

LOCAL GRADIENT ESTIMATE FOR p -HARMONIC FUNCTIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. For positive p -harmonic functions on Riemannian manifolds, we derive a gradient estimate and Harnack inequality with constants depending only on the lower bound of the Ricci curvature, the dimension n , p and the radius of the ball on which the function is defined. Our approach is based on a careful application of the Moser iteration technique and is different from Cheng-Yau's method [2] employed by Kostchwar and Ni [5], in which a gradient estimate for positive p -harmonic functions is derived under the assumption that the sectional curvature is bounded from below.

1. INTRODUCTION

The study of harmonic functions on Riemannian manifolds has been one of the central subjects in geometric analysis. In their classical work Cheng-Yau [15] derived the following gradient estimate for positive harmonic functions on Riemannian manifolds:

Theorem A (Cheng-Yau) *Let M be an n -dimensional complete Riemannian manifold with $\text{Ric} \geq -(n-1)\kappa$, where $\kappa \geq 0$ is a constant. Suppose that u is a positive harmonic function on a geodesic ball $B(o, R)$. Then*

$$(1.1) \quad \sup_{B(o, R/2)} \frac{|\nabla u|}{u} \leq C_n \frac{1 + R\sqrt{\kappa}}{R},$$

where C_n is a constant depending only on n .

An important feature of Cheng-Yau's estimate is that the RHS (which stands for the right hand side) of (1.1) depends only on n , k and R , it does not depend on the lower bound of the injectivity radius or a global coordinate system. From PDE viewpoints, deriving a Harnack inequality requires some bounds on the coefficients in some fixed coordinate system, thus not suitable for many problems defined on manifolds. We also observe that the RHS of (1.1) is optimal in the sense that even for $\kappa > 0$, the bound stays bounded when $R \rightarrow \infty$.

There are two major ingredients in the proof of Theorem A. First a Bochner formula is used to derive a lower bound of the Laplacian of $|\nabla u|^2$ for

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a harmonic function u in terms of the lower bound of the Ricci tensor. The second major ingredient is a clever application of the maximum principle. The trick is to multiply $|\nabla u|^2$ by a cut-off function, derive a new differential inequality for the product and then apply the maximum principle. The cut-off function is constructed using the distance function. As a result, the new differential inequality involves the Laplacian of the distance function. As is well known, the Riemannian distance function is uniformly Lipschitz and its Laplacian has an upper bound depending on the lower bound of the Ricci tensor.

Cheng-Yau's approach turned out to be very useful and some important results for other problems are deeply influenced by Theorem A. For example, P. Li [8] obtained the sharp lower bound for the first eigenvalue of a manifold, which was later generalized by Li-Yau [9]. Similar results were also obtained by Li-Yau [10] for heat equations. S. Y. Cheng [1] and H. I. Choi [3] obtained gradient estimates for harmonic mappings, etc. We refer to [15, 14] and the more recent survey [7] for an overview of the subject.

p -harmonic functions are natural extensions of harmonic functions from a variational point of view. It has been extensively studied because of its various interesting features and applications. Compared with the theory for harmonic functions the study of p -harmonic functions is generally harder because the equation, even though elliptic, is degenerate and the regularity results are far weaker (see, for example [16]). Recently, there has been renewed interest in p -harmonic functions. In particular R. Moser [11] established a nice connection between p -harmonic functions and the inverse mean curvature flow. In a recent paper [5] Kotschwar and Ni derived, among other things, a local gradient estimate for p -harmonic functions under the assumption that the sectional curvature is bounded from below. It is remarkable that the constant in their estimate does not blowup when $p \rightarrow 1$, which leads to interesting results on the inverse mean curvature flow problems. Their proof follows the same strategy introduced by Cheng-Yau [2] for harmonic functions (i.e. $p = 2$). However for general p -harmonic functions, the computation involves the full Hessian of the distance function when the cut-off function is introduced. As a result, a lower bound on the sectional curvature has to be assumed in [5].

Kotschwar and Ni speculated that their estimate may hold if only a lower bound on the Ricci tensor is assumed. The main result of this paper is to establish the following theorem:

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric} \geq -(n-1)\kappa$. Assume that v is a positive p -harmonic function on the ball*

$B(o, R) \subset M$. Then there exists a constant $C_{p,n}$ such that

$$\frac{|\nabla v|}{v} \leq C_{p,n} (1 + \sqrt{\kappa}R) / R \quad \text{on } B(o, R/2).$$

The proof of Theorem 1.1 will be presented in section two. As far as the second major ingredient of Cheng-Yau's proof is concerned, our approach follows a different strategy by carefully using the Moser iteration technique. This approach only involves differentiating the distance function once and hence bypasses the difficulty of handling the full Hessian of the distance function. In the special case $p = 2$, when p -harmonic function are just harmonic functions, Theorem 1.1 is exactly Cheng-Yau's theorem. An immediate consequence of Theorem 1.1 is the following Harnack inequality.

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric} \geq -(n-1)\kappa$. Assume that v is a positive p -harmonic function on the ball*

$B(o, R) \subset M$. Then there exists a constant $C_{p,n}$ such that for any $x, y \in B(o, R/2)$,

$$v(x)/v(y) \leq e^{C_{p,n}(1+\sqrt{\kappa}R)}.$$

It follows that if $\text{Ric} \geq 0$, then we have a uniform constant $c_{p,n}$ (independent of R) s.t.

$$(1.2) \quad \sup_{B(o, R/2)} v \leq c_{p,n} \inf_{B(o, R/2)} v.$$

This was already proved by Rigoli, Salvatori, and Vignati [12]. In fact, they proved the stronger result that (1.2) holds provided that the volume is doubling and a weak Poincare inequality holds. See also [4].

Another standard application of Theorem 1.1 is the following Liouville theorem, which was also deduced from the Harnack inequality in [12].

Let u be a p -harmonic function bounded from above or below on a complete Riemannian manifold with non-negative Ricci tensor, then u is constant.

Finally we point out that our constant $C_{p,n}$ in Theorem 1.1 becomes unbounded as $p \rightarrow 1$, while in Kotschwar-Ni's result, all the constants stay bounded when $p \rightarrow 1$. We do not know if the method can be tweaked to remove this defect.

2. THE GRADIENT ESTIMATE FOR THE p -HARMONIC FUNCTIONS

p -harmonic functions arise naturally as critical points of the L^p ($p > 1$) norm of the gradient. Let (M^n, g) be a complete Riemannian manifold and $\Omega \subset M$ an open set. A function $v \in W_{loc}^{1,p}(\Omega)$ is p -harmonic if

$$\text{div} \left(|\nabla v|^{p-2} \nabla v \right) = 0$$

in the weak sense, i.e.

$$\int_U |\nabla v|^{p-2} \langle \nabla v, \nabla \xi \rangle = 0$$

for all $\xi \in W_0^{1,p}(\Omega)$. By [16] for example, v must be $C^{1,\alpha}$. Moreover $v \in W_{loc}^{2,2}$ if $p \geq 2$; $v \in W_{loc}^{2,p}$ if $1 < p < 2$. Away from $\{\nabla v = 0\}$, v is in fact smooth.

Suppose that v is positive. Set $u = -(p-1) \log v$. Then u satisfies

$$(2.1) \quad \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) = |\nabla u|^p.$$

Let $f = |\nabla u|^2$. We define

$$\mathcal{L}(\psi) = \operatorname{div} \left(f^{p/2-1} A(\nabla \psi) \right) - p f^{p/2-1} \langle \nabla u, \nabla \psi \rangle,$$

where

$$A = id + (p-2) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}.$$

We need the following lemma from [5] and the proof is by direct calculation.

Lemma 2.1.

$$\mathcal{L}(f) = 2f^{p/2-1} \left(|D^2 u|^2 + \operatorname{Ric}(\nabla u, \nabla u) \right) + \left(\frac{p}{2} - 1 \right) |\nabla f|^2 f^{p/2-2}.$$

Remark 2.1. *Lemma 2.1 holds point-wisely in $\{x : f(x) > 0\}$. From the gradient estimate of [16] we know that $f = |\nabla u|^2 \in C^\alpha$ for some $\alpha > 0$ and $f \in W_{loc}^{1,\beta}$ for some $\beta > 1$.*

We choose a local orthonormal frame $\{e_i\}$ with $e_1 = \nabla u / |\nabla u|$. Then (2.1) takes the following form

$$(p-1)u_{11} + \sum_{i=2}^n u_{ii} = f.$$

Therefore

$$\begin{aligned} |D^2 u|^2 &\geq u_{11}^2 + 2 \sum_{i=2}^n u_{1i}^2 + \sum_{i=2}^n u_{ii}^2 \\ &\geq u_{11}^2 + 2 \sum_{i=2}^n u_{1i}^2 + \frac{1}{n-1} \left(\sum_{i=2}^n u_{ii} \right)^2 \\ &= u_{11}^2 + 2 \sum_{i=2}^n u_{1i}^2 + \frac{1}{n-1} (f - (p-1)u_{11})^2 \\ &= \frac{1}{n-1} f^2 - \frac{2(p-1)}{n-1} f u_{11} + \left(1 + \frac{(p-1)^2}{n-1} \right) u_{11}^2 + 2 \sum_{i=2}^n u_{1i}^2 \\ &\geq \frac{1}{n-1} f^2 - \frac{2(p-1)}{n-1} f u_{11} + a_0 \sum_{i=1}^n u_{1i}^2, \end{aligned}$$

where $a_0 = 1 + \min\left(\frac{(p-1)^2}{n-1}, 1\right) > 1$. Using the identities

$$\begin{aligned} 2fu_{11} &= \langle \nabla u, \nabla f \rangle, \\ \sum_{i=1}^n u_{1i}^2 &= \frac{1}{4} \frac{|\nabla f|^2}{f}, \end{aligned}$$

we end up with

$$|D^2u|^2 \geq \frac{1}{n-1} f^2 - \frac{(p-1)}{n-1} \langle \nabla u, \nabla f \rangle + \frac{a_0}{4} \frac{|\nabla f|^2}{f}.$$

Assume that $\text{Ric} \geq -(n-1)\kappa$. Therefore

$$\begin{aligned} (2.2) \quad \mathcal{L}(f) &\geq -2(n-1)\kappa f^{p/2} + \left(\frac{p+a_0}{2} - 1\right) |\nabla f|^2 f^{p/2-2} \\ &\quad + \frac{2}{n-1} f^{p/2+1} - \frac{2(p-1)}{n-1} f^{p/2-1} \langle \nabla u, \nabla f \rangle \\ &\geq -2(n-1)\kappa f^{p/2} + \frac{2}{n-1} f^{p/2+1} - \frac{2(p-1)}{n-1} f^{p/2-1} \langle \nabla u, \nabla f \rangle \end{aligned}$$

Equation (2.2) holds wherever f is strictly positive. Let $K = \{x \in \Omega : f(x) = 0\}$. Then for any nonnegative function ψ with compact support in $\Omega \setminus K$, we have

$$\begin{aligned} (2.3) \quad &\int_{\Omega} \left\langle f^{p/2-1} \nabla f + (p-2) f^{p/2-2} \langle \nabla u, \nabla f \rangle \nabla u, \nabla \psi \right\rangle \\ &+ p \int_{\Omega} f^{p/2-1} \langle \nabla u, \nabla f \rangle \psi + \frac{2}{n-1} \int_{\Omega} f^{p/2+1} \psi \\ &\leq 2(n-1)\kappa \int_{\Omega} f^{p/2} \psi + \frac{2(p-1)}{n-1} \int_{\Omega} f^{p/2-1} \langle \nabla u, \nabla f \rangle \psi. \end{aligned}$$

In particular, let $\epsilon > 0$ and $\psi = f_{\epsilon}^b \eta^2$ where $f_{\epsilon} = (f - \epsilon)^+$, $\eta \in C_0^{\infty}(B_R)$ is non-negative, $b > 1$ is to be determined later. Then direct computation yields

$$\nabla \psi = b f_{\epsilon}^{b-1} \nabla f \eta^2 + 2 f_{\epsilon}^b \eta \nabla \eta.$$

Using the above in (2.3) we obtain

$$\begin{aligned} (2.4) \quad &b \int_{B_R} \left(f^{p/2-1} f_{\epsilon}^{b-1} |\nabla f|^2 + (p-2) f^{p/2-2} \langle \nabla u, \nabla f \rangle^2 f_{\epsilon}^{b-1} \right) \eta^2 \\ &+ 2(p-2) \int_{B_R} f^{p/2-2} \langle \nabla u, \nabla f \rangle f_{\epsilon}^b \eta (\nabla u, \nabla \eta) + 2 \int_{\Omega} f^{p/2-1} f_{\epsilon}^b \eta \langle \nabla f, \nabla \eta \rangle \\ &+ p \int_{\Omega} f^{p/2-1} \langle \nabla u, \nabla f \rangle f_{\epsilon}^b \eta^2 + \frac{2}{n-1} \int_{\Omega} f^{p/2+1} f_{\epsilon}^b \eta^2 \\ &\leq 2(n-1)\kappa \int_{\Omega} f^{p/2-1} f_{\epsilon}^b \eta^2 + \frac{2(p-1)}{n-1} \int_{\Omega} f^{p/2-1} \langle \nabla u, \nabla f \rangle f_{\epsilon}^b \eta^2. \end{aligned}$$

Since $u \in C^{1,\alpha}$, $f \in C^\alpha$ and $\nabla f \in L^\beta(\Omega)$ for some $\alpha > 0$ and $\beta > 1$, we see that except for the first term, all the other terms converge to the corresponding form without ϵ . For the first term, observe that

$$f^{p/2-1} f_\epsilon^{b-1} |\nabla f|^2 + (p-2) f^{p/2-2} \langle \nabla u, \nabla f \rangle^2 f_\epsilon^{b-1} \geq a_1 f^{p/2-1} f_\epsilon^{b-1} |\nabla f|^2$$

where $a_1 = 1$ if $p \geq 2$ and $a_1 = (p-1)$ if $p \in (1, 2)$. Thus by passing ϵ to 0 we have

$$\begin{aligned} (2.5) \quad & ba_1 \int_{\Omega} f^{p/2+b-2} |\nabla f|^2 \eta^2 \\ & + 2(p-2) \int_{\Omega} f^{\frac{p-4}{2}+b} \langle \nabla u, \nabla f \rangle \eta \langle \nabla u, \nabla \eta \rangle + 2 \int_{\Omega} f^{\frac{p-2}{2}+b} \eta \langle \nabla f, \nabla \eta \rangle \\ & + p \int_{\Omega} f^{\frac{p-2}{2}+b} \langle \nabla u, \nabla f \rangle \eta^2 + \frac{2}{n-1} \int_{\Omega} f^{\frac{p+2}{2}+b} \eta^2 \\ & \leq 2(n-1)k \int_{\Omega} f^{\frac{p-2}{2}+b} \eta^2 + \frac{2(p-1)}{n-1} \int_{\Omega} f^{\frac{p-2}{2}+b} \langle \nabla u, \nabla f \rangle \eta^2. \end{aligned}$$

From now on we use a_1, a_2, \dots etc. to denote constants depending only on p and n . Combining terms in (2.5) using the definition of f we have

$$\begin{aligned} (2.6) \quad & a_1 b \int_{\Omega} f^{p/2+b-2} |\nabla f|^2 \eta^2 + \frac{2}{n-1} \int_{\Omega} f^{p/2+1+b} \eta^2 \\ & \leq 2(n-1)\kappa \int_{\Omega} f^{p/2+b} \eta^2 + a_2 \int_{\Omega} f^{\frac{p-1}{2}+b} |\nabla f| \eta^2 \\ & \quad + a_3 \int_{\Omega} f^{p/2+b-1} |\nabla f| |\nabla \eta| \eta. \end{aligned}$$

For R_3 (the third term on the RHS, L_1, L_2, R_1 etc. are understood similarly) in (2.6) we have

$$|R_3| \leq \frac{a_1 b}{4} \int_{\Omega} f^{b+p/2-2} |\nabla f|^2 \eta^2 + \frac{a_4}{b} \int_{\Omega} |\nabla \eta|^2 f^{b+p/2}.$$

Also by Cauchy's inequality R_2 of (2.6) can be estimated as

$$|R_2| \leq \frac{a_1 b}{4} \int_{\Omega} f^{p/2+b-2} |\nabla f|^2 \eta^2 + \frac{a_5}{b} \int_{\Omega} f^{b+p/2+1} \eta^2.$$

With these two inequalities we obtain

$$\begin{aligned} (2.7) \quad & \frac{a_1 b}{2} \int_{\Omega} f^{p/2+b-2} |\nabla f|^2 \eta^2 + \frac{2}{n-1} \int_{\Omega} f^{p/2+1+b} \eta^2 \\ & \leq \int_{\Omega} \left(2(n-1)\kappa \eta^2 + \frac{a_4}{b} |\nabla \eta|^2 \right) f^{p/2+b} + \frac{a_5}{b} \int_{\Omega} f^{b+p/2+1} \eta^2. \end{aligned}$$

By requiring

$$(2.8) \quad \frac{a_5}{b} < \frac{1}{n-1}$$

we see that the last term in on the RHS of (2.7) is majorized by the last term on the LHS. Therefore we have

$$\begin{aligned} & \frac{a_1 b}{2} \int_{\Omega} f^{p/2+b-2} |\nabla f|^2 \eta^2 + \frac{1}{n-1} \int_{\Omega} f^{p/2+1+b} \eta^2 \\ & \leq 2(n-1)\kappa \int_{\Omega} f^{p/2+b} \eta^2 + \frac{a_4}{b} \int_{\Omega} |\nabla \eta|^2 f^{p/2+b}. \end{aligned}$$

For the first term on the LHS we use

$$|\nabla(f^{p/4+b/2}\eta)|^2 \leq \frac{1}{2} \left(\frac{p}{2} + b\right)^2 f^{p/2+b-2} |\nabla f|^2 \eta^2 + 2f^{p/2+b} |\nabla \eta|^2.$$

From the above we obtain

$$(2.9) \quad \int_{\Omega} |\nabla(f^{p/4+b/2}\eta)|^2 + d_1 \int_{\Omega} f^{p/2+1+b} \eta^2 \leq \kappa d_2 \int_{\Omega} f^{p/2+b} \eta^2 + a_7 \int_{\Omega} |\nabla \eta|^2 f^{b+p/2}.$$

where $d_1 \sim b, d_2 \sim b$ (recall $b > 1$, $d_1 \sim b$ means d_1 is comparable to b , $d_2 \sim b$ is understood the same way).

The following Sobolev embedding theorem of Saloff-Coste plays an important role in our approach:

Theorem B (Theorem 3.1 of [13]) Let (M^n, g) be a complete Riemannian manifold with $\text{Ric} \geq -(n-1)\kappa$. For $n > 2$, there exists C , depending only on n , such that for all $B \subset M$ of radius R and volume V we have for $f \in C_0^\infty(B)$

$$\left(\int |f|^{2q} \right)^{1/q} \leq e^{C(1+\sqrt{\kappa}R)} V^{-2/n} R^2 \left(\int |\nabla f|^2 + R^{-2} f^2 \right),$$

where $q = n/(n-2)$. For $n = 2$, the above inequality holds with n replaced by any fixed $n' > 2$.

From now on, we assume $\Omega = B(o, R)$. Theorem B gives

$$(2.10) \quad \begin{aligned} & \left(\int_{\Omega} f^{\frac{n(p/2+b)}{n-2}} \eta^{\frac{2n}{n-2}} \right)^{(n-2)/n} \\ & \leq e^{c_0(1+\sqrt{\kappa}R)} V^{-\frac{2}{n}} \left(R^2 \int_{\Omega} |\nabla(f^{p/4+b/2}\eta)|^2 + \int_{\Omega} f^{p/2+b} \eta^2 \right). \end{aligned}$$

where $c_0(n, p) > 0$ depends only on n, p . Let $b_0 = c_1(n, p)(1 + \sqrt{\kappa}R)$ with $c_1(n, p)$ large enough to make b_0 satisfy (2.8), then (2.9) and (2.10) combined gives

$$(2.11) \quad \begin{aligned} & \left(\int_{\Omega} f^{\frac{n(p/2+b)}{n-2}} \eta^{\frac{2n}{n-2}} \right)^{(n-2)/n} + a_8 b R^2 e^{c_2 b_0} V^{-2/n} \int_{\Omega} f^{p/2+1+b} \eta^2 \\ & \leq a_9 b_0^2 e^{c_2 b_0} V^{-2/n} \int_{\Omega} f^{p/2+b} \eta^2 + a_{10} e^{c_2 b_0} V^{-2/n} R^2 \int_{\Omega} |\nabla \eta|^2 f^{p/2+b} \end{aligned}$$

Lemma 2.2. *Let $b_1 = (b_0 + \frac{p}{2})\frac{n}{n-2}$. Then there exists $c_3(n, p) > 0$ such that*

$$(2.12) \quad \|f\|_{L^{b_1}(B_{3R/4})} \leq c_3 \frac{b_0^2}{R^2} V^{1/b_1}.$$

Proof of Lemma 2.2: Let $b = b_0$ in (2.11), then by comparing L_2 and R_1 of (2.11) we observe that

$$a_9 b_0^3 f^{p/2+b_0} < \frac{1}{2} a_8 b_0 R^2 f^{p/2+1+b_0}$$

if $f > a_{11} b_0^2 R^{-2}$. Thus in the evaluation of R_1 we decompose Ω into two subregions, one over the places where $f \leq a_{11} b_0^2 R^{-2}$ and the second region is the complement of the first region. With this decomposition we have

$$R_1 \leq a_{12} b_0^3 \left(\frac{b_0}{R}\right)^{p+2b_0} e^{c_2 b_0} V^{1-2/n} + \frac{L_2}{2}.$$

Now (2.11) with $b = b_0$ can be written as

$$(2.13) \quad \left(\int_{\Omega} f^{\frac{n(p/2+b_0)}{n-2}} \eta^{\frac{2n}{n-2}}\right)^{(n-2)/n} + \frac{a_8}{2} b_0 R^2 e^{c_2 b_0} V^{-2/n} \int_{\Omega} f^{p/2+1+b_0} \eta^2 \\ \leq a_{12} b_0^3 \left(\frac{b_0}{R}\right)^{p+2b_0} e^{c_2 b_0} V^{1-2/n} + a_{10} e^{c_2 b_0} V^{-2/n} R^2 \int_{\Omega} |\nabla \eta|^2 f^{p/2+b_0}$$

Now we choose η to make R_2 in (2.13) dominated by the LHS. Let $\eta_1 \in C_0^\infty(B_R)$ satisfy

$$0 \leq \eta_1 \leq 1, \quad \eta_1 \equiv 1 \quad \text{in } B_{3R/4}, \quad |\nabla \eta_1| \leq C(n)/R.$$

Let $\eta = \eta_1^m$ where $m = b_0 + \frac{p}{2} + 1$. Direct computation shows

$$(2.14) \quad R^2 |\nabla \eta|^2 \leq a_{13} b_0^2 \eta^{\frac{2b_0+p}{b_0+p/2+1}}.$$

By (2.14) and Young's inequality, the R_2 of (2.13) can be written as

$$a_{10} R^2 \int_{\Omega} |\nabla \eta|^2 f^{b_0+p/2} \\ \leq a_{14} b_0^2 \int_{\Omega} f^{b_0+p/2} \eta^{\frac{2b_0+p}{b_0+p/2+1}} \\ \leq a_{14} b_0^2 \left(\int_{\Omega} f^{b_0+p/2+1} \eta^2\right)^{\frac{b_0+p/2}{b_0+p/2+1}} V^{\frac{1}{b_0+p/2+1}} \\ \leq \frac{a_8 b_0}{2} R^2 \int_{\Omega} f^{b_0+p/2+1} \eta^2 + a_{15}^{b_0+p/2} \frac{b_0^{b_0+p/2+2}}{R^{2b_0+p}} V.$$

With the estimates on R_1, R_2 we arrive at

$$(2.15) \quad \left(\int_{B_{3R/4}} f^{(b_0+p/2)n/(n-2)}\right)^{(n-2)/n} \leq a_{16}^{b_0} e^{c_2 b_0} V^{1-2/n} b_0^3 \left(\frac{b_0}{R}\right)^{p+2b_0}.$$

Recall $b_1 = (b_0 + \frac{p}{2}) \frac{n}{n-2}$. Taking the $1/(b_0 + p)$ root on both sides of (2.15) we have

$$\|f\|_{L^{b_1}(B_{3R/4})} \leq a_{17} V^{\frac{1}{b_1}} b_0^2 / R^2.$$

Lemma 2.2 is established. \square

Now we go back to (2.11), by ignoring L_2 we have

$$(2.16) \quad \left(\int_{\Omega} f^{\frac{n(p/2+b)}{n-2}} \eta^{\frac{2n}{n-2}} \right)^{(n-2)/n} \leq a_{17} \frac{e^{c_2 b_0}}{V^{2/n}} \int_{\Omega} \left((b_0^2 b \eta^2 + R^2 |\nabla \eta|^2) \right) f^{b+p/2}.$$

To apply the Moser iteration we set

$$b_{l+1} = b_l \frac{n}{n-2}, \quad \Omega_l = B(o, \frac{R}{2} + \frac{R}{4^l}), \quad l = 1, 2, \dots$$

and choose $\eta_l \in C_0^\infty(\Omega)$ s.t.

$$\eta_l \equiv 1 \text{ in } \Omega_{l+1}, \quad \eta_l \equiv 0 \text{ in } \Omega \setminus \Omega_l, \quad |\nabla \eta_l| \leq \frac{C 4^l}{R}, \quad 0 \leq \eta_l \leq 1.$$

Then in (2.16), by letting $b + \frac{p}{2} = b_l$, $\eta = \eta_l$ we have

$$\left(\int_{\Omega_{l+1}} f^{b_{l+1}} \right)^{\frac{1}{b_{l+1}}} \leq \left(a_{17} \frac{e^{c_2 b_0}}{V^{\frac{2}{n}}} \right)^{\frac{1}{b_l}} \left(\int_{\Omega_l} (b_0^2 b_l + R^2 |\nabla \eta_l|^2) f^{b_l} \right)^{\frac{1}{b_l}}.$$

By the estimate of $|\nabla \eta_l|$

$$(2.17) \quad \|f\|_{L^{b_{l+1}}(\Omega_{l+1})} \leq \left(a_{17} \frac{e^{c_2 b_0}}{V^{2/n}} \right)^{1/b_l} \left(b_0^2 b_l + 16^l \right)^{1/b_l} \|f\|_{L^{b_l}(\Omega_l)}.$$

Notice that $\sum_{l=1}^{\infty} \frac{1}{b_l} = \frac{n}{2b_1}$, then (2.17) leads to

$$(2.18) \quad \begin{aligned} & \|f\|_{L^\infty(B_{R/2})} \\ & \leq \left(a_{18} \frac{e^{c_2 b_0}}{V^{2/n}} \right)^{\sum_{l=1}^{\infty} 1/b_l} \prod_{l=1}^{\infty} \left(b_0^3 \left(\frac{n}{n-2} \right)^l + 16^l \right)^{1/b_l} \|f\|_{L^{b_1}(B_{3R/4})} \\ & \leq a_{19} \frac{e^{\frac{nc_2 b_0}{2b_1}}}{V^{1/b_1}} b_0^{\frac{3n}{2b_1}} \|f\|_{L^{b_1}(B_{3R/4})}. \end{aligned}$$

Using Lemma 2.2 in (2.18) we obtain

$$(2.19) \quad \|f\|_{L^\infty(B_{R/2})} \leq a_{20} b_0^2 / R^2.$$

Thus Theorem 1.1 is established. \square

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