $\epsilon > 0$ and $\delta > 0$ with the following property. Suppose $g_{ij}(t)$ is a smooth solution to the Ricci flow on $[0,(\epsilon r_0)^2]$ and assume that at $t = 0$ we have $|\mathrm{Rm}|(x) \le r_0^{-2}$ in $B(x_0,r_0)$ and $\mathrm{Vol}B(x_0,r_0) \ge (1-\delta)w_n r_0^n$, where w_n is the volume of the unit ball in R^n. Then the estimate $|\mathrm{Rm}|(x,t) \le (\epsilon r_0)^2$ holds whenever $0 \le t \le (\epsilon r_0)^2$ and $\mathrm{dist}_t(x,x_0) \le \epsilon r_0$.*

Proof. It is useful to point out that the proof of this theorem will be almost the same as the proof of theorem 10.1. One possible simplification in the proof might be at the point when in step 15.4 we try to prove that the limit metric $\hat{g}(t)$ at the point $t = 0$ is flat, since now we have the condition that $|\mathrm{Rm}_i|(x) \le 1$ (after scaling we may assume that $r_0 = 1$). When we rescale our metrics g_i by factors $\frac{1}{2}\bar{t}_i$ to get metrics $\hat{g}_i$, we get that $|\hat{\mathrm{Rm}_i}| \le 2\bar{t}_i \to 0$ as $i \to \infty$. To get the lower bound on the injectivity radii of $\hat{g}_i$ at time $t = 0$ we proceed as above, since for that we do not need the isoperimetric inequality, but the lower bound on the volume that is given as a condition in our theorem. From what we have just said, when we pass to a limit of metrics $\hat{g}_i$, we immediatelly get that $\hat{g}(0)$ is flat. That is the metric we want to continue to deal with and we proceed as in the proof of the theorem 10.1 to get a contradiction with the Gaussian logarithmic Sobolev inequality.

□

Corollary 10.1. *Under the assumptions of theorem 10.1, we also have at time t where $0 < t \le (\epsilon r_0)^2$ an estimate $\mathrm{Vol}B(x, \sqrt{t}) \ge C(\sqrt{t})^n$ for $x \in B(x_0, \epsilon r_0)$, where $C = C(n)$ is a universal constant.*

Proof. Fix $\alpha > 0$. Assume $r_0 = 1$. Choose ϵ and δ as in the theorem 10.1. The proof of this corollary is by contradiction. Assume that the statement is not true, i.e.that there exists a sequence $(\bar{x}_i, \bar{t}_i)$, with $\bar{x}_i \in B(x_0, \epsilon)x$ and $\bar{t}_i \in [0, \epsilon^2]$, such that

$$\mathrm{Vol}B(\bar{x}_i, \sqrt{\bar{t}_i})(\sqrt{\bar{t}_i})^{-n} \to 0 \tag{62}$$

as $i \to \infty$. We have (by theorem 10.1):

$$|\mathrm{Rm}|(x, \bar{t}_i) \le \frac{\alpha}{\bar{t}_i} + \frac{1}{\epsilon^2} \le \frac{1}{(\sqrt{\frac{\bar{t}_i}{\alpha+1}})^2} \tag{63}$$

for $x \in B_{\bar{t}_i}(x_0, \epsilon)$. Since $r_i = \sqrt{\frac{\bar{t}_i}{\alpha+1}} \le \epsilon$, the curvature bounds 63 hold in the balls $B_{\bar{t}_i}(x_0, r_i)$ as well. $\frac{r_i^2}{\bar{t}_i} = \frac{1}{\alpha+1}$ is bounded for all i. Since $\bar{t}_i \to t_0 \in [0, \epsilon^2]$ as $i \to \infty$, by noncollapsing theorem in section 4, metric g is not locally collapsed at t_0. Therefore, there exists some constant C, such that

$$\mathrm{Vol}B(\bar{x}_i, \sqrt{\frac{\bar{t}_i}{\alpha+1}}) \ge C(\sqrt{\frac{\bar{t}_i}{\alpha+1}})^n$$

$\mathrm{Vol}B(\bar{x}_i, \bar{t}_i) \ge \mathrm{Vol}B(\bar{x}_i, \sqrt{\frac{\bar{t}_i}{\alpha+1}})$ and therefore we get that $\mathrm{Vol}B(\bar{x}_i, \bar{t}_i) \ge \tilde{C}(\sqrt{\bar{t}_i})^n$ which contradicts our assumption 62.

□

Remark 16. *It is interesting to point out the different proof of the step 15.4, by using the spherical symmetric function. This proof will not work in the case of theorem 10.2, since in that case we do not have the isoperimetric inequality.*

Proof. We will adopt the notation from the proof of the pseudolocality theorem 10.1. As in the step 15.4 we get a sequence of metrics (solutions to a Ricci flow), $\hat{g}_i(t)$ with uniformly bounded curvatures and the uniform lower bound on the injectivity radii at time $t = 0$. By Hamilton's compactness theorem, we can extract a convergent subsequence that converges to a metric $\hat{g}(t)$. We will consider metric $\hat{g} = \hat{g}(0)$. As before, we have functions u and f as the limit functions of $\tilde{u}_i$ and $\tilde{f}_i$, respectively, where the function $u = (2\pi)^{-\frac{n}{2}} e^{-f}$ is compactly supported, and $\int_{\bar{M}} u = 1$, where $\bar{M}$ is a limit manifold. We also have:

$$\int_{\bar{M}} [-\frac{1}{2}|\nabla f|^2 - f + n]u \geq C > 0$$

Let $F = e^{-\frac{f}{2}}$. The previous inequality implies:

$$\int_{\bar{M}} \frac{1}{2}|\nabla F|^2 - F^2 \ln F^2 + F^2 \leq -\beta < 0 \tag{64}$$

Construct a nonnegative spherical symmetric function G on R^n, such that:

$$\mu(\{F \geq a\}) = \mu(\{G \geq a\})$$

for every $a \geq 0$, where we denote by the same symbol $\mu(\cdot)$ a volume of a set in R^n and a volume of a set in $\bar{M}$. Since F has compact support, we can assume that G has a compact support as well, that is a ball $B(0, R)$.

By co-area formula we have:

$$\frac{d\mathrm{Vol}(F \geq C)}{dC} = \int_{F=C} \frac{1}{|\nabla F|}$$

$$\frac{d\mathrm{Vol}(G \geq C)}{dC} = \int_{G=C} \frac{1}{|\nabla G|}$$

Since the right hand sides of the above equalities are equal, we get that

$$\int_{M_a} |\nabla F|^{-1} d\mathrm{Vol} = \int_{N_a} |\nabla G|^{-1} d\mathrm{Vol}$$

where M_a and N_a are the level sets of F and G respectively. Also:

$$\int_{\bar{M}} F^2 = \int_0^\infty \mathrm{Vol}(F^2 \geq C) dC = \int_0^\infty \mathrm{Vol}(G^2 \geq C) dC = \int_{R^n} G^2$$

Similarly we can show that the above equality is true when we replace functions F^2 and G^2 by $F^2 \ln F^2$ and $G^2 \ln G^2$ respectively.

Isoperimetric inequality for $\bar{M}$ will give us:

$$|M_a| := \mathrm{Vol}(M_a) \geq c_n(\mu(\{F \geq C\}))^{\frac{n-1}{n}} = c_n(\mu(\{G \geq C\}))^{\frac{n-1}{n}} = |N_a|$$

We will be done once we prove the following claim:

Claim 17. *For any $p > 1$, $\int_{M_a} |\nabla F|^p \le \int_{N_a} |\nabla G|^p$.*

Proof. Let $\bar{p} > 1$, $\bar{q} > 1$ and b be such that $b\bar{p} = p$ and $b\bar{q} = 1$. We choose $b = \frac{p}{p+1}$ to satisfy the requirements. Using the Holder inequality we get:

$$\begin{aligned} |M_a| &= \int_{M_a} 1 = \int_{M_a} |\nabla F|^b |\nabla F|^{-b} \\ &\le (\int_{M_a} |\nabla F|^{b\bar{p}})^{\frac{1}{\bar{p}}} (\int_{M_a} |\nabla F|^{-b\bar{q}})^{\frac{1}{\bar{q}}} \\ &= (\int_{M_a} |\nabla F|^p)^{\bar{p}^{-1}} (\int_{N_a} |\nabla G|^{-1})^{\bar{q}^{-1}} \end{aligned}$$

which is equivalent to:

$$|M_a|^{\frac{p+1}{p}} \le (\int_{M_a} |\nabla F|^{\frac{1}{p}})^{\frac{1}{p}} \int_{N_a} |\nabla F|^{-1}$$

Since ∇G is constant along N_a we have that

$$|N_a|^{\frac{p+1}{p}} = (\int_{N_a} |\nabla G|^p)^{\frac{1}{p}} \int_{N_a} |\nabla G|^{-1}$$

From the previous, since $|M_a| \ge |N_a|$, it follows:

$$\int_{N_a} |\nabla G|^p \le \int_{M_a} |\nabla F|^p$$

□

From equation 64 we get that

$$\int_{R^n} \frac{1}{2} |\nabla G|^2 - G^2 \ln G^2 + G^2 \le -\beta < 0$$

where G is a function with compact support on R^n and with $\int_{R^n} G^2 = \int_{\bar{M}} F^2 = \int_{\bar{M}} u = 1$. Therefore, we get a contradiction with the Gaussian logarithmic Sobolev inequality.

□

11 Argument for section 11

In this section we will consider smooth solutions to the Ricci flow $(g_{ij})_t = -2R_{ij}$ for $-\infty < t \le 0$, such that for each t the metric $g_{ij}(t)$ is a complete non-flat metric of bouded curvature and nonnegative curvature operator. Hamilton discovered a remarkable differential inequality for such solutions; we need only its trace version

$$R_t + 2\langle X, \nabla R\rangle + 2\mathrm{Ric}(, X) \ge 0 \quad (65)$$

Its corollary is that $R_t \geq 0$.In particular, the scalar curvature at some time t_0 controls the curvatures for all $t \leq t_0$.

We impose one more requirement on the solutions; namely, we fix some $\kappa > 0$ and require that $g_{ij}(t)$ be κ - noncollapsed on all scales.

Pick an arbitrary point (p, t_0) and define $\tilde{V}(\tau)$, $l(q, \tau)$ for $\tau(t) = t_0 - t$. Recall from section 7 that for each $\tau > 0$ we can find $q = q(\tau)$, such that $l(q, \tau) \leq \frac{n}{2}$.

Proposition 11.1. *The scalings of $g_{ij}(t_0 - \tau)$ at $q(\tau)$ with factors τ^{-1} converge along a subsequence of $\tau \to \infty$ to a non-flat gradient shrinking soliton.*

Proof.

Claim 18. $\forall \epsilon > 0$*, there exists $\delta > 0$, such that $l(q, \tau)$ and $\tau R(q, t_0 - \tau)$ do not exceed δ^{-1}, whenever $\frac{1}{2}\bar{\tau} \leq \tau \leq \bar{\tau}$ and $\mathrm{dist}_{t_0 - \bar{\tau}}(q, q(\tau)) \leq \frac{\bar{\tau}}{\epsilon}$ for some $\bar{\tau} > 0$.*

Proof. Fix $\epsilon > 0$. Let t_1 and t_2 be such that $t_0 - t_1 = \tau$ and $t_0 - t_2 = \bar{\tau}$. Let $\gamma(s)$ be a geodesic between q and $q(\bar{\tau})$, such that $s \in [t_2, t_1]$ and $\gamma(t_1) = q$, $\gamma(t_2) = q(\bar{\tau})$. Consider $l(\gamma(t), t_0 - t)$.

$$\frac{d}{dt} l(\gamma(t), t_0 - t) = \langle \nabla l, \dot{\gamma} \rangle - l_t(\gamma(t), t_0 - t)$$

Integrate the above equation over t, from t_2 to t_1.

$$l(q, \tau) = l(q(\bar{\tau}), \bar{\tau}) + \int_{t_2}^{t_1} \langle \nabla l, \dot{\gamma} \rangle dt - \int_{t_2}^{t_1} l_t dt$$

$$\int_{t_2}^{t_1} l_t(\gamma(t), t_0 - t) dt \overset{t_0 - t = u}{=} - \int_{\tau}^{\bar{\tau}} l_\tau(\gamma(t_0 - u), u) du$$

We know that $l_\tau(\gamma(t_0 - u), u) = -\frac{1}{2} u^{-\frac{3}{2}} \int_0^u H(X) \tilde{\tau}^{\frac{3}{2}} d\tilde{\tau}$, where $X(\tau) = \dot{\gamma}(\tau)$ and $H(X)$ is just a trace of Harnack differential expression for our flow $(g_{ij})_t = -2R_{ij}$. Since the curvature operator is positive, $H(X) \geq 0$. Therefore, $l_\tau \leq 0$. Therefore:

$$l(q, \tau) \leq \frac{n}{2} + \int_{t_0 - \bar{\tau}}^{t_0 - \tau} \langle \nabla l, \dot{\gamma} \rangle ds$$

From section 7 we also have that

$$|\nabla l|^2 \leq |\nabla l|^2 + R \leq \frac{Cl}{\tau} \tag{66}$$

Since the curvature is nonnegative, the norms of vectors are ninincreasing in t, so $|\dot{\gamma}|_s \leq |\dot{\gamma}|_{t_0 - \bar{\tau}}$. Therefore:

$$
\begin{aligned}
I &= \int_{t_0-\bar{\tau}}^{t_0-\tau} \langle \nabla l, \dot{\gamma} \rangle ds \\
&\leq \int_{t_0-\bar{\tau}}^{t_0-\tau} |\nabla l|_s |\dot{\gamma}|_{t_0-\bar{\tau}} \\
&\leq C \int_{\tau}^{\bar{\tau}} \frac{\sqrt{l}(u)}{\sqrt{u}} |\dot{\gamma}|_{t_0-\bar{\tau}} \\
&\leq C\sqrt{l(q,\tau)}\sqrt{\frac{2}{\tau}} \mathrm{dist}_{t_0-\bar{\tau}}(q, q(\bar{\tau})) \leq C\sqrt{l(q,\tau)}\sqrt{\frac{2}{\epsilon}}
\end{aligned}
$$

where we have used the fact that $l_\tau \leq 0$ and that $\mathrm{dist}_{t_0-\bar{\tau}}(q, q(\bar{\tau})) \leq \frac{\bar{\tau}}{\epsilon}$. Using Cauchy-Schwartz inequality gives us:

$$l(q,\tau) \leq \frac{n}{2} + \tilde{C} + \eta l(q,\tau)$$

for some $\eta < 1$ Finally we get that $l(q,\tau) \leq \delta^{-1}$ for some δ depending on ϵ.

- If $\tau \geq \frac{t_0}{2}$ then $t_0 - \tau \leq \tau$ and since R is nondecreasing, $R(t_0 - \tau) \leq R(\tau)$. This implies $\tau R(t_0 - \tau) \leq Cl(\tau) \leq \frac{C}{\delta}$.
- If $\tau \leq \frac{t_0}{2}$ then $t_0 - \tau \geq \tau$ and $R(t_0 - \tau)\tau \leq (t_0 - \tau)R(t_0 - \tau) \leq Cl(t_0 - \tau) \leq \frac{C}{\delta}$.

In any case, we get that $\tau R(t_0 - \tau) \leq C\delta^{-1}$ for some $\delta > 0$.

□

We have our original flow $g_{ij}(t)$. If we make translation by t_0, i.e. if $\tau = \tau(t) = t_0 - t$, we get the flow $g_{ij}(\tau)$ that satisfies $(g_{ij})_\tau = 2R_{ij}$. Let $\bar{g}_{ij}(s) = \frac{1}{\bar{\tau}} g_{ij}(t_0 - s\bar{\tau})$, where $s \in [\frac{1}{2}, 1]$. Also, for all $s \in [\frac{1}{2}, 1]$ and all q such that $\mathrm{dist}^2_{\bar{g}(1)}(q, q(\bar{\tau})) \leq \frac{1}{\epsilon}$, we have that $\bar{R}(s) = \bar{\tau} R(t_0 - s\bar{\tau}) \leq \delta^{-1}$ by the claim 18. By κ noncollapsing assumption:

$$\mathrm{Vol}_{g(t)} B(q(\bar{\tau}), \sqrt{\frac{\bar{\tau}}{\epsilon}}) \geq (\sqrt{\frac{\bar{\tau}}{\epsilon}})^n$$

After scaling metrics by factors $\bar{\tau}^{-1}$ we get that

$$\mathrm{Vol}_{\bar{g}} B(q(\bar{\tau}), \epsilon^{-\frac{1}{2}}) \geq \epsilon^{-\frac{n}{2}}$$

This volume estimate, together with the curvature estimate give us the uniform lower bound on the injectivity radii at $q(\bar{\tau})$. Take a sequence $\bar{\tau}_i \to \infty$. Consider the sequence of manifolds $\{B_{\bar{g}(1)}(q(\bar{\tau}_i), \epsilon^{-\frac{1}{2}}), \bar{g}(s), q(\bar{\tau}_i)\}$, for $s \in [\frac{1}{2}, 1]$. By Hamilton's compactness theorem there exists a subsequence $\bar{\tau}_i$, such that the corresponding sequence of manifolds converge to a manifold $(N, \bar{g}(s), \bar{q})$, where $\bar{g}(s)$ is a Ricci flow on N. Functions $\bar{l}_i$ tend to a function $\bar{l}$, that is locally Lipshitz function (this will be justified below). $\bar{l}_i$ is a reduced distance for a metric $\bar{g}_i$.

We know that $\bar{l}_i$ satisfy the following inequalities:

$$(\bar{l}_i)_\tau - \Delta\bar{l}_i + |\nabla\bar{l}_i|^2 - R(\bar{g}_i) + \frac{n}{2\tau} \geq 0 \tag{67}$$

$$2\Delta\bar{l}_i - |\nabla\bar{l}_i|^2 + R + \frac{\bar{l}_i - n}{\tau} \leq 0 \tag{68}$$

Claim 19. *$\bar{l}$ satisfies the inequalities 67 and 68 in the sense of distributions.*

Proof. It is easy to see that $\bar{l}_i(q,s) = l(q, \bar{\tau}_i s)$, for $s \in [\frac{1}{2}, 1]$. After rescaling the metrics we also have that $|\nabla_i\bar{l}_i|^2 + R_i \leq C\bar{l}_i$. The uniform estimates on $l(q,\tau)$ and $\tau R(q, t_0 - \tau)$ in claim 18 will give us that there exists some constant $\tilde{C}$, such that $|\nabla_i\bar{l}_i|^2 \leq \tilde{C}$, and $\bar{l}_i \leq \tilde{C}$, for all i. That means that functions $\bar{l}_i$ tend to a function $\bar{l}$ that is a Lipshitz function.

We will check that $\bar{l}$ satisfies the inequality 68 in the sense of distributions. Let ϕ be a positive test function with a compact support. Since $\bar{l}_i \leq C$ and $|\nabla\bar{l}_i|^2 \leq C$ for all i, we have that $\bar{l}_i \to l$ in $C^{0,\alpha}$ norm. It is easy to check that

$$\lim_{i\to\infty} \int \phi(2\Delta\bar{l}_i + R_i + \frac{\bar{l}_i - n}{\tau}) = \int \phi(2\Delta\bar{l} + R + \frac{\bar{l} - n}{\tau})$$

Therefore it is enough to check that

$$\lim_{i\to\infty} \int \phi(|\nabla\bar{l}_i|^2 - |\nabla\bar{l}|^2) = 0$$

Fatou's lemma gives us that the left hand side of the above equality is greater or equal to zero. Choose $\epsilon_i \to 0$ such that $\bar{l} - \bar{l}_i - \epsilon_i \geq 0$ for all i. Then:

$$\begin{aligned} 0 &\leq \lim_{i\to\infty} \int \phi(|\nabla\bar{l}_i|^2 - |\nabla\bar{l}|^2) \qquad (69) \\ &= \lim_{i\to\infty} \int \nabla(\bar{l}_i - \bar{l})\phi\nabla\bar{l} + \lim_{i\to\infty} \nabla(\bar{l}_i - \bar{l})\phi\nabla\bar{l}_i \\ &= \lim_{i\to\infty} \int \nabla(\bar{l}_i - \bar{l} + \epsilon_i)\phi\nabla\bar{l}_i \\ &= \lim_{i\to\infty} \int (\bar{l} - \bar{l}_i - \epsilon_i)\nabla\phi\nabla\bar{l}_i + \lim_{\to\infty} \int (\bar{l} - \bar{l}_i - \epsilon)\phi\Delta\bar{l}_i \end{aligned}$$

The first limit in 69 is zero, since $|\nabla\bar{l}_i|$ is uniformly bounded and $\bar{l}_i$ converge uniformly to $\bar{l}$ as $i \to \infty$. Since $\Delta\bar{l}_i \leq \frac{1}{2}(|\nabla\bar{l}_i|^2 - R_i - \frac{\bar{l}_i - n}{\tau}$, we have

$$0 \leq \lim_{i\to\infty} \int \phi(|\nabla\bar{l}_i|^2 - |\nabla\bar{l}|^2) \leq \lim_{i\to\infty} \frac{1}{2} \int (\bar{l} - \bar{l}_i - \epsilon_i)\phi(|\nabla\bar{l}_i|^2 - R_i - \frac{\bar{l}_i - n}{\tau}) = 0$$

□

Claim 19 gives us that $\bar{V}(s) = \int_M s^{-\frac{n}{2}} e^{-l} dV_s \downarrow$ in s. Define $h(s) = g(t_0 - s)$, for $s \in [\frac{\bar{\tau}}{2}, \bar{\tau}]$. $h(s)$ satisfies $\frac{d}{ds}h_{jk} = 2R_{jk}$.

Claim 20. *$\tilde{V}_i(s) = \tilde{V}(\bar{\tau}_i s)$, where the first quantity is the reduced volume for $\bar{g}_i$ and the second one is the reduced volume for flow h.*

Proof.

$$\tilde{V}_i(s) = \int_{M_i} s^{-\frac{n}{2}} e^{-l_i(s)} dV_{\bar{g}_i(s)} = \int_{M_i} (\bar{\tau}_i s)^{-\frac{n}{2}} e^{-l_i(s)} dV_{h(\bar{\tau}_i s)}$$

Since

$$\begin{aligned} \frac{1}{2\sqrt{s}}\int_0^s \sqrt{u}(\bar{R}_i(\gamma(u),u) + |\dot{\gamma}|_i^2)du &= \frac{1}{2\sqrt{s}}\int_0^s \sqrt{u}(\bar{\tau}_i R_{h(\bar{\tau}_i s)} + |\dot{\gamma}|^2_{h(\bar{\tau}_i s)}\bar{\tau}_i)du \\ &\overset{\bar{\tau}_i u = v}{=} \frac{1}{2\sqrt{\bar{\tau}_i s}}\int_0^{\bar{\tau}_i s} \sqrt{v}(R_{h(\bar{\tau_i s})} + |\dot{\gamma}|^2)dv \end{aligned}$$

we get that $l_i(q,s) = l(q, \bar{\tau}_i s)$ where the second term is the reduced distance for h. Now our claim follows.

□

Since $\tilde{V}(u)$ is decreasing in u (this quantity is realted to h), there exists $\lim_{u\to\infty}\tilde{V}(u)$. For $s \in [\frac{1}{2}, 1]$, $\bar{\tau}_i s \nearrow \infty$ for every s and therefore, by claim 20 we get that there exists $\lim_{i\to\infty}\tilde{V}_i(s) = C$ for every $s \in [\frac{1}{2}, 1]$. Since $\lim_{i\to\infty}\tilde{V}_i(s) = \bar{V}(s)$, where $\bar{V}(s) = \int s^{-\frac{n}{2}} e^{-\bar{l}} dV_s$, we can conclude that $\bar{V}(s) = \bar{V}$ for all $s \in [\frac{1}{2}, 1]$.

The discussion in section 7 gives us that $\bar{V}(s)$ is constant in s only if

$$2\Delta\bar{l} - |\nabla l|^2 + R + \frac{l-n}{s} = 0 \tag{70}$$

The previous implies that $\bar{l}$ is actually smooth. Now we can conclude (from section 7) that $\bar{V}(s) = const$ can happen only if $\bar{g}$ is the gradient Ricci soliton, i.e. more precisely:

$$\bar{R}_{ij} + \bar{\nabla}_i\bar{\nabla}_j\bar{l} - \frac{1}{2\tau} = 0 \tag{71}$$

Since the scaled metrics $\bar{g}_i$ are noncollapsed, with uniformly bounded geometries, there exists a uniform constant C, such that $l_i(s) \le C$ for all i. This will give us that $\bar{V} \ge c > 0$.

Claim 21. *$\bar{V} < (4\pi)^{\frac{n}{2}}$*

Proof. Compute $\lim_{s\to 0}\tilde{V}(s)$, where $\tilde{V}(s)$ is a reduced volume for h. Using the $\mathcal{L}$ - exponential map, when we compute $\tilde{V}(s)$, we can integrate over T_p (remeber that we choose (p, t_0) at the beginning as a reference point for defining $\tilde{V}$ and l), i.e.

$$\tilde{V}(s) = \int_{U\subset T_pM} s^{-\frac{n}{2}} e^{-l}\mathcal{J}(s)dX$$

We have that for small s, $|\sqrt{s}\dot{\gamma}(s)| \sim |X|$ and therefore $\frac{1}{2\sqrt{s}}\int_0^s \sqrt{u}|\dot{\gamma}|^2 du \sim |X|^2$. Now we have, since the curvature is positive and since from section 7 we know that $\lim_{s\to 0} s^{-\frac{n}{2}}\mathcal{J}(s) = 1$:

$$\lim_{s\to 0}\tilde{V}(s) \leq \lim_{s\to 0}\int_{R^n} s^{-\frac{n}{2}}\mathcal{J}(s)e^{-|X|^2}dX = \int_{R^n} e^{-|X|^2}dX = (4\pi)^{\frac{n}{2}}$$

From the monotonicity of $\tilde{V}(s)$ we get that $\tilde{V}(s) \leq (4\pi)^{-\frac{n}{2}}$. It is not dificult to show that for every s, $\tilde{V}(s) < (4\pi)^{-\frac{n}{2}}$. $\tilde{V}(s)$ is nonincreasing and $\lim_{i\to\infty}\tilde{V}(\bar{\tau}_i s) = \bar{V}$ so we get that $0 < \bar{V} < (4\pi)^{\frac{n}{2}}$.

Claim 22. *For every s,$\tilde{V}(s) < (4\pi)^{-\frac{n}{2}}$.*

Proof. If for some $s > 0$, $\tilde{V}(s) = (4\pi)^{-\frac{n}{2}}$, by monotonicity of $\tilde{V}(t)$, it would have to be $(4\pi)^{-\frac{n}{2}}$ for all $t \in [0, s]$. By similar reasoning as above $h(t)$ would have to be a gradient shrinking Ricci soliton with $\Delta l = \frac{n}{2t} - R < \frac{n}{2t}$, since the curvature is positive. The function $F = \frac{r^2}{4t}$ satifies the equation $\Delta F = \frac{n}{2t}$, $F(0) = l(p,t)$ (p corresponds to the origin after applying the $mathcalL$ - exponential map), l is uniformly bounded (by claim 18 applied to a sequence of reduced distances converging to l), so by the maximum principle applied to a subharmonic map $F - l$ we get that $l \leq \frac{r^2}{4t}$. On the other hand, by our assumption

$$(4\pi)^{\frac{n}{2}} = \int_{R^n} t^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}dx = \int_{T_pM} t^{-\frac{n}{2}}e^{-l}dX = (4\pi)^{\frac{n}{2}}$$

Keeping in mind the fact that $l \leq \frac{r^2}{4t}$, this is possible only if $l = \frac{r^2}{4t}$. In that case $R = 0$, which is not possible since $g_{ij}(t)$ is not a flat metric for any t.

□

□

We have that $\bar{g}$ is a Ricci soliton, satisfying the equation 71. That implies $\Delta\bar{l} = \frac{n}{2\tau} - R$.From equation 70 we get that $|\nabla\bar{l}|^2 = \frac{\bar{l}}{\tau} - R$. If this soliton were flat, $\bar{l}$ would be uniquely determined by $|\nabla\bar{l}|^2 = \frac{\bar{l}}{\tau}$. We are now in a euclidean case and $\frac{r^2}{4\tau}$ would satisfy this equation (r is a distance from the origin). In this case $\bar{V} = (4\pi)^{\frac{n}{2}}$ which was ruled out before. Therefore, gradient shrinking soliton $\bar{g}$ is not flat.

□

Corollary 11.1. *There is only one oriented two-dimensional solution, satisfying the assumptions at the beginning of this section, the round sphere.*

Proof. Hamilton proved that round sphere is the only non-flat oriented nonnegatively curved gradient shrinking soliton in dimension two. Thus the scalings of our ancientsolution must converge to a round sphere.Hamilton has also showed that an almost round sphere is getting more round under the Ricci flow, therefore our ancient solution must be flat. □

For any non-compact complete Riemannian manifold M of nonnegative Ricci curvature and a point $p \in M$, the function $F(r) = \mathrm{Vol}B(p,r)r^{-n}$ is nondecreasing in $r > 0$.

Definition 11.1. *The asymptotic volume ratio is* $\nu = \lim_{r\to\infty} = \mathrm{Vol}B(p,r)r^{-n}$.

Proposition 11.2. *Under the assumtions at the beginning of this section,* $\nu = 0$ *for each* t.

Proof. We will prove the proposition by the induction on dimension. In dimension two the statement is vacuous, as it follows from corollary 11.1. Let $n \geq 3$. Suppose that $\nu > 0$ for some $t = t_0$. Consider the asymptotic scalar curvature ratio $\mathcal{R} = \limsup_{d(x)\to\infty} R(x,t_0)d^2(x)$. $d(x)$ denotes the distance at time t_0, from x to some fixed point x_0. We will consider two cases:

1. $\mathcal{R} = \infty$

   In the case the asymptotic scalar curvature ratio is infinite and therefore by standard argument we can find a sequence of points x_k and radii $r_k > 0$, such that $\frac{r_k}{d(x_k)} \to 0$, $R(x_k)r_k^2 \to \infty$ and $R(x) \leq 2R(x_k)$ whenever $x \in B(x_k, r_k)$. Let x_0 be a fixed point. Following Gromov's argument, by taking a subsequence, we may assume that the angle between geodesics x_0x_k and x_0x_{k+1} at x_0 is very small for every k (that it tends to zero as $k \to \infty$) and that $d(x_0, x_{k+1}) \gg d(x_0, x_k)$. We have the triangles $\triangle_k$ in M with vertices x_0, x_k and x_{k+1}. Topogonov's comparison theorem gives us the existence of triangles $\bar{\triangle}_k \subset R^n$ (since $Rm \geq 0$), whose sides have the same lengths as the lengths of the sides of corresponding triangles $\triangle_k$. Denote the corresponding vertices of triangles $\bar{\triangle}_k$ by z_0, z_k, z_{k+1}. Topogonov's comparison theorem tells us that the angle at z_0, between geodesics z_0z_k and z_0z_{k+1} is smaller than the corresponding angle of the triangle $\triangle_k$ at vertex x_0. The cosine theorem applied to a triangle $\bar{\triangle}_k \subset R^n$ gives us that the angle at z_k tends to π as $k \to \infty$. Applying Topogonov's theorem once again to triangles $\triangle_k$ and $\bar{\triangle}_k$ we get (since the corresponding angle of $\triangle_k$ at x_k is bigger than the angle of $\bar{\triangle}_k$ at z_k) that the angle of $\triangle_k$ at x_k tends to π as $k \to \infty$. That is how we get a line in a limiting process, when we take a blow-up limit of $g_{ij}(t)$ at (x_k, t_0) with factors $R(x_k)$. Since as a limit we get a smooth non-flat ancient solution, satisfying the assumptions at the beginning of this section, which splits off a line, we can do a dimension reduction as in Hamilton's survey paper [1].

2. $\mathcal{R} < \infty$

   In this case $R(x) \leq \frac{c}{d(x)^2}$, for some constant c. Again by a standard argument we can get a sequence of points $x_k \in M$ and sequence of radii r_k, such that $R(x) \leq 2R(x-k)$ whenever $x \in B(x_k, r_k)$, at time $t = t_0$. Let $d_k = \mathrm{dist}_{t_0}(x_0, x_k)$. Consider $\tilde{g}_k(t) = \frac{1}{d_k^2}g(td_k^2)$. We have that $\tilde{R}(x) \leq 2\tilde{R}(x_k) = d(x_k)^2R(x_k) \leq c$ for all k. It follows from Gromov's compactness theorem [2] that the sequence of pointed rescaled manifolds $(B(x_k, \frac{r_k}{d_k}, x_k, \tilde{g}_k)$, has a subsequence that converges in the pointed Gromov-Hausdorff topology to a length space, $(M_\infty, \bar{x}, g_\infty)$.

Claim 23. *M_∞ is a metric cone.*

Proof. Take a ball $B(\bar{x}, r) \subset M_\infty$. We work at time $t = t_0$. Then:

$$\begin{aligned} \frac{\mathrm{Vol}B(\bar{x}, r)}{r^n} &= \lim_{k\to\infty} \frac{\mathrm{Vol}_{\tilde{g}_k} B(x_k, r)}{r^n} \\ &= \lim_{k\to\infty} \frac{\mathrm{Vol}_g B(x_k, rd_k)}{(d_k r)^n} = \nu(g) \end{aligned}$$

We get that $\mathrm{Vol}B(\bar{x}, r) = \nu(g) r^n$ for all r, with the curvature operator nonnegative. Cheeger and Colding proved in [3] that this fact implies M_∞ is a metric cone with metric $dr^2 + r^2 h$. Metric cone is a warped product with function $f(r) = r$, so from the formulas for the curvature of warped product we can get that the sectional curvature will be zero in any plane containing the radial direction. Furthermore, Hamilton showed in [5] that we can write the evolution equation of a curvature operator in the following form

$$\frac{d}{dt} M = \Delta M + Q$$

where $Q = Q(M)$ is a quadratic term which has the property that $Q(M) \geq 0$ for all $M \geq 0$. In the plane where M vanishes (in the case of a metric cone such a plane always exists), $0 = \Delta M + Q$. Since $M \geq 0$, $Q \geq 0$ and $\Delta M \geq 0$ where it vanishes, we get that $\Delta M = 0$ and $Q = 0$. In [5] Hamilton proved that the null space of M is invariant under parallel translation, so we get that $\Delta M \equiv 0$ on M_∞. Therefore $M \equiv 0$, i.e. our metric cone would be a flat one which is ruled out by our assumption of non-flat solutions.

□

□

Corollary 11.2. *For every $\epsilon > 0$ there exists $A < \infty$ with the following property. Suppose we have a sequence of (not necessarily complete) solutions $(g_k)_{ij}(t)$ with nonnegative curvature operator, defined on $M_k \times [t_k, 0]$, such that for each k, the ball $B(x_k, r_k)$ at time $t = 0$ is compactly contained in M_k, $\frac{1}{2}R(x,t) \leq R(x_k, 0) = Q_k$ for all (x,t), $t_k Q_k \to -\infty$, $r_k^2 Q_k \to \infty$ as $k \to \infty$. Then $\mathrm{Vol}B(x_k, \frac{A}{\sqrt{Q_k}}) \leq \epsilon(\frac{A}{\sqrt{Q_k}})^n$ at $t = 0$, if k is large enough.*

Proof. The argument is by contradiction. Assume there exists some $\epsilon > 0$, such that for all $A_k \to \infty$ there exist x_k such that at time $t = 0$

$$\mathrm{Vol}B(x_k, \frac{A_k}{\sqrt{Q_k}}) \geq \epsilon(\frac{A_k}{\sqrt{Q_k}})^n$$

Take $\tilde{g}_k = Q_k g_k(tQ_k^{-1})$. We have that $\tilde{R}_k \leq 2$ for $x \in M_k$, i.e. the curvatures of rescaled metrics are uniformly bounded. The previous inequality becomes $\mathrm{Vol}_{\tilde{g}_k} B(x_k, A_k) \geq \epsilon A_k^n$. This, together with the curvature estimates give us the uniform lower bound on the injectivity radii of $\tilde{g}_k$ at x_k, at $t = 0$. Take a limit of $(M_k, x_k, \tilde{g}_k)$ to get $(N, \bar{x}, \tilde{g})$, by Hamilton's compactness argument in [6]. $\tilde{g}$ is a non-flat, ancient solution with nonnegative curvature operator, such that

$$\lim_{k\to\infty} \frac{\mathrm{Vol}B(\bar{x}, A_k)}{A_k^n} \geq \epsilon$$

i.e. $\nu_{\tilde{g}}(0) \geq \epsilon$. It is easy to see that $\tilde{g}$ satisfies κ - noncollapsing assumption. If this assumption were violated for each $\kappa > 0$, then $\nu(t)$ would not be bounded away from zero as $t \to -\infty$, i.e. we could find a sequence $t_i \to -\infty$ such that $\nu_{\tilde{g}}(t_i) \leq \frac{1}{i}$. On the other hand we have that $\nu_{\tilde{g}}(0) \geq \epsilon$. This is not possible becuase of the following claim:

Claim 24. *$\nu(t)$ is nonincreasing in t.*

Proof. Denote by K the bound on the curvature of $\tilde{g}(t)$. If $t \geq s$, we have that $d_t(p,q) - d_s(p,q) \geq C\sqrt{K}(t-s)$. This implies that $B_t(p,r) \subset B_s(r - C\sqrt{K}(t-s))$. Since the curvature is nonnegative and since $\frac{d}{dt}\ln \mathrm{Vol}_t = -R$, we have that $\mathrm{Vol}_t \leq \mathrm{Vol}_s$ and therefore

$$\frac{\mathrm{Vol}_t B(p,r)}{r^n} \leq \frac{\mathrm{Vol}_s B_s(r - C\sqrt{K}(t-s))}{(r - C\sqrt{K}(t-s))^n} \cdot \frac{(r - C\sqrt{K}(t-s))^n}{r^n}$$

Let $r \to \infty$ in the above inequality. As a result we get that $\nu(t) \leq \nu(s)$.

□

□

Corollary 11.3. *For every $\omega > 0$ there exists $B = B(\omega) < \infty$, $C = C(\omega) < \infty$,$\tau_0 = \tau_0(\omega)$, with the following properties.*

1. *Suppose we have a (not necessarily complete) solution $g_{ij}(t)$ to the Ricci flow, defined on $M \times [t_0, 0]$, so that at time $t = 0$ the metric ball $B(x_0, r_0)$ is compactly contained in M. Suppoes that at each time t, $t_0 \leq t \leq 0$, the metric $g_{ij}(t)$ has nonnegative curvature operator and $\mathrm{Vol}B(x_0, r_0) \geq \omega r_0^n$. Then we have an estimate $R(x,t) \leq Cr_0^{-2} + B(t - t_0)^{-1}$ whenever $\mathrm{dist}_t(x, x_0) \leq \frac{1}{4}r_0$.*

2. *If, rather than assuming a lower bound on volume for all t, we assume it only for $t = 0$, then the same conclusion holds with $-\tau_0 r_0^2$ in place of t_0, provided that $-t_0 \geq \tau_0 r_0^2$.*

Proof. 1. By scaling assume that $r_0 = 1$. Argue by contradiction. Assume that there exist sequences B_i, C_i tending to infinity, metrics $g_i(t)$ and points (x_i, t_i), such that

$\text{dist}_{t_i}(x_i, x_0) \le \frac{1}{4}$ and $R(x_i, t_i) > C_i + \frac{B_i}{t_i - t_0}$. Arguing as in the proof of pseudolocality theorem in section 10, we can find points $(\bar{x}_i, \bar{t}_i)$ satisfying $\text{dist}_{\bar{t}_i}(\bar{x}_i, x_0) < \frac{1}{3}$, $Q_i = R(\bar{x}_i, \bar{t}_i) > C_i + \frac{B_i}{\bar{t}_i - t_0}$, such that $R(x', t') \le 2Q_i$ whenever $\bar{t}_i - A_i Q_i^{-1} \le t' \le \bar{t}_i$, $\text{dist}_{\bar{t}_i}(x', \bar{x}_i) < A_i Q_i^{-\frac{1}{2}}$, where A_i tends to infinity, as $ito\infty$.

Fix $\epsilon > 0$. Find A as in the corollary 11.2. We want to apply this corollary to the sequence of solutions $g_k(t)$ on $M_k \times [\bar{t}_k - A_k Q_k^{-1}, \bar{t}_k]$, at points $(\bar{x}_i, \bar{t}_i)$. $M_k = B_{\bar{t}_k}(\bar{x}_k, A_k Q_k^{-\frac{1}{2}})$ are closed balls. Let $r_k = \min\{\frac{1}{3}, A_k Q_k^{-\frac{1}{2}}\}$. In order to apply corollary 11.2 we will check few things.

- $-A_k Q_k^{-1} Q_k = -A_k \to -\infty$ as $i \to \infty$
- $B_{\bar{t}_i}(\bar{x}_i, r_i) \subset B_{\bar{t}_i}(x_0, 1)$, since $\text{dist}_{\bar{t}_i}(x_0, \bar{x}_i) < \frac{1}{3}$. Since the curvature is nonnegative, the distances shrink and therefore $B_{\bar{t}_i}(x_0, 1) \subset B_0(x_0, 1)$. The last ball is compactly contained in M, so the balls $B_{\bar{t}_i}(\bar{x}_i, r_i)$ are compactly contained in M_i, since M_i are closed sets in M.
- $r_k^2 Q_k \to \infty$ as $k \to \infty$.

Now we have that $\text{Vol}B(\bar{x}_k, \frac{A}{\sqrt{Q_k}}) \le \epsilon(\frac{A}{\sqrt{Q_k}})^n$ at time t_k. Choose k big enough such that $\frac{A}{Q_k} < \frac{1}{3} + 1$. Bishop comparison principle will give us

$$\frac{\text{Vol}B(\bar{x}_k, \frac{1}{3} + 1)}{(1 + \frac{1}{3})^n} \le \frac{\text{Vol}B(\bar{x}_k, \frac{A}{\sqrt{Q_k}})}{(\frac{A}{\sqrt{Q_k}})^n} \le \epsilon$$

$\text{dist}_{\bar{t}_k}(x_0, \bar{x}_k) < \frac{1}{3}$ implies that $B(x_0, 1) \subset B(\bar{x}_k, \frac{1}{3} + 1)$ at time $\bar{t}_k$ and therefore $\text{Vol}_{\bar{t}_k} B(x_0, 1) \le \epsilon(\frac{4}{3})^n$. On the other hand, by the assumption of the corollary $\omega \le \text{Vol}_{\bar{t}_k} B(x_0, 1)$, so we get $\omega \le C\epsilon$. We can choose arbitrary small ϵ at the beginning to get a contradiction.

2. Let $B(\omega)$ and $C(\omega)$ be good for the first part of the corollary. We will show that $B = B(5^{-n}\omega)$, $C = C(5^{-n}\omega)$ are good for the second part of the corollary, for an appropriate $\tau_0(\omega) > 0$.

   Let $[\tau, 0]$ be the maximal time interval where the assumption of the first part of the corollary still holds with $5^{-n}\omega$ in place of ω and with $-\tau$ in place of t_0. Then $\text{Vol}B(x_0, 1) \le 5^{-n}\omega$ at time $t = -\tau$. The first part of the corollary gives us that $R(x, t) \le C + \frac{B}{\tau + t}$ whenever $\text{dist}_t(x, x_0) \le \frac{1}{4}$.

   Let $r_0(t) = \frac{\sqrt{t+\tau}}{4\sqrt{\tau}}$. For all $t \in (-\tau, 0]$, $r_0(t) \le \frac{1}{4}$, so we have our curvature bound whenever $\text{dist}_t(x, x_0) \le r_0(t)$. Lemma 8.1 from section 8, after intgeration over $t \in (-\tau, 0]$ will give us the following estimate

$$\begin{aligned}\text{dist}_{-\tau}(x,x_0) &\leq \text{dist}_0(x,x_0)+2(n-1)\int_{-\tau}^{0}(\frac{2}{3}\frac{\sqrt{t+\tau}}{4\sqrt{\tau}}(C+\frac{B}{t+\tau})+\frac{\sqrt{t+\tau}}{4\sqrt{\tau}})dt \\ &= \text{dist}_0(x,x_0)+2(n-1)(\frac{C\tau}{4}+\frac{B\tau}{3}+8\tau)\end{aligned}$$

If $\text{dist}_0(x,x_0)\leq\frac{1}{4}-10(n-1)\tau(C+B+8)$ then $\text{dist}_{-\tau}(x,x_0)\leq\frac{1}{4}$, i.e. the ball $B(x_0,\frac{1}{4})$ at time $t=-\tau$ contains the ball $B(x_0,\frac{1}{4}-10(n-1)\tau(C+B+8))$ at time $t=0$ and the volume of the former is at least as large as the volume of the latter.Choose $\tau_0=\tau_0(\omega)$ in such a way that the radius of the latter ball is $>\frac{1}{5}$. It will follow that

$$\text{Vol}_{-\tau_0}B(x_0,1)\geq\text{Vol}_{-\tau_0}B(x_0,\frac{1}{4})>\text{Vol}_0B(x_0,\frac{1}{5})\geq 5^{-n}\omega$$

where the last inequality follows from Bishop comparison principle and the assumtion that $\text{Vol}_0B(x_0,1)\geq\omega$. As a result, we get that $\text{Vol}_{-\tau_0}>5^{-n}\omega$, i.e. our curvature estimate with coeficients $B(5^{-n}\omega)$ and $C(5^{-n}\omega)$ will continue to hold at time $t=\tau_0$ as well.

□

From now on we restrict our attention to oriented manifolds of dimension three.

Theorem 11.1. *The set of non-compact ancient solutions, satisfying the assumptions at the beginning of this section is compact modulo scaling. That is, from any sequence of such solutions and points $(x_k,0)$ with $R(x_k,0)=1$, we can extract a smoothly converging subsequence and the limit satisfies the same conditions.*

Proof. To ensure a converging subsequence, it is enough to show that whenever $R(y_k,0)\to\infty$, the distances at $t=0$ between x_k and y_k go to infinity as well. We will argue by a contradiction.

Define a sequence z_k by the requirement that z_k is the closest point to x_k at $t=0$, satisfying $R(z_k,0)\text{dist}_0^2(x_k,z_k)=1$. Denote by $Q_k=R(z_k,0)$.

Step 24.1. $\frac{R(z,0)}{R(z_k,0)}$ *is uniformly bounded for $z\in B(z_k,2R(z_k,0)^{-\frac{1}{2}})$.*

Proof. We work at time $t=0$. Assume the contraray. Let $r_k=R(z_k,0)^{-\frac{1}{2}}$ and let $F_k(z)=R(z,0)(4r_k-\text{dist}_0(z,z_k))$. Since $\sup_{B(z_k,4r_k)}F_k(z)\geq\sup_{B(z_k,2r_k)}F_k(z)\geq\sup_{B(z_k,2r_k)}\frac{2R(z,0)}{r_k}$ and since the last term tends to ∞ by our assumption, we get that

$$\lim\sup_{B(z_k,4r_k)}R(z,0)(4r_k-\text{dist}_0(z,z_k))^2=\infty$$

By standard argument that can be found for example in [1] we get the sequence of points z_k', such that $R(z_k',0)(4r_k-\text{dist}(z_k',z_k))^2\to\infty$ as $k\to\infty$ and $R(z,0)\leq 4R(z_k',0)$.

Let $4r'_k = 4r_k - d(z'_k, z_k)$. Then $r'_k \leq r_k$. Since $r'^2_k Q_k \to \infty$, we can apply corollary 11.2 to points z'_k to conclude that for every ϵ there exists A such that $\mathrm{Vol}B(z'_k, \frac{A}{\sqrt{Q'_k}}) \leq \epsilon(\frac{A}{\sqrt{Q'_k}})^n$, for big enough k, where $Q'_k = R(z'_k, 0)$. $Q'_k r'^2_k \to \infty$, $r'_k \leq r_k$ and $Q_k r_k^2 = 1 \Rightarrow \lim_{k\to\infty} \frac{Q'_k}{Q_k} = \infty$. Since $\mathrm{dist}(z_k, z'_k) \leq Q_k^{\frac{1}{2}}$, first by Bishop comparison theorem we have

$$\frac{\mathrm{Vol}B(z'_k, Q_k^{\frac{1}{2}} + \frac{A}{\sqrt{Q'_k}})}{(Q_k^{\frac{1}{2}} + \frac{A}{\sqrt{Q'_k}})^n} \leq \epsilon$$

We know that $B(z_k, \frac{A}{\sqrt{Q'_k}}) \subset B(z'_k, Q_k^{\frac{1}{2}} + \frac{A}{\sqrt{Q'_k}})$. Since $\frac{Q_k}{Q_k} \to \infty$, for big enough k $\frac{A}{\sqrt{Q'_k}} \leq Q_k^{\frac{1}{2}}$, so Bishop comparison principle, now applied to balls centered at z_k will give us

$$\mathrm{Vol}B(z_k, Q_k^{\frac{1}{2}}) \leq C\epsilon Q_k^{\frac{n}{2}}$$

for big enough k. Therefore, if we choose $\epsilon_k \to 0$, we will be able to find z_k (that is actually a subsequence of our original sequence $\{z_k\}$ that we will call by the same name), such that

$$\mathrm{Vol}B(z_k, Q_k^{\frac{1}{2}}) \leq C\epsilon_k Q_k^{\frac{n}{2}}$$

That means the balls $B(z_k, Q_k^{\frac{1}{2}})$ are collapsing on the scale of their radii, but this is not possible by our κ noncollapsing assumption at the beginning of this section.

□

Step 24.2. $R(z_k, 0)$ *is bounded.*

Proof. We have differential Harnack inequality in the following form

$$\ln \frac{R(x_k, \eta)}{R(z_k, -cQ_k^{-1})} \geq \ln \frac{-cQ_k^{-1}}{\eta} - \frac{\mathrm{dist}(x_k, z_k, -cQ_k^{-1})}{2(\eta + cQ_k^{-1})} \tag{72}$$

for $-cQ_k^{-1} < \eta < 0$. For each fixed k let $\eta \to 0$. We have that $\ln \frac{-cQ_k^{-1}}{\eta} \to \infty$ as $\eta \to 0$. By step 24.1 we have a curvature bound in terms of Q_k and therefore

$$d(x_k, z_k, 0) \geq d(x_k, z_k, -cQ_k^{-1}) - Cc\sqrt{Q_k}Q_k^{-1}$$

Since $d(x_k, z_k, 0) \leq Q_k^{-\frac{1}{2}}$, this will give us that $-d^2(x_k, z_k, -cQ_k^{-1}) \geq -\tilde{C}Q_k^{-1}$. From equation 72 we get

$$R(x_k, 0) \geq C_1 R(z_k, -cQ_k^{-1}) \tag{73}$$

Furthermore,

$$R(z_k, -cQ_k^{-1}) = R(z_k, 0) - R_t c Q_k^{-1}$$

R_t can be bounded of order $R(z_,0)^2$, due to X. Shi (see [1]) so that we have $R(z_k, -cQ_k^{-1}) \geq R(z_k, 0) - cCR(z_k, 0)$. By choosing c small we can assume that $1 - cC < 1$, so that we have

$$R(z_k, -cQ_k^{-1}) \geq \tilde{C} R(z_k, 0) \tag{74}$$

From equations 73 and 74 we have that $1 = R(x_k, 0) \geq \bar{C} R(z_k, 0)$. So, $R(z_k, 0)$ are uniformly bounded.

□

Step 24.3. *If y_k is a sequence such that $R(y_k, 0) \to \infty$, then* $\mathrm{dist}_0(x_k, y_k) \to \infty$ *as $k \to \infty$.*

Proof. Assume the contrarary, that $R(y_k, 0) \to \infty$, but that $\mathrm{dist}_0(x_k, y_k) \leq r$ for all k. If $y_k \in B(z_k, 2Q_k^{-\frac{1}{2}})$, by steps 24.1 and 24.2 we would have that $R(y_k, 0) \leq C$ which contradicts our assumption that $R(y_k, 0)$ is unbounded. Therefore $d(y_k, z_k) \geq 2Q_k^{-\frac{1}{2}}$, which implies that $d(x_k, y_k) \geq Q_k^{-\frac{1}{2}} \geq C$, again by step 24.2. That means $R(y_k, 0) d^2(x_k, y_k) \to \infty$ as $k \to \infty$. By a standard argument, we can now find a sequence of points y'_k and a sequence of radii r'_k, such that $R(y'_k, 0) r'^2_k \to \infty$ and $R(y, 0) \leq 2R(y'_k, 0)$ in the ball $B(y'_k, r'_k)$, where y'_k are still at the bounded distance from x_k. We can now apply corollary 11.2 to our sequencey'_k and similarly as in the proof of step 24.1, using Bishop comparison principle, we can get that the balls $B(x_k, c)$ are collapsing and that is a contradiction.

□

To finish the proof of the theorem 11.1 we have to show that the limit has bounded curvature at $t = 0$. If this were not the case, then we could find a sequence y_i going to infinity, such that $R(y_i, 0) \to \infty$ and $R(y, 0) \leq 2R(y_i, 0)$ for $y \in B(y_i, A_i R(y_i, 0)^{-\frac{1}{2}})$, where $A_i \to \infty$. Then the limit of scalings at $(y_i, 0)$ with factors $R(y_i, 0)$ satisfies the assumptions at the beginning of this section and splits off a line (by a similar argument as in proposition 11.2). Thus, it must be a round infinite cylinder.

Let $R_i = R(y_i, 0)^{-\frac{1}{2}}$. We have that $R_i \to 0$ as $i \to \infty$. We want to rule out the existence of long cylinder-like annuli $A_i = B_{R_i+100}(y_i) \backslash B_{R_i-100}(y_i)$. Let γ_i be a geodesic ray from y_i to infinity that goes through A_i. Let $B_i(x) = \lim_{t\to\infty}(t - d(x, \gamma_i(t)))$. This is a Lipschitz function with $|\nabla B_i| \leq 1$ and $\Delta B_i \geq 0$. Let ϕ_i be a cut-off function such that $\phi_i(t) = 1$ for $x \in [R_1 + 1, R_i]$ and $\phi_i(t) = 0$ for t outside $[R_1, R_i + 1]$. Now

$$0 \leq \int \phi_i(B_i) \Delta B_i = -\int \phi'_i(B_i) |\nabla B_i|^2 \tag{75}$$
$$\leq \mathrm{Vol}(B_{R_i+1}(y_i) \backslash B_{R_i}(y_i) - \mathrm{Vol}(B_{R_1+1}(y_i) \backslash B_{R_1}(y_i)) \tag{76}$$

where we used the fact that $\phi_i' = -1$ whenever $t \in (R_1, R_1+1)$ or $t \in (R_i, R_i+1)$ and the fact that $|\nabla B_i|^2 = 1$ almost everywhere. κ noncollapsing assumption and Bishop-Gromov volume comparison principle give us that $\mathrm{Vol}(B_{R_1+1}(y_i) \backslash B_{R_1}(y_i)) \geq C(R_1)$, where $C(R_1)$ is a constant depending on R_1. If $R_i \to 0$ as $i \to \infty$, the inequality 75 becomes impossible when we let $i \to \infty$.

□

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