****

Lecture Notes for
Partial Differential Equations I
(Math 847, Fall 2016)

by

Baisheng Yan

Department of Mathematics
Michigan State University
yan@math.msu.edu

Contents

Introduction

A **partial differential equation** (**PDE**) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Let U be an open subset of $\mathbb{R}^n$ with closure $\bar{U}$. Let $k \geq 0$ be an integer. A function $u\colon U \to \mathbb{R}$ is in $C^k(U)$ if all of its partial derivatives up to order k exist and are continuous in U. In this case, for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, with each α_i nonnegative integer and $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$, we denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For each $j = 0, 1, \cdots, k$, we denote

$$D^j u = \{D^\alpha u \mid |\alpha| = j\}$$

to be the (certain ordered) set of all partial derivatives of j. Note that without distinguishing equal derivatives, $D^k u$ is a set of n^k elements and thus can be considered as a subset of $\mathbb{R}^{n^k}$.

A PDE for unknown function u in U can be written in the form

$$F(D^k u(x), D^{k-1} u(x), \cdots, Du(x), u(x), x) = 0 \quad (x \in U), \tag{1.1}$$

where

$$F\colon \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

is a given function that depends non-trivially on the first variable in $\mathbb{R}^{n^k}$. In this case the PDE (1.1) is called of the **k-th order**.

Depending on the specific form of the function F involved, the PDE (1.1) can be characterized further as follows.

(1) The PDE (1.1) is called **linear** if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x),$$

where functions a_α and f are given. This linear equation is called **homogeneous** if $f \equiv 0$.

(2) The PDE (1.1) is called **semilinear** if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + b(D^{k-1}u, \cdots, Du, u, x) = 0.$$

(3) The PDE (1.1) is called **quasilinear** if it has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, Du, u, x) D^\alpha u(x) + b(D^{k-1}u, \cdots, Du, u, x) = 0.$$

(4) The PDE (1.1) is called **fully nonlinear** if it depends nonlinearly on the highest order derivatives.

A **system of PDEs** is a collection of several PDEs for more than one unknown functions. If we write these unknown functions as a vector-valued function $\mathbf{u}\colon U \to \mathbb{R}^m$, then a system of PDEs can be written in the general form

$$\mathbf{F}(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \cdots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0},$$

where $\mathbf{F}$ is a given vector-valued function (of many variables) into some $\mathbb{R}^N$. Note that m is the number of unknown functions and N is the number of PDEs, and that N may not be the same as m.

Sometimes, the variables of unknown function u are separated; for example, we may have $u = u(x,t)$, where $x \in U \subset \mathbb{R}^n$ denotes the space variable and $t \in \mathbb{R}$ denotes the time variable. We will always denote

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}$$

to be the (spatial) **Laplacian** of u. Many important second-order PDEs in applications are related to the Laplacian of unknown functions; e.g.,

- **Laplace's equation**:

$$\Delta u(x) = 0 \quad (x \in U).$$

- **Heat equation**:

$$u_t - \Delta u = 0 \quad (x \in U,\ t \geq 0).$$

- **Wave equation**:

$$u_{tt} - \Delta u = 0 \quad (x \in U,\ t \in \mathbb{R}).$$

- **System of linear elasticity equations**:

$$\mu\Delta\mathbf{u} + (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

Here for a vector-valued function $\mathbf{u}\colon U \to \mathbb{R}^n$ with $\mathbf{u} = (u^1, \cdots, u^n)$, we denote the **divergence** of $\mathbf{u}$ by

$$\operatorname{div}\mathbf{u} = \sum_{i=1}^n u^i_{x_i} = u^1_{x_1} + \cdots + u^n_{x_n}.$$

The first three types of equations and their generalization will be the main subject of this course.

To solve a PDE, it is ideal to find all solutions of the PDE, possibly among those that satisfy certain additional conditions in terms of some given data. However, it is most likely that the solutions cannot be found in an explicit form. Therefore, the study of a

PDE problem is to understand some of the important issues concerning the problem; most importantly, we should address at least the following three issues:

(1) (**existence**) the problem in fact has a solution;

(2) (**uniqueness**) this solution is unique; that is, it is the only solution to the problem;

(3) (**stability** or **continuous dependence** on given data) the solution depends continuously on the data given in the problem.

If a PDE problem has all these three properties, then we say that the problem is **well-posed**. Most of the PDE problems are concerned with the study of the well-posedness of the problem and certain properties of the solution because usually it is impractical to write down a formula for the solution even we know the unique solution exists.

Finally, it comes to the issue concerning the sense of a solution to a PDE. In the classical sense, a solution to a k-th order PDE is required to have all orders of (continuous) partial derivatives up to order k in the domain. This course deals with only such a solution called a **classical solution**. However, the modern theory and methods of PDE are built upon the notion of **weak solutions** to the PDE problem; we shall not study such a notion in this course. Please continue to the next semester's course to learn these subjects.

Chapter 2

Single First Order Equations

2.1. Transport equation

$$u_t + B \cdot Du(x,t) = 0,$$

where $B = (b_1, \ldots, b_n)$ is a constant vector in $\mathbb{R}^n$.

(a) $(n = 1)$ Find all solutions to $u_t + cu_x = 0$.

Geometric method: (u_x, u_t) is perpendicular to $(c, 1)$. The directional derivative along the direction $(c, 1)$ is zero, hence the function along the straight line $x = ct + d$ is constant. i.e., $u(x,t) = f(d) = f(x - ct)$. Here $x - ct = d$ is called a characteristic line.

Coordinate method: Change variable. $x_1 = x - ct$, $x_2 = cx + t$, then $u_x = u_{x_1} + u_{x_2}c$, $u_t = u_{x_1}(-c) + u_{x_2}$, hence $u_t + cu_x = u_{x_2}(1 + c^2) = 0$; i.e., $u(x,t) = f(x_1) = f(x - ct)$.

(b) (general n) Let us consider the initial value problem

$$\begin{cases} u_t + B \cdot Du(x,t) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x). \end{cases}$$

As in (a), given a point (x,t), the line through (x,t) with the direction $(B, 1)$ is represented parametrically by $(x + Bs, t + s)$, $s \in \mathbb{R}$. This line hits the plane $t = 0$ when $s = -t$ at $(x - Bt, 0)$. Since

$$\frac{d}{ds} u(x + Bs, t + s) = Du \cdot B + u_t = 0,$$

and hence u is constant on the line, we have that

$$u(x,t) = u(x - Bt, 0) = g(x - Bt).$$

If $g \in C^1$, then u is a classical solution. But if g is not in C^1, u is not a classical solution, but it is a weak solution as we will see later.

(c) Non-homogeneous problem

$$\begin{cases} u_t + B \cdot Du(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x). \end{cases}$$

This problem can be solved as in part (b). Let $z(s) = u(x+Bs, t+s)$. Then $z'(s) = f(x+Bs, t+s)$. Hence

$$u(x,t) = z(0) = z(-t) + \int_{-t}^{0} z'(s)ds = g(x-Bt) + \int_0^t f(x+(s-t)B, s)ds.$$

2.2. Linear first order equation

$$\begin{cases} a(x,t)u_t(x,t) + b(x,t)u_x(x,t) = c(x,t)u + d(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = g(x). \end{cases}$$

The idea is to find a curve $(x(s), t(s))$ so that the values of $z(s) = u(x(s), t(s))$ on this curve can be calculated easily. Note that

$$\frac{d}{ds} z(s) = u_x \dot{x} + u_t \dot{t}.$$

(Here and below, the "dot" means $\frac{d}{ds}$.) We see that if $x(s)$, $t(s)$ satisfy

$$\dot{x} = b(x,t), \quad \dot{t} = a(x,t), \tag{2.1}$$

which is called the **characteristic ODEs** for the PDE, then we would have

$$\frac{d}{ds} z(s) = \tilde{c}(s) z(s) + \tilde{d}(s), \tag{2.2}$$

where $\tilde{c}(s) = c(x(s), t(s))$ and $\tilde{d}(s) = d(x(s), t(s))$; this is a linear ODE for $z(s)$ and can be solved easily if we know the initial data.

Now given a point (x,t) in $\mathbb{R}\times(0,\infty)$, we want to find the curve $(x(s), t(s))$ starting at some point $(x^0, 0)$ when $s = 0$ passes (x,t) at some $s = s^0$. Since $z(0) = g(x^0)$, $u(x,t) = z(s^0)$ would be known by solving the equation (2.2) for $z(s)$.

Therefore, let $x = x(\tau, s)$ and $t = t(\tau, s)$ be the solution to the characteristic equations (2.1) with initial data $x(\tau, 0) = \tau,\ t(\tau, 0) = 0$ at $s = 0$. Let $z = z(\tau, s)$ be the associated solution of (2.2) with initial data $z(\tau, 0) = g(\tau)$. If we can solve (τ, s) in terms of (x,t) from $x(\tau, s) = x$ and $t(\tau, s) = t$ (e.g., by inverse function theorem), then we plug them into $z(\tau, s)$ to obtain the solution defined by $u(x,t) = z(\tau, s)$ as a function of (x,t).

Example 2.1. Solve the initial value problem

$$u_t + x u_x = u, \quad u(x,0) = x^2.$$

Solution. The characteristic ODEs with the initial data are given by

$$\begin{cases} \dot{x} = x, & x(0) = \tau, \\ \dot{t} = 1, & t(0) = 0, \\ \dot{z} = z, & z(0) = \tau^2. \end{cases}$$

Hence the solutions are given by $x = e^s\tau,\ t = s$ and $z = e^s\tau^2$. This means that each smooth solution $u(x,t)$ satisfies

$$u(e^s\tau, s) = e^s\tau^2.$$

Solve (τ, s) in terms of (x,t) to obtain that $s = t,\ \tau = xe^{-t}$. Therefore,

$$u(x,t) = \tau^2 e^s = e^{-t}x^2.$$

It is easy to check that this function $u = e^{-t}x^2$ is the solution to the given problem. $\square$

2.3. The Cauchy problem for general first order equations

We now consider the general first-order PDE of the form

$$F(Du, u, x) = 0, \quad x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n, \tag{2.3}$$

where $F = F(p, z, x)$ is smooth in $p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in \mathbb{R}^n$. Let

$$D_x F = (F_{x_1}, \cdots, F_{x_n}),\ D_z F = F_z,\ D_p F = (F_{p_1}, \cdots, F_{p_n}).$$

The **Cauchy problem** of (2.3) is to find a solution $u(x)$ such that the hypersurface $z = u(x)$ in the xz-space passes a prescribed n-dimensional manifold S in xz-space.

Assume the projection of S onto the x-space is a smooth $(n-1)$-dimensional surface Γ. We are then concerned with finding a solution u in some domain Ω in $\mathbb{R}^n$ containing Γ with a given **Cauchy data**:

$$u(x) = g(x) \quad \text{for } x \in \Gamma.$$

Let Γ be parametrized by a parameter $y \in D \subset \mathbb{R}^{n-1}$ with $x = f(y)$. The Cauchy data are then given by

$$u(f(y)) = g(f(y)) := h(y) \quad (y \in D). \tag{2.4}$$

2.3.1. Derivation of characteristic ODE.

The method to solve the Cauchy problem is motivated from the first-order linear PDE considered above, and is called the **method of characteristics**.

The idea is the following: To calculate $u(x)$ for some fixed point $x \in \Omega$, we try to find a curve connecting this x with a point $x_0 \in \Gamma$, along which we can compute u easily. How do we choose such a curve so that all this will work?

Suppose that u is a smooth (C^2) solution and $x(s)$ is our curve with $x(0) \in \Gamma$ defined on an interval I containing 0 as interior point. Let $z(s) = u(x(s))$ and $p(s) = Du(x(s))$; namely $p^i(s) = u_{x_i}(x(s))$ for $i = 1, 2, \cdots, n$. Hence

$$F(p(s), z(s), x(s)) = 0. \tag{2.5}$$

We now attempt to choose the function $x(s)$ so that we can compute $z(s)$ and $p(s)$.

First, differentiate $p^i(s) = u_{x_i}(x(s))$ to get

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \dot{x}^j(s), \quad (i = 1, \dots, n). \tag{2.6}$$

This expression is not very promising since it involves the second derivatives of u.

Second, we differentiate $F(Du, u, x) = 0$ with respect to x_i to obtain

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_i x_j} + \frac{\partial F}{\partial z}(Du, u, x) u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0, \tag{2.7}$$

and we evaluate this identity along the curve $x = x(s)$.

We now assume that the curve $x = x(s)$ is chosen so that

$$\dot{x}^j(s) = \frac{\partial F}{\partial p_j}(p(s), z(s), x(s)) \quad \forall\, j = 1, 2, \cdots, n, \tag{2.8}$$

where $z(s) = u(x(s))$ and $p(s) = Du(x(s))$. When the solution $u(x)$ is known, (2.8) is a complicated ODE for $x(s)$; so, *theoretically*, we can solve it. Then we combine (2.6)-(2.8) to obtain

$$\dot{p}^i(s) = -\frac{\partial F}{\partial z}(p(s), z(s), x(s))p^i - \frac{\partial F}{\partial x_i}(p(s), z(s), x(s)). \tag{2.9}$$

Finally, we differentiate $z(s) = u(x(s))$ to obtain

$$\dot{z}(s) = \sum_{j=1}^{n} u_{x_j}(x(s))\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s)\frac{\partial F}{\partial p_j}(p(s), z(s), x(s)). \tag{2.10}$$

What we have just demonstrated above proves the following theorem.

Theorem 2.2. *Let $u \in C^2(\Omega)$ solve $F(Du, u, x) = 0$ in Ω. Assume $x(s)$ solves* (2.8)*, where $p(s) = Du(x(s))$, $z(s) = u(x(s))$. Then $p(s)$ and $z(s)$ solve the ODEs* (2.9) *and* (2.10)*, for those s where $x(s) \in \Omega$.*

We rewrite the ODEs (2.8), (2.9) and (2.10) into a vector form:

$$\begin{aligned} &(a) \quad \dot{p}(s) = -D_x F(p, z, x) - D_z F(p, z, x)p, \\ &(b) \quad \dot{z}(s) = D_p F(p, z, x) \cdot p, \\ &(c) \quad \dot{x}(s) = D_p F(p, z, x). \end{aligned} \tag{2.11}$$

Definition 2.1. The system (2.11) of $2n + 1$ first-order ODEs together with the nonlinear equation

$$F(p(s), z(s), x(s)) = 0 \tag{2.12}$$

is called the **characteristic equations** for $F(Du, u, x) = 0$.

The solution $(p(s), z(s), x(s))$ is called the **full characteristics strip** and its projection $x(s)$ is called the **projected characteristics**.

Remark 2.1. (i) Condition (2.12) must hold if it is satisfied at $s = 0$; this follows from the fact that $F(p(s), z(s), x(s))$ must be constant if $(p(s), z(s), x(s))$ is a solution of ODEs (2.11). Therefore, the characteristic equations are essentially the system of ODEs (2.11).

(ii) The characteristic equations are truly remarkable because they form a closed autonomous system of $(2n + 1)$ ODEs for unknown vector function $X(s) = (p(s), z(s), x(s))$ of $2n + 1$ functions, which can be written in the form of

$$\dot{X} = \mathcal{A}(X), \tag{2.13}$$

where $\mathcal{A}\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is a smooth function given in terms of $F(X) = F(p, z, x)$.

(iii) If we know the initial data $X(0) = (p(0), z(0), x(0))$ when $s = 0$, then we can solve the system (2.13) at least for small s. As the initial data $(x(0), z(0))$ lie on the initial surface S, we assume that

$$x(0) = f(y), \quad z(0) = h(y).$$

In general, we need to choose the initial data $p(0) = q(y)$ to satisfy $F(q(y), h(y), f(y)) = 0$ and some *admissible conditions* in order to build the solution u to (2.3) and (2.4), in a compatible way that $z(s) = u(x(s))$ and $p(s) = Du(x(s))$; this is the important step for the **characteristics method** to be discuss later.

2.3.2. Examples. Before continuing our investigation of the characteristics method, we pause to consider some special cases for which the structure of the characteristics equations is especially simple. We illustrate as well how we can actually compute solutions of certain first-order PDEs by solving their characteristics ODE equations, subject to appropriate Cauchy conditions.

(a) **Linear equations**: $B(x) \cdot Du(x) + c(x)u = d(x)$.
In this case,

$$F(p, z, x) = B(x) \cdot p + c(x)z - d(x)$$

and hence $D_pF = B(x)$, $D_zF = c(x)$. Hence (2.11)(b)-(c), together with (2.12), yield that

$$\begin{aligned} &\dot{x}(s) = B(x(s)), \\ &\dot{z}(s) = d(x(s)) - c(x(s))z(s). \end{aligned} \tag{2.14}$$

One can solve the first set of ODEs to obtain $x(s)$, and then solve z. In this case, p **is not needed.**

Example 2.3. Solve

$$\begin{cases} xu_y - yu_x = u, & x > 0,\ y > 0, \\ u(x, 0) = g(x), & x > 0. \end{cases}$$

Solution. Equation (2.14) becomes

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = z.$$

The initial data are $x(0) = x^0,\ y(0) = 0,\ z(0) = g(x^0)$, where $x^0 > 0$ is any fixed number. Accordingly we have

$$x(s) = x^0 \cos s, \quad y(s) = x^0 \sin s, \quad z(s) = g(x^0)e^s.$$

Note that this simply means that

$$u(x^0 \cos s, x^0 \sin s) = g(x^0)e^s$$

for all x^0, s. Now given any $x > 0,\ y > 0$, we solve for $x^0 > 0,\ s > 0$ from $(x, y) = (x^0 \cos s, x^0 \sin s)$. This yields that

$$x^0 = (x^2 + y^2)^{1/2}, \quad s = \arctan(y/x);$$

therefore,

$$u(x, y) = z(s) = g((x^2 + y^2)^{1/2})e^{\arctan(y/x)}.$$

□

(b) **Quasilinear equations**: $B(x, u) \cdot Du + c(x, u) = 0$.
In this case, $F(p, z, x) = B(x, z) \cdot p + c(x, z)$; so $D_pF = B(x, z)$. Hence (2.11)(b)-(c) and (2.12) becomes

$$\dot{x}(s) = B(x, z), \quad \dot{z}(s) = B(x, z) \cdot p = -c(x, z),$$

which are *autonomous* ODEs for x, z. Once again, p **is not needed.**

Example 2.4. Solve

$$\begin{cases} u_x + u_y = u^2, & x \in \mathbb{R},\ y > 0, \\ u(x, 0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Solution. The characteristics equations become

$$\dot{x} = 1, \quad \dot{y} = 1, \quad \dot{z} = z^2.$$

The initial data are $x(0) = x^0,\ y(0) = 0,\ z(0) = g(x^0)$, where $x^0 \in \mathbb{R}$ is any fixed number. Accordingly we have

$$x(s) = x^0 + s, \quad y(s) = s, \quad z(s) = \frac{g(x^0)}{1 - sg(x^0)}.$$

Now given any $x \in \mathbb{R},\ y > 0$, we select $x^0 \in \mathbb{R},\ s > 0$ so that $(x, y) = (x(s), y(s)) = (x^0 + s, s)$. This yields that $x^0 = x - y,\ s = y$ and hence

$$u(x, y) = z(s) = \frac{g(x - y)}{1 - yg(x - y)}.$$

Note that this solution makes sense only if $yg(x - y) \neq 1$, which is the case for all $y \geq 0$ if $g(x) \leq 0$ for all $x \in \mathbb{R}$. $\square$

(c) **Fully nonlinear equations.** We now consider an example of fully nonlinear equation where we do need to solve the full characteristics ODEs in order to build a possible solution.

Example 2.5. Solve

$$\begin{cases} u_x u_y = u, & x > 0,\ y \in \mathbb{R}, \\ u(0, y) = y^2, & y \in \mathbb{R}. \end{cases}$$

Solution. In this case, $F(p, z, x, y) = p_1 p_2 - z$ and so Equation (2.11) becomes

$$\dot{x} = F_{p_1} = p_2, \quad \dot{y} = F_{p_2} = p_1, \quad \dot{z} = p_1 F_{p_1} + p_2 F_{p_2} = 2p_1 p_2 = 2z,$$

$$\dot{p_1} = -F_x - F_z p_1 = p_1, \quad \dot{p_2} = -F_y - F_z p_2 = p_2.$$

Therefore, the equations for p_1, p_2 are needed. The initial data are selected to be $x(0) = 0,\ \ y(0) = \tau,\ \ z(0) = \tau^2$, where $\tau \in \mathbb{R}$ is any fixed number. To find the initial data $p_1(0) = q_1(\tau)$ and $p_2(0) = q_2(\tau)$, we have, from $F(p(0), z(0), x(0), y(0)) = 0$, that

$$q_1(\tau) q_2(\tau) = \tau^2.$$

Since $u(0, \tau) = \tau^2$, differentiating with respect to τ, we have

$$p_2(0) = u_y(0, \tau) = 2\tau = q_2(\tau), \quad \text{so } p_1(0) = q_1(\tau) = \tfrac{1}{2}\tau.$$

Using these initial data, solve the characteristics ODEs to obtain

$$z = z(\tau, s) = \tau^2 e^{2s}, \quad p_1 = p_1(\tau, s) = \frac{1}{2}\tau e^s, \quad p_2 = p_2(\tau, s) = 2\tau e^s,$$

$$x = x(\tau, s) = \int_0^s p_2(\tau, s)\, ds = 2\tau(e^s - 1),$$

$$y = y(\tau, s) = \tau + \int_0^s p_1(\tau, s)\, ds = \frac{1}{2}\tau(e^s + 1).$$

Given (x, y) with $x > 0$ and $y \in \mathbb{R}$ we solve $x(\tau, s) = x$ and $y(\tau, s) = y$ for (τ, s); namely

$$2\tau(e^s - 1) = x, \quad \frac{1}{2}\tau(e^s + 1) = y. \tag{2.15}$$

From this we have $x + 4y = 4\tau e^s$ and hence

$$u(x, y) = u(x(\tau, s), y(\tau, s)) = z(\tau, s) = \tau^2 e^{2s} = \frac{1}{16}(x + 4y)^2.$$

(Check this is a true solution!) Note that we don't need to solve (2.15) explicitly for (τ, s). □

2.3.3. The characteristics method for Cauchy problems. We now discuss the **characteristics method** for solving the Cauchy problem:

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \Gamma, \end{cases} \tag{2.16}$$

where Γ is a given hypersurface in $\bar{\Omega}$ parametrized by $x = f(y)$ with parameter y in some open set $D \subset \mathbb{R}^{n-1}$, and f is a smooth function on D. Let

$$u(f(y)) = g(f(y)) := h(y), \quad y \in D \subset \mathbb{R}^{n-1}.$$

Fix $y_0 \in D$. Let $x_0 = f(y_0) \in \Gamma,\ z_0 = g(x_0) = h(y_0)$. We assume that $p_0 \in \mathbb{R}^n$ is given such that

$$F(p_0, z_0, x_0) = 0. \tag{2.17}$$

In order that $p_0 = Du(x_0)$, it is necessary that

$$h_{y_j}(y_0) = \sum_{i=1}^n f^i_{y_j}(y_0) p^i_0 \quad \forall\, j = 1, 2, \cdots, n-1. \tag{2.18}$$

Definition 2.2. Given $y_0 \in D$, we say that a vector $p_0 \in \mathbb{R}^n$ is *admissible* at y_0 (or the pair (p_0, y_0) is *admissible*) for the Cauchy problem if (2.17) and (2.18) are satisfied.

Note that an admissible p_0 may or may not exist; even when it exists, it may not be unique.

Conditions (2.17) and (2.18) can be written in terms of a map $\mathcal{F}(p, y)$ from $\mathbb{R}^n \times D$ to $\mathbb{R}^n$ defined by $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_n)$, where

$$\begin{aligned} \mathcal{F}_j(p, y) &= \sum_{i=1}^n f^i_{y_j}(y) p^i - h_{y_j}(y) \quad (j = 1, 2, \cdots, n-1); \\ \mathcal{F}_n(p, y) &= F(p, h(y), f(y)), \quad y \in D, \ \ p \in \mathbb{R}^n. \end{aligned}$$

Note that (p_0, y_0) is an admissible pair if and only if $\mathcal{F}(p_0, y_0) = 0$.

Definition 2.3. We say that an admissible pair (p_0, y_0) is **non-characteristic** if

$$\det \frac{\partial \mathcal{F}(p, y)}{\partial p}\Big|_{p=p_0,\, y=y_0} = \det \begin{pmatrix} \frac{\partial f^1(y_0)}{\partial y_1} & \cdots & \frac{\partial f^n(y_0)}{\partial y_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial f^1(y_0)}{\partial y_{n-1}} & \cdots & \frac{\partial f^n(y_0)}{\partial y_{n-1}} \\ F_{p_1}(p_0, z_0, x_0) & \cdots & F_{p_n}(p_0, z_0, x_0) \end{pmatrix} \neq 0.$$

In this case, by the **implicit function theorem**, there exists a smooth function $p = q(y)$ in a neighborhood J of y_0 in D such that $q(y_0) = p_0$ and

$$\mathcal{F}(q(y), y) = 0 \quad (y \in J), \tag{2.19}$$

that is, $(q(y), y)$ is admissible for all $y \in J$.

Remark 2.2. The determinant above is actually equal to $\kappa\nu(x_0) \cdot D_p F(p_0, z_0, x_0)$, where κ is a nonzero number and $\nu(x_0)$ is the outward unit normal to Γ at $x_0 \in \Gamma$. Therefore, the non-characteristic condition at the admissible pair (p_0, y_0) can be written as

$$D_p F(p_0, z_0, x_0) \cdot \nu(x_0) \neq 0, \tag{2.20}$$

where $x_0 = f(y_0), z_0 = h(y_0) = g(x_0)$. In particular, if (2.16) is quasilinear where $F(p, z, x) = B(x, z) \cdot p + c(x, z)$, then the non-characteristic condition (2.20) becomes

$$B(x_0, g(x_0)) \cdot \nu(x_0) \neq 0. \tag{2.21}$$

In what follows, we assume the pair (p_0, y_0) is admissible and non-characteristic. Let $q(y)$ be determined by (2.19) in a neighborhood J of y_0 in D. Let $(p(s), z(s), x(s))$ be the solution to Eq. (2.10) with initial data

$$p(0) = q(y), \quad z(0) = h(y), \quad x(0) = f(y)$$

for given $y \in J$. Since these solutions depend on the parameter y, we denote them by $p = P(y, s)$, $z = Z(y, s)$ and $x = X(y, s)$, where

$$P(y, s) = (p^1(y, s), p^2(y, s), \cdots, p^n(y, s)),$$
$$X(p, y) = (x^1(y, s), x^2(y, s), \cdots, x^n(y, s)),$$

to display the dependence on y. By the ODE theory of continuous dependence, P, Z, X are C^2 in (y, s).

Lemma 2.6. *Let (p_0, y_0) be admissible and non-characteristic. Let $x_0 = f(y_0)$. Then there exists an open interval I containing 0, a neighborhood J of y_0, and a neighborhood V of x_0 such that for each $x \in V$ there exist unique $s = s(x) \in I$, $y = y(x) \in J$ such that $x = X(y(x), s(x))$. Moreover, $s(x)$ and $y(x)$ are C^2 on $x \in V$.*

Proof. We have $X(y_0, 0) = f(y_0) = x_0$. Then the result follows from the **inverse function theorem**. In fact, using $\frac{\partial X}{\partial y}|_{s=0} = \frac{\partial f}{\partial y}$ and $\frac{\partial X}{\partial s}|_{s=0} = F_p(p_0, z_0, x_0)$, we easily see that the Jacobian determinant

$$\det \frac{\partial X(y, s)}{\partial(y, s)}|_{y=y_0,\, s=0} = \det \begin{pmatrix} \frac{\partial f^1(y_0)}{\partial y_1} & \cdots & \frac{\partial f^1(y_0)}{\partial y_{n-1}} & F_{p_1}(p_0, z_0, x_0) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^n(y_0)}{\partial y_1} & \cdots & \frac{\partial f^n(y_0)}{\partial y_{n-1}} & F_{p_n}(p_0, z_0, x_0) \end{pmatrix} \neq 0$$

from the noncharacteristic assumption. Furthermore, since $X(y, s)$ is C^2, we also have that $y(x), s(x)$ are C^2. $\square$

Finally, define

$$u(x) = Z(y(x), s(x)) \quad \forall\, x \in V.$$

Then we have the following:

Theorem 2.7. *Let (p_0, y_0) be admissible and non-characteristic. Then the function $u(x)$ defined above solves the equation $F(Du, u, x) = 0$ in V with the Cauchy data $u(x) = g(x)$ on $\Gamma \cap V$.*

Proof. (1) If $x \in \Gamma \cap V$ then $x = f(y) = X(y, 0)$ for some $y \in J$. So $s(x) = 0$, $y(x) = y$ and hence $u(x) = Z(y(x), s(x)) = Z(y, 0) = h(y) = g(f(y)) = g(x)$. The Cauchy data follow.

(2) The function

$$\psi(y, s) = F(P(y, s), Z(y, s), X(y, s)) = 0$$

for all $s \in I$ and $y \in J$. In fact $\psi(y, 0) = 0$ since $(q(y), y)$ is admissible, and

$$\psi_s(y, s) = D_pF \cdot P_s + (D_zF)Z_s + D_xF \cdot X_s \equiv 0,$$

from the characteristic ODEs.

(3) Note that

$$Z_s(y,s) = \dot{Z}(y,s) = \sum_{j=1}^{n} p^j(y,s)\dot{x}^j(y,s),$$

and hence, for each $i = 1, \ldots, n-1$,

$$\frac{\partial^2 Z}{\partial y_i \partial s} = \sum_{j=1}^{n} p^j \frac{\partial^2 x^j}{\partial y_i \partial s} + \sum_{j=1}^{n} \frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s}. \tag{2.22}$$

Furthermore, upon differentiating $F(P(y,s), Z(y,s), X(y,s)) = 0$ with respect to y_i, we obtain that

$$\sum_{j=1}^{n}\left(\frac{\partial F}{\partial x_j}\frac{\partial x^j}{\partial y_i} + \frac{\partial F}{\partial p_j}\frac{\partial p^j}{\partial y_i}\right) = -\frac{\partial F}{\partial z}\frac{\partial Z}{\partial y_i}. \tag{2.23}$$

We claim that

$$\frac{\partial Z}{\partial y_i}(y,s) = \sum_{j=1}^{n} p^j(y,s)\frac{\partial x^j}{\partial y_i}(y,s) \quad \forall\, i = 1, \ldots, n-1. \tag{2.24}$$

To prove this, let

$$r(s) = \frac{\partial Z}{\partial y_i}(y,s) - \sum_{j=1}^{n} p^j(y,s)\frac{\partial x^j}{\partial y_i}(y,s).$$

Then $r(0) = h_{y_i}(y) - \sum_{j=1}^{n} q^j(y)\frac{\partial f^j}{\partial y_i}(y) = 0$ by the choice of $q(y) = P(y,0)$. Moreover, by (2.22) and (2.23), it follows that

$$\begin{aligned}
\dot{r}(s) &= \frac{\partial^2 Z}{\partial y_i \partial s} - \sum_{j=1}^{n} p^j \frac{\partial^2 x^j}{\partial y_i \partial s} - \sum_{j=1}^{n} \frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i} \\
&= \sum_{j=1}^{n}\left(\frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s} - \frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i}\right) \\
&= \sum_{j=1}^{n}\left[\frac{\partial p^j}{\partial y_i}\frac{\partial F}{\partial p_j} + \left(\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial z}p^j\right)\frac{\partial x^j}{\partial y_i}\right] \\
&= \sum_{j=1}^{n}\left(\frac{\partial F}{\partial x_j}\frac{\partial x^j}{\partial y_i} + \frac{\partial F}{\partial p_j}\frac{\partial p^j}{\partial y_i}\right) + \frac{\partial F}{\partial z}\sum_{j=1}^{n} p^j \frac{\partial x^j}{\partial y_i} \\
&= -\frac{\partial F}{\partial z}(P(y,s), Z(y,s), X(y,s))r(s).
\end{aligned}$$

This shows that $r(s)$ solves a homogeneous linear first-order ODE with zero initial condition; consequently $r(s) \equiv 0$, proving the claim.

(4) Finally let $w(x) = P(y(x), s(x))$ for $x \in V$; namely, $p^k(y(x), s(x)) = w^k(x)$. From the definition of $u(x)$ we have

$$F(w(x), u(x), x) = 0 \quad \forall\, x \in V.$$

To finish the proof, we show that $w(x) = Du(x)$. Since $u(x) = Z(y(x), s(x))$ and $x = X(y(x), s(x))$ for all $x \in V$, we have, by (2.24) evaluated at $(y(x), s(x))$, that

$$\begin{aligned} u_{x_j} &= Z_s \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} Z_{y_i} \frac{\partial y^i}{\partial x_j} \\ &= \sum_{k=1}^{n} w^k \dot{x}^k \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \left(\sum_{k=1}^{n} w^k \frac{\partial x^k}{\partial y_i} \right) \frac{\partial y^i}{\partial x_j} \\ &= \sum_{k=1}^{n} w^k \left(\dot{x}^k \frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x^k}{\partial y_i} \frac{\partial y^i}{\partial x_j} \right) \\ &= \sum_{k=1}^{n} w^k \frac{\partial x^k}{\partial x_j} = \sum_{k=1}^{n} w^k \delta_{kj} = w^j. \end{aligned}$$

So $Du(x) = w(x)$ on V. This completes the proof. □

Example 2.8. What happens if the non-characteristic condition fails? Let us look at an example in this case. Solve

$$\begin{cases} xu_x + yu_y = u, \\ u(\tau, \tau) = g(\tau) \ \ \forall\, \tau \in \mathbb{R}. \end{cases}$$

Solution. In this case, the non-characteristic condition (2.21) fails for all $x_0 = (\tau, \tau)$. If we still solve the characteristic ODEs

$$\dot{x} = x, \quad \dot{y} = y, \quad \dot{z} = z$$

with initial data $x(0) = \tau,\ \ y(0) = \tau,\ \ z(0) = g(\tau)$, then we obtain

$$X(\tau, s) = \tau e^s, \quad Y(\tau, s) = \tau e^s, \quad Z(\tau, s) = g(\tau)e^s.$$

However, we cannot solve for (τ, s) from $X(\tau, s) = x$ and $Y(\tau, s) = y$. In this case, the Cauchy problem may or may not have a solution.

If the Cauchy problem does have a smooth solution $u(x, t)$, then

$$g'(\tau) = u_x(\tau, \tau) + u_y(\tau, \tau) = g(\tau)/\tau,$$

and hence we must have $g(\tau) = \alpha\tau$ for some constant $\alpha \in \mathbb{R}$.

For such a Cauchy datum $g(\tau) = \alpha\tau$, we can find *infinitely many solutions* by solving a Cauchy with **any non-characteristic** Cauchy data; for example, with $u(\tau, \tau^2) = h(\tau)$, where $h(0) = g(0) = 0$ and $h(1) = g(1)$ to be compatible with $u(\tau, \tau) = g(\tau)$. Doing so, we obtain infinitely many solutions

$$u(x, y) = h(\frac{y}{x}) \frac{x^2}{y},$$

as long as $h(\tau)$ is a smooth function with $h(0) = 0$ and $h(1) = g(1) = \alpha$. □

Example 2.9. Solve

$$\begin{cases} \sum_{j=1}^{n} x_j u_j = \alpha u, \\ u(x_1, \ldots, x_{n-1}, 1) = h(x_1, \ldots, x_{n-1}). \end{cases}$$

Solution. The characteristic ODEs:

$$\dot{x}_j = x_j, \quad j = 1, 2, \ldots, n; \quad \dot{z} = \alpha z$$

with initial data $x_j(0) = y_j$ for $j = 1, \ldots, n-1$, $x_n(0) = 1$ and $z(0) = h(y_1, \ldots, y_n)$. So

$$x_j(y, s) = y_j e^s, \ j = 1, \ldots, n-1; \quad x_n(y, s) = e^s; \quad z(y, s) = e^{\alpha s} h(y).$$

Solving (y, s) in terms of x, we have $e^s = x_n$, $y_j = x_j/x_n$ $(j = 1, \cdots, n-1)$. Hence

$$u(x) = z(y, s) = x_n^\alpha h(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}).$$

□

Example 2.10. Solve

$$\begin{cases} u_t + B(u) \cdot Du = 0, \\ u(x, 0) = g(x), \end{cases} \tag{2.25}$$

where $u = u(x, t)$ for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $Du = (u_{x_1}, \cdots, u_{x_n})$, with $B \colon \mathbb{R} \to \mathbb{R}^n$, $g \colon \mathbb{R}^n \to \mathbb{R}$ being given smooth functions.

Solution. In this case, let $q = (p, p_{n+1})$ and $F(q, z, x) = p_{n+1} + B(z) \cdot p$, and so the characteristic ODEs are

$$\begin{cases} \dot{x} = B(z), & x(y, 0) = y, \\ \dot{t} = 1, & t(y, 0) = 0, \\ \dot{z} = 0, & z(y, 0) = g(y). \end{cases}$$

The solution is given by

$$Z(y, s) = g(y), \quad X(y, s) = y + sB(g(y)), \quad T(y, s) = s.$$

The projected characteristics C_y in the xt-space passing through a point $(y, 0)$ is the line $\{(x, t) = (y + tB(g(y)), t) \mid t \in \mathbb{R}\}$, along which u is a constant $g(y)$. Hence, the solution $u = u(x, t)$ is given *implicitly* by

$$u(x, t) = g(x - tB(u(x, t))).$$

Furthermore, two distinct projected characteristic curves C_{y_1}, C_{y_2} (with $y_1 \neq y_2$) intersect at a point (x, t) if and only if

$$y_1 - y_2 = t(B(g(y_2)) - B(g(y_1))). \tag{2.26}$$

At the intersection point (x, t) we have $g(y_1) \neq g(y_2)$; hence the solution u becomes singular (undefined). Therefore $u(x, t)$ becomes singular for some $t > 0$ if and only if there exist $y_1 \neq y_2$ such that (2.26) holds for $t > 0$.

When $n = 1$, $u(x, t)$ becomes singular for some $t > 0$ unless $B(g(y))$ is a *nondecreasing* function. In fact, if $B(g(y))$ is not nondecreasing then there exist $y_1 < y_2$ such that $B(g(y_1)) > B(g(y_2))$. Then with

$$t = -\frac{y_2 - y_1}{B(g(y_2)) - B(g(y_1))} > 0,$$

the projected characteristic lines C_{y_1} and C_{y_2} intersect at some point (x, t). If u were smooth up to the point (x, t) then $u(x, t)$ would be equal to $g(y_k)$ for both $k = 1, 2$. However, $g(y_1) \neq g(y_2)$ since $B(g(y_1)) > B(g(y_2))$. The same argument also shows that the problem cannot have a regular (smooth) solution defined on *whole* $\mathbb{R}^2$ unless $B(g(y))$ is a constant function. □

Laplace's Equation

3.1. Green's identities

For a smooth vector field $\vec{F} = (f^1, \ldots, f^n)$, we define the **divergence** of $\vec{F}$ by

$$\operatorname{div} \vec{F} = \sum_{j=1}^{n} \frac{\partial f^j}{\partial x_j} = \sum_{j=1}^{n} f^j_{x_j}.$$

Lemma 3.1 (**Divergence Theorem**). *Assume Ω is a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Let $\vec{F} \in C^1(\bar{\Omega}; \mathbb{R}^n)$. Then*

$$\int_\Omega \operatorname{div} \vec{F}(x)\, dx = \int_{\partial\Omega} \vec{F}(x) \cdot \nu(x)\, dS, \tag{3.1}$$

where $\nu(x)$ is the **outer unit normal** *to the boundary $\partial\Omega$ at x.*

For a smooth function $u(x)$, we denote the **gradient** of u by $Du = \nabla u = (u_{x_1}, \cdots, u_{x_n})$ and define the **Laplacian** of u by

$$\Delta u = \operatorname{div}(\nabla u) = \sum_{k=1}^{n} u_{x_k x_k}. \tag{3.2}$$

Lemma 3.2 (**Green's Identities**). *Assume Ω is a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Let $u \in C^2(\bar{\Omega}), v \in C^1(\bar{\Omega})$. Then*

$$\int_\Omega v \Delta u\, dx = -\int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, dS. \tag{3.3}$$

If $u, v \in C^2(\bar{\Omega})$, then

$$\int_\Omega (v \Delta u - u \Delta v)\, dx = \int_{\partial\Omega} (v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu})\, dS. \tag{3.4}$$

Equation (3.3) *is called* **Green's first identity**, *while Equation* (3.4) *is called* **Green's second identity**.

Proof. Assume $u \in C^2(\bar{\Omega}), v \in C^1(\bar{\Omega})$ and let $\vec{F} = v \nabla u$. Then $\vec{F} \in C^1(\bar{\Omega}; \mathbb{R}^n)$ and $\operatorname{div} \vec{F} = \nabla u \cdot \nabla v + v \Delta u$, and hence, by the Divergence Theorem,

$$\int_\Omega v \Delta u\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx = \int_{\partial\Omega} v \nabla u \cdot \nu\, dS = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, dS,$$

from which (3.3) follows. Now, assume $u, v \in C^2(\bar{\Omega})$. Exchanging u and v in (3.3), we obtain

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} dS. \tag{3.5}$$

Combining (3.3) and (3.5) yields (3.4). □

Definition 3.1. The equation $\Delta u = 0$ is called **Laplace's equation**. A C^2 function u satisfying $\Delta u = 0$ in an open set $\Omega \subseteq \mathbb{R}^n$ is called a **harmonic function** in Ω.

Dirichlet and Neumann (boundary) problems. The **Dirichlet (boundary) problem** for Laplace's equation is:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

The **Neumann (boundary) problem** for Laplace's equation is:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Here f and g are given (boundary) functions.

Theorem 3.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$.*

(i) *Given any f, there can be at most one $C^2(\bar{\Omega})$-solution to the Dirichlet problem.*

(ii) *If the Neumann problem (3.7) has a $C^2(\bar{\Omega})$-solution u then $\int_{\partial\Omega} g\, dS = 0$; moreover, if Ω is connected, any two $C^2(\bar{\Omega})$-solutions to the Neumann problem must differ by a constant.*

Proof. In the Green's first identity (3.3), choose $u = v$, and we have

$$\int_\Omega u\Delta u\, dx = -\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u\frac{\partial u}{\partial \nu}\, dS. \tag{3.8}$$

(i) Let $u_1,\ u_2 \in C^2(\bar{\Omega})$ be any two solutions to the Dirichlet problem and let $u = u_1 - u_2$. Then $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$; hence, by (3.8), $\nabla u = 0$; consequently $u \equiv$ constant on each connected component of Ω; however, since $u = 0$ on $\partial\Omega$, the constant must be zero. So $u \equiv 0$ and hence $u_1 = u_2$ in Ω.

(ii) Using (3.3) with $v \equiv 1$, we have

$$\int_\Omega \Delta u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS.$$

Hence, if there is a $C^2(\bar{\Omega})$ solution u to (3.7) then $\int_{\partial\Omega} g(x) dS = 0$. Let $u_1,\ u_2 \in C^2(\bar{\Omega})$ be any two solutions to the Neumann problem (3.7) and let $u = u_1 - u_2$. Then $\Delta u = 0$ in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$; hence, by (3.8), $\nabla u = 0$ in Ω. If Ω is connected, then u must be a constant in Ω; therefore, any two C^2-solutions of the Neumann problem must differ by a constant. □

3.2. Fundamental solutions and Green's function

We try to seek a harmonic function $u(x)$ that depends only on the radius $r = |x|$, i.e., $u(x) = v(r)$, $r = |x|$ (**radial function**). Computing Δu for such a function leads to an ODE for v:

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

If $n = 1$, then $v(r) = r$ is a solution. For $n \geq 2$, let $s(r) = v'(r)$, and we have $s'(r) = -\frac{n-1}{r} s(r)$, which is a first-order linear ODE for $s(r)$; upon solving this ODE, we have that $s(r) = cr^{1-n}$. Consequently, we obtain a solution for $v(r)$ as follows:

$$v(r) = \begin{cases} Cr, & n = 1, \\ C \ln r, & n = 2, \\ Cr^{2-n}, & n \geq 3. \end{cases}$$

Note that $v(r)$ is well-defined for $r > 0$, but is singular at $r = 0$ when $n \geq 2$.

3.2.1. Fundamental solutions.

Definition 3.2. We call the function $\Phi(x) = \phi(|x|)$ the **fundamental solution** of Laplace's equation in $\mathbb{R}^n$, where

$$\phi(r) = \begin{cases} -\frac{1}{2} r, & n = 1, \\ -\frac{1}{2\pi} \ln r, & n = 2, \\ \frac{1}{n(n-2)\alpha_n} r^{2-n}, & n \geq 3. \end{cases} \tag{3.9}$$

Here, for $n \geq 3$, α_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 3.1. With the number α_n, it follows that a ball of radius ρ in $\mathbb{R}^n$ has the volume $\alpha_n \rho^n$ and the surface area $n\alpha_n \rho^{n-1}$. The constant appearing in the fundamental solution $\Phi(x)$ exactly assures the following theorem.

Theorem 3.4. *For any $f \in C_c^2(\mathbb{R}^n)$, define*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad (x \in \mathbb{R}^n).$$

Then $u \in C^2(\mathbb{R}^n)$ and solves the **Poisson's equation**

$$-\Delta u(x) = f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{3.10}$$

Proof. We only prove this for the case $n \geq 2$; the proof of the case $n = 1$ (where $\Phi(x) = -\frac{1}{2}|x|$) is left as an exercise.

1. Let $f \in C_c^2(\mathbb{R}^n)$. Fix any bounded open set $V \subset \mathbb{R}^n$ and take $x \in V$. Then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy = \int_{B(0,R)} \Phi(y) f(x-y)\, dy,$$

where $B(0,R)$ is a large ball in $\mathbb{R}^n$ such that $f(x-y) = 0$ for all $x \in V$ and $y \notin B(0, R/2)$. Since $\Phi(y)$ is integrable near $y = 0$ (see (3.11) below), by differentiation under the integral, we have

$$u_{x_i}(x) = \int_{B(0,R)} \Phi(y) f_{x_i}(x-y)\, dy, \quad u_{x_i x_j}(x) = \int_{B(0,R)} \Phi(y) f_{x_i x_j}(x-y)\, dy.$$

This proves that $u \in C^2(V)$. Since V is arbitrary, it follows that $u \in C^2(\mathbb{R}^n)$. Moreover

$$\Delta u(x) = \int_{B(0,R)} \Phi(y)\Delta_x f(x-y)\,dy = \int_{B(0,R)} \Phi(y)\Delta_y f(x-y)\,dy.$$

2. Fix $0 < \epsilon < R$. Write

$$\Delta u(x) = \int_{B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,dy + \int_{B(0,R)\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,dy =: I_\epsilon + J_\epsilon.$$

Now

$$|I_\epsilon| \le C\|D^2 f\|_{L^\infty} \int_{B(0,\epsilon)} |\Phi(y)|\,dy \le \begin{cases} C\int_0^\epsilon r|\ln r|\,dr & (n=2) \\ C\epsilon^2 & (n\ge 3). \end{cases} \tag{3.11}$$

Hence $I_\epsilon \to 0$ as $\epsilon \to 0^+$. For J_ϵ we apply the Green's second identity (3.4) with $\Omega = B(0,R)\setminus B(0,\epsilon)$ to have

$$\begin{aligned} J_\epsilon &= \int_\Omega \Phi(y)\Delta_y f(x-y)\,dy \\ &= \int_\Omega f(x-y)\Delta\Phi(y)\,dy + \int_{\partial\Omega}\left[\Phi(y)\frac{\partial f(x-y)}{\partial\nu_y} - f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}\right]dS_y \\ &= \int_{\partial\Omega}\left[\Phi(y)\frac{\partial f(x-y)}{\partial\nu_y} - f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}\right]dS_y \\ &= \int_{\partial B(0,\epsilon)}\left[\Phi(y)\frac{\partial f(x-y)}{\partial\nu_y} - f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}\right]dS_y, \end{aligned}$$

where $\nu_y = -\frac{y}{\epsilon}$ is the outer unit normal of $\partial\Omega$ on the sphere $\partial B(0,\epsilon)$. Now

$$\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial\nu_y}\right| \le C|\phi(\epsilon)|\|Df\|_{L^\infty}\int_{\partial B(0,\epsilon)} dS \le C\epsilon^{n-1}|\phi(\epsilon)| \to 0$$

as $\epsilon \to 0^+$. Furthermore, $\nabla\Phi(y) = \phi'(|y|)\frac{y}{|y|}$; hence

$$\frac{\partial\Phi(y)}{\partial\nu_y} = \nabla\Phi(y)\cdot\nu_y = -\phi'(\epsilon) = \begin{cases} \frac{1}{2\pi}\epsilon^{-1} & (n=2) \\ \frac{1}{n\alpha_n}\epsilon^{1-n} & (n\ge 3), \end{cases} \qquad \text{for all } y \in \partial B(0,\epsilon).$$

That is, $\frac{\partial\Phi(y)}{\partial\nu_y} = \frac{1}{\int_{\partial B(0,\epsilon)} dS}$ and hence

$$\int_{\partial B(0,\epsilon)} f(x-y)\frac{\partial\Phi(y)}{\partial\nu_y}dS_y = ⨍_{\partial B(0,\epsilon)} f(x-y)\,dS_y \to f(x),$$

as $\epsilon \to 0^+$. Combining all the above, we finally prove that

$$-\Delta u(x) = f(x) \quad \forall\, x \in \mathbb{R}^n.$$

□

Remark 3.2. The reason the function $\Phi(x)$ is called a fundamental solution of Laplace's equation is as follows. The function $\Phi(x)$ formally satisfies

$$-\Delta_x\Phi(x) = \delta_0 \quad \text{on } x \in \mathbb{R}^n,$$

where δ_0 is the **Dirac measure** concentrated at 0:

$$\langle\delta_0, f\rangle = f(0) \quad \forall\, f \in C_c^\infty(\mathbb{R}^n).$$

If $u(x)=\int_{\mathbb{R}^n}\Phi(x-y)f(y)\,dy$, we can formally compute that (in terms of distributions)

$$
\begin{aligned}
-\Delta u(x) &= -\int_{\mathbb{R}^n}\Delta_x\Phi(x-y)f(y)\,dy = -\int_{\mathbb{R}^n}\Delta_y\Phi(x-y)f(y)\,dy\\
&= -\int_{\mathbb{R}^n}\Delta_y\Phi(y)f(x-y)\,dy = \langle\delta_0, f(x-\cdot)\rangle = f(x)
\end{aligned}
$$

3.2.2. Green's function. Suppose $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $h\in C^2(\bar\Omega)$ be any harmonic function in Ω.

Given any function $u\in C^2(\bar\Omega)$, fix $x\in\Omega$ and $0<\epsilon<\operatorname{dist}(x,\partial\Omega)$. Let $\Omega_\epsilon=\Omega\setminus B(x,\epsilon)$. Apply Green's second identity

$$
\int_{\Omega_\epsilon}(u(y)\Delta v(y)-v(y)\Delta u(y))\,dy=\int_{\partial\Omega_\epsilon}\left(u(y)\frac{\partial v}{\partial\nu}(y)-v(y)\frac{\partial u}{\partial\nu}(y)\right)dS
$$

to functions $u(y)$ and $v(y)=\Gamma(x,y)=\Phi(y-x)-h(y)$ on Ω_ϵ, where $\Phi(y)=\phi(|y|)$ is the fundamental solution above, and since $\Delta v(y)=0$ on Ω_ϵ, we have

$$
\begin{aligned}
-\int_{\Omega_\epsilon}\Gamma(x,y)\Delta u(y)\,dy &= \int_{\partial\Omega_\epsilon}u(y)\frac{\partial\Gamma}{\partial\nu_y}(x,y)dS-\int_{\partial\Omega_\epsilon}\Gamma(x,y)\frac{\partial u}{\partial\nu_y}(y)dS\\
&= \int_{\partial\Omega}u(y)\frac{\partial\Gamma}{\partial\nu_y}(x,y)dS-\int_{\partial\Omega}\Gamma(x,y)\frac{\partial u}{\partial\nu_y}(y)dS\\
&\quad+\int_{\partial B(x,\epsilon)}u(y)\left(\frac{\partial\Phi}{\partial\nu}(y-x)-\frac{\partial h}{\partial\nu}(y)\right)dS\\
&\quad-\int_{\partial B(x,\epsilon)}(\Phi(y-x)-h(y))\frac{\partial u}{\partial\nu}(y)dS,
\end{aligned}
\tag{3.12}
$$

where $\nu=\nu_y$ is the outer unit normal at $y\in\partial\Omega_\epsilon=\partial\Omega\cup\partial B(x,\epsilon)$. Note that $\nu_y=-\frac{y-x}{\epsilon}$ at $y\in\partial B(x,\epsilon)$. Hence $\frac{\partial\Phi}{\partial\nu_y}(y-x)=-\phi'(\epsilon)=\frac{1}{\int_{\partial B(x,\epsilon)}dS}$ for $y\in\partial B(x,\epsilon)$. So, in (3.12), letting $\epsilon\to0^+$ and noting that

$$
\begin{aligned}
&\int_{\partial B(x,\epsilon)}u(y)\left(\frac{\partial\Phi}{\partial\nu_y}(y-x)-\frac{\partial h}{\partial\nu_y}(y)\right)dS_y\\
&= ⨍_{\partial B(x,\epsilon)}u(y)dS_y-\int_{\partial B(x,\epsilon)}u(y)\frac{\partial h}{\partial\nu_y}(y)dS_y\to u(x)
\end{aligned}
$$

and

$$
\int_{\partial B(x,\epsilon)}(\Phi(y-x)-h(y))\frac{\partial u}{\partial\nu_y}(y)dS\to0,
$$

we deduce

Theorem 3.5 (Representation formula)**.** *Let $\Gamma(x,y)=\Phi(y-x)-h(y)$, where $h\in C^2(\bar\Omega)$ is harmonic in Ω. Then, for all $u\in C^2(\bar\Omega)$,*

$$
u(x)=\int_{\partial\Omega}\left[\Gamma(x,y)\frac{\partial u}{\partial\nu_y}(y)-u(y)\frac{\partial\Gamma}{\partial\nu_y}(x,y)\right]dS-\int_\Omega\Gamma(x,y)\Delta u(y)\,dy\quad(x\in\Omega).
\tag{3.13}
$$

This formula permits us to solve for u *if* we know the values of Δu in Ω and *both* u and $\frac{\partial u}{\partial\nu}$ on $\partial\Omega$. However, for Poisson's equation with Dirichlet boundary condition, $\partial u/\partial\nu$ is not known (and cannot be prescribed arbitrarily). We must modify this formula to remove the boundary integral term involving $\partial u/\partial\nu$.

Given $x \in \Omega$, we assume that there exists a **corrector function** $h = h^x \in C^2(\bar{\Omega})$ solving the special Dirichlet problem:

$$\begin{cases} \Delta_y h^x(y) = 0 & (y \in \Omega), \\ h^x(y) = \Phi(y - x) & (y \in \partial\Omega). \end{cases} \tag{3.14}$$

Definition 3.3. We define **Green's function** for domain Ω to be the function

$$G(x, y) = \Phi(y, x) - h^x(y) \quad (x \in \Omega,\ y \in \bar{\Omega},\ x \neq y).$$

Then $G(x, y) = 0$ for $y \in \partial\Omega$ and $x \in \Omega$; hence, by (3.13),

$$u(x) = -\int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu_y}(x, y) dS - \int_{\Omega} G(x, y) \Delta u(y)\, dy. \tag{3.15}$$

The function

$$K(x, y) = -\frac{\partial G}{\partial \nu_y}(x, y) \quad (x \in \Omega,\ y \in \partial\Omega)$$

is called **Poisson's kernel** for domain Ω. Given a function g on $\partial\Omega$, the function

$$K[g](x) = \int_{\partial\Omega} K(x, y) g(y) dS_y \quad (x \in \Omega)$$

is called the **Poisson integral of** g **with kernel** K.

Remark 3.3. A corrector function h^x, if exists for bounded domain Ω, must be unique. Here we require the corrector function h^x exist in $C^2(\bar{\Omega})$, which may not be possible for general bounded domains Ω. However, for bounded domains Ω with smooth boundary, existence of h^x in $C^2(\bar{\Omega})$ is guaranteed by the general existence and regularity theory and consequently for such domains the Green's function always exists and is unique; we do not discuss these issues in this course.

Theorem 3.6 (Representation by Green's function)**.** *If $u \in C^2(\bar{\Omega})$ solves the Dirichlet problem*

$$\begin{cases} -\Delta u(x) = f(x) & (x \in \Omega), \\ u(x) = g(x) & (x \in \partial\Omega), \end{cases}$$

then

$$u(x) = \int_{\partial\Omega} K(x, y) g(y) dS + \int_{\Omega} G(x, y) f(y)\, dy \quad (x \in \Omega). \tag{3.16}$$

Theorem 3.7 (Symmetry of Green's function)**.** *$G(x, y) = G(y, x)$ for all $x, y \in \Omega,\ x \neq y$.*

Proof. Fix $x, y \in \Omega$, $x \neq y$. Let

$$v(z) = G(x, z), \quad w(z) = G(y, z) \qquad (z \in U).$$

Then $\Delta v(z) = 0$ for $z \neq x$ and $\Delta w(z) = 0$ for $z \neq y$ and $v|_{\partial\Omega} = w|_{\partial\Omega} = 0$. For sufficiently small $\epsilon > 0$, we apply Green's second identity on $\Omega_\epsilon = \Omega \setminus (B(x, \epsilon) \cup B(y, \epsilon))$ for functions $v(z)$ and $w(z)$ to obtain

$$\int_{\partial\Omega_\epsilon} \left(v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z) \right) dS = 0.$$

This implies

$$\int_{\partial B(x,\epsilon)} \left(v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z) \right) dS = \int_{\partial B(y,\epsilon)} \left(w(z) \frac{\partial v}{\partial \nu}(z) - v(z) \frac{\partial w}{\partial \nu}(z) \right) dS, \tag{3.17}$$

where ν denotes the *inward* unit normal vector on $\partial B(x, \epsilon) \cup \partial B(x, \epsilon)$.

We compute the limits of two terms on both sides of (3.17) as $\epsilon \to 0^+$. For the term on LHS, since $w(z)$ is smooth near $z = x$,

$$\left| \int_{\partial B(x,\epsilon)} v(z) \frac{\partial w}{\partial \nu}(z) dS \right| \le C\epsilon^{n-1} \sup_{z\in\partial B(x,\epsilon)} |v(z)| = o(1).$$

Also, $v(z) = \Phi(x - z) - h^x(z) = \Phi(z - x) - h^x(z)$, where the corrector h^x is smooth in Ω. Hence

$$\lim_{\epsilon\to 0^+} \int_{\partial B(x,\epsilon)} w(z) \frac{\partial v}{\partial \nu}(z) dS = \lim_{\epsilon\to 0^+} \int_{\partial B(x,\epsilon)} w(z) \frac{\partial \Phi}{\partial \nu}(z - x) dS = w(x).$$

So

$$\lim_{\epsilon\to 0^+} \text{LHS of (3.17)} = -w(x).$$

Similarly,

$$\lim_{\epsilon\to 0^+} \text{RHS of (3.17)} = -v(y),$$

proving $w(x) = v(y)$, which exactly shows that $G(y, x) = G(x, y)$. □

Remark 3.4. (1) Strong Maximum Principle below implies that $G(x, y) > 0$ for all $x, y \in \Omega$, $x \neq y$. (**Homework!**) Since $G(x, y) = 0$ for $y \in \partial\Omega$, it follows that $\frac{\partial G}{\partial \nu_y}(x, y) \le 0$, where ν_y is outer unit normal of Ω at $y \in \partial\Omega$. (In fact, we have $\frac{\partial G}{\partial \nu_y}(x, y) < 0$ for all $x \in \Omega$ and $y \in \partial\Omega$.)

(2) Since $G(x, y)$ is harmonic in $y \in \Omega \setminus \{x\}$, by the symmetry property, we know that $G(x, y)$ is also harmonic in $x \in \Omega \setminus \{y\}$. In particular, $G(x, y)$ is harmonic in $x \in \Omega$ for all $y \in \partial\Omega$; hence Poisson's kernel $K(x, y) = -\frac{\partial G}{\partial \nu_y}(x, y)$ is harmonic in $x \in \Omega$ for all $y \in \partial\Omega$.

(3) We always have that $K(x, y) \ge 0$ for all $x \in \Omega$ and $y \in \partial\Omega$ and that, by Green's representation theorem,

$$\int_{\partial\Omega} K(x, y)\, dS_y = 1 \quad (x \in \Omega).$$

The following general theorem implies that the Poisson integral gives a solution to the Dirichlet problem to Laplace's equation.

Theorem 3.8. *Let Ω be an open set in $\mathbb{R}^n$. Assume a function $K : \Omega \times \partial\Omega \to \mathbb{R}$ satisfies*

(i) $K(x, y) \ge 0$ *for all* $x \in \Omega$ *and* $y \in \partial\Omega$;

(ii) $K(\cdot, y)$ *is harmonic in* Ω *for each* $y \in \partial\Omega$;

(iii) $D^\alpha_x K(x, \cdot) \in L^1(\partial\Omega)$ *for all* $x \in \Omega$ *and multi-indexes* α *with* $|\alpha| \le 2$;

(iv) $\int_{\partial\Omega} K(x, y)\, dS_y = 1$ *for all* $x \in \Omega$;

(v) *for each* $x^0 \in \partial\Omega$ *and* $\delta > 0$,

$$\lim_{x\to x^0,\, x\in\Omega} \int_{\partial\Omega\setminus B(x^0,\delta)} K(x, y)\, dS_y = 0.$$

Let $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ *and define*

$$u(x) = \int_{\partial\Omega} K(x, y) g(y)\, dS_y \quad (x \in \Omega).$$

Then u *is harmonic in* Ω *and satisfies*

$$\lim_{x\to x^0,\, x\in\Omega} u(x) = g(x^0) \quad (x^0 \in \partial\Omega). \tag{3.18}$$

Proof. That u is harmonic in Ω follows easily from (ii) and (iii). Let $M = \|g\|_{L^\infty}$. For $\varepsilon > 0$, let $\delta > 0$ be such that
$$|g(y) - g(x^0)| < \varepsilon \quad \forall\, y \in \partial\Omega,\ \ |y - x^0| < \delta.$$
Then, by (i) and (iv),
$$\begin{aligned}
|u(x) - g(x^0)| &= \left| \int_{\partial\Omega} K(x,y)(g(y) - g(x^0))\, dS_y \right| \le \int_{\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y \\
&\le \int_{B(x^0,\delta)\cap\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y + \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\, dS_y \\
&\le \int_{B(x^0,\delta)\cap\partial\Omega} K(x,y)|g(y) - g(x^0)|\, dS_y + 2M \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y \\
&\le \varepsilon + 2M \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y.
\end{aligned}$$
Hence, by (v),
$$\limsup_{x\to x^0,\, x\in\Omega} |u(x) - g(x^0)| \le \varepsilon + 2M \limsup_{x\to x^0,\, x\in\Omega} \int_{\partial\Omega\setminus B(x^0,\delta)} K(x,y)\, dS_y = \varepsilon.$$
This proves (3.18) and completes the proof. □

3.2.3. Green's functions for half spaces and balls. Although Green's function is defined above for a bounded domain with smooth boundary, it can be similarly defined for unbounded domains or domains with nonsmooth boundaries; however, the representation formula may not be valid for such domains. Green's functions for certain special domains Ω can be explicitly found from the fundamental solution $\Phi(x)$.

Case 1. Green's function for a half-space. Let
$$\Omega = \mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, then its **reflection** with respect to the hyper-plane $x_n = 0$ is defined to be the point
$$\hat{x} = (x_1, x_2, \cdots, x_{n-1}, -x_n).$$
Clearly $\hat{\hat{x}} = x\,(x \in \mathbb{R}^n)$, $\hat{x} = x\,(x \in \partial\mathbb{R}^n_+)$ and $\Phi(x) = \Phi(\hat{x})\,(x \in \mathbb{R}^n)$. In this case, we can easily see that the corrector can be chosen as $h^x(y) = \Phi(y - \hat{x})$.

Definition 3.4. Green's function for half-space $\mathbb{R}^n_+$ is defined by
$$G(x,y) = \Phi(y - x) - \Phi(y - \hat{x}) \quad (x \in \mathbb{R}^n_+,\ y \in \bar{\mathbb{R}}^n_+,\ x \ne y).$$

Note that
$$\begin{aligned}
\frac{\partial G}{\partial y_n}(x,y) &= \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \hat{x}) \\
&= \frac{-1}{n\alpha_n}\left[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \hat{x}|^n} \right].
\end{aligned}$$
So, the corresponding **Poisson's kernel of half-space** $\mathbb{R}^n_+$ is given by
$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) = \frac{\partial G}{\partial y_n}(x,y) = \frac{2x_n}{n\alpha_n |x - y|^n} \quad (x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+).$$
If, for $y \in \partial\mathbb{R}^n_+$, we write $y = (y', 0)$ with $y' \in \mathbb{R}^{n-1}$, then
$$K(x,y) = \frac{2x_n}{n\alpha_n (|x' - y'|^2 + x_n^2)^{n/2}} := H(x,y')$$

and the Poisson integral $u = K[g]$ of a function $g \in C(\partial\mathbb{R}^n_+)$ can be written as

$$u(x) = \int_{\mathbb{R}^{n-1}} H(x,y')g(y')dy' = \frac{2x_n}{n\alpha_n}\int_{\mathbb{R}^{n-1}} \frac{g(y')\,dy'}{(|x'-y'|^2+x_n^2)^{n/2}} \quad (x \in \mathbb{R}^n_+). \tag{3.19}$$

Theorem 3.9 (Poisson's formula for half-space). *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and $u = K[g]$ is defined by (3.19). Then,*

(i) *$u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ is harmonic in $\mathbb{R}^n_+$,*

(ii) *for all $x^0 \in \partial\mathbb{R}^n_+$,*

$$\lim_{x\to x^0,\, x\in\mathbb{R}^n_+} u(x) = g(x^0).$$

Proof. It is easily verified that $D^\alpha_x H(x,y') \in L^1(\mathbb{R}^{n-1}_{y'})$ for all $x \in \mathbb{R}^n_+$ and multi-indexes α and that $H(x,y')$ is harmonic in $x \in \mathbb{R}^n_+$ for each $y' \in \mathbb{R}^{n-1}$; also, a complicated computation shows that

$$\int_{\mathbb{R}^{n-1}} H(x,y')\,dy' = 1 \quad (x \in \mathbb{R}^n_+).$$

Hence the conclusion (i) follows easily. Conclusion (ii) will follow from the general theorem Theorem 3.8 if we verify the condition (v) there. So let $x^0 \in \partial\mathbb{R}^n_+$ and $\delta > 0$. Then , if $|x-x^0| < \delta/2$ and $|y'-x^0| \geq \delta$, then we have

$$|y'-x^0| \leq |y'-x| + \delta/2 \leq |y'-x| + \frac{1}{2}|y'-x^0|;$$

so $|y'-x| \geq \frac{1}{2}|y'-x^0|$ and hence $H(x,y') \leq \frac{2^{n+1}x_n}{n\alpha_n}|y'-x^0|^{-n}$. Therefore

$$\int_{\mathbb{R}^{n-1}\setminus B(x^0,\delta)} H(x,y')\,dy' \leq \frac{2^{n+1}x_n}{n\alpha_n}\int_{\mathbb{R}^{n-1}\setminus B(x^0,\delta)} |y'-x^0|^{-n}\,dy'$$

$$= \frac{2^{n+1}x_n}{n\alpha_n}\left((n-1)\alpha_{n-1}\int_\delta^\infty r^{-n}r^{n-2}\,dr\right) = 2^{n+1}\frac{(n-1)\alpha_{n-1}}{n\alpha_n\delta}\,x_n \to 0,$$

as $x_n \to 0^+$ if $x \to x^0$ in $\mathbb{R}^n_+$, which proves (v) of Theorem 3.8. ☐

Case 2. Green's function for a ball. Let

$$\Omega = B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

be the unit ball in $\mathbb{R}^n$. If $x \in \mathbb{R}^n \setminus \{0\}$, the point

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the **inversion point** of x with respect to unit sphere $\partial B(0,1)$. The mapping $x \mapsto \tilde{x}$ is called the **inversion with respect to unit sphere**.

Given $x \in B(0,1)$, $x \neq 0$, we try to find the corrector $h^x(y)$ in the form of

$$h^x(y) = \Phi(b(x)(y-\tilde{x})).$$

For this we need to have

$$|b(x)||y-\tilde{x}| = |y-x| \quad (y \in \partial B(0,1)).$$

Note that, if $y \in \partial B(0,1)$ then $|y| = 1$ and

$$|y-\tilde{x}|^2 = 1 - 2y\cdot\tilde{x} + |\tilde{x}|^2 = 1 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2} = \frac{|y-x|^2}{|x|^2}.$$

So we can choose $b(x) = |x|$. Consequently, for $x \neq 0$, the corrector h^x is given by

$$h^x(y) = \Phi(|x|(y - \tilde{x})) \quad (y \in \bar{B}(0,1)).$$

Definition 3.5. Green's function for unit ball $B(0,1)$ is given by

$$G(x,y) = \begin{cases} \Phi(y-x) - \Phi(|x|(y-\tilde{x})) & (x \neq 0, \;\; x \neq y), \\ G(y,0) = \Phi(y) - \Phi(|y|\tilde{y}) = \Phi(y) - \phi(1) & (x = 0, \;\; y \neq 0). \end{cases}$$

(Note that $G(0,y)$ cannot be given by the first formula since $\tilde{0}$ is undefined, but it is found from the symmetry of G: $G(0,y) = G(y,0)$ for $y \neq 0$.)

Since $\Phi_{y_i}(y) = \phi'(|y|)\frac{y_i}{|y|} = \frac{-y_i}{n\alpha_n |y|^n}$ ($y \neq 0$ and $n \geq 2$), we deduce that, if $x \neq 0$, $x \neq y$, then

$$\begin{aligned} G_{y_i}(x,y) &= \Phi_{y_i}(y-x) - \Phi_{y_i}(|x|(y-\tilde{x}))|x| \\ &= \frac{1}{n\alpha_n}\left[\frac{x_i - y_i}{|y-x|^n} - \frac{|x|^2((\tilde{x})_i - y_i)}{(|x||y-\tilde{x}|)^n}\right] \\ &= \frac{1}{n\alpha_n}\left[\frac{x_i - y_i}{|y-x|^n} - \frac{x_i - |x|^2 y_i}{(|x||y-\tilde{x}|)^n}\right] \end{aligned}$$

So, if $y \in \partial B(0,1)$, since $|x||y-\tilde{x}| = |y-x|$ and $\nu_y = y$, we have

$$\begin{aligned} \frac{\partial G}{\partial \nu_y}(x,y) = \sum_{i=1}^n G_{y_i}(x,y) y_i &= \frac{1}{n\alpha_n}\sum_{i=1}^n \left[\frac{x_i y_i - y_i^2}{|y-x|^n} - \frac{x_i y_i - |x|^2 y_i^2}{(|x||y-\tilde{x}|)^n}\right] \\ &= \frac{1}{n\alpha_n}\frac{|x|^2 - 1}{|y-x|^n} \quad (x \in B(0,1) \setminus \{0\}). \end{aligned}$$

The same formula also holds for $x = 0$ and $y \in \partial B(0,1)$.

Therefore, the **Poisson's kernel for unit ball** $B(0,1)$ is given by

$$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y) = \frac{1 - |x|^2}{n\alpha_n |y-x|^n} \quad (x \in B(0,1), \;\; y \in \partial B(0,1)).$$

Given $g \in C(\partial B(0,1))$, its Poisson integral $u = K[g]$ is given by

$$u(x) = \int_{\partial B(0,1)} K(x,y) g(y)\, dS_y = \frac{1 - |x|^2}{n\alpha_n}\int_{\partial B(0,1)} \frac{g(y)\, dS_y}{|y-x|^n} \quad (x \in B(0,1)). \tag{3.20}$$

By Green's representation formula, the $C^2(\bar{B}(0,1))$-solution u of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1), \\ u = g & \text{on } \partial B(0,1), \end{cases}$$

is given by the formula (3.20).

Suppose u is the C^2-solution to the Dirichlet problem on a closed ball $\bar{B}(a,R)$, of center a and radius R:

$$\begin{cases} \Delta u = 0 & \text{in } B(a,R), \\ u = g & \text{on } \partial B(a,R). \end{cases}$$

Let $\tilde{u}(x) = u(a + Rx)$ and $\tilde{g}(x) = g(a + Rx)$ for $x \in \bar{B}(0,1)$. Then $\tilde{u}$ solves the Dirichlet problem on unit ball $B(0,1)$ with boundary data $\tilde{g}$. By formula (3.20) with $\tilde{g}$ we have, for all $x \in B(a,R)$,

$$u(x) = \tilde{u}(\frac{x-a}{R}) = \frac{1 - |\frac{x-a}{R}|^2}{n\alpha_n}\int_{\partial B(0,1)} \frac{g(a+Ry)\, dS_y}{|y - \frac{x-a}{R}|^n}$$

$$= \frac{R^2 - |x-a|^2}{n\alpha_n R^2} \int_{\partial B(a,R)} \frac{g(z) R^{1-n}\, dS_y}{R^{-n}|z-x|^n} \quad (z = a + Ry).$$

Hence, changing z back to y,

$$u(x) = \frac{R^2 - |x-a|^2}{n\alpha_n R} \int_{\partial B(a,R)} \frac{g(y)\, dS_y}{|y-x|^n} = \int_{\partial B(a,R)} K(x,y;a,R) g(y)\, dS_y, \tag{3.21}$$

where

$$K(x,y;a,R) = \frac{R^2 - |x-a|^2}{n\alpha_n R |y-x|^n} \quad (x \in B(a,R), \ \ y \in \partial B(a,R))$$

is the **Poisson's kernel for general ball** $B(a,R)$.

The formula (3.21) is called the **Poisson's formula on ball** $B(a,R)$. This formula has a special consequence if we take $x = a$, which gives

$$u(a) = \frac{R}{n\alpha_n} \int_{\partial B(a,R)} \frac{g(y)}{|y-a|^n}\, dS_y = ⨍_{\partial B(a,R)} g(y)\, dS_y.$$

(Note that $|\partial B(a,R)| = n\alpha_n R^{n-1}$.) Therefore, if u is harmonic in a domain Ω and $B(a,r) \subset\subset \Omega$ (this means $\bar{B}(a,r) \subset \Omega$), then

$$u(a) = ⨍_{\partial B(a,r)} u(y)\, dS_y. \tag{3.22}$$

This is the **mean-value property** for harmonic functions; we will give another proof in the next section.

Theorem 3.10 (Poisson's formula for a ball)**.** *Assume $g \in C(\partial B(a,R))$ and $u = K[g]$ is defined by* (3.21)*. Then,*

(i) *$u \in C^\infty(B(a,R))$ is harmonic in $B(a,R)$;*

(ii) *for each $x^0 \in \partial B(a,R)$,*

$$\lim_{x \to x^0,\, x \in B(a,R)} u(x) = g(x^0).$$

Proof. The result (i) follows since the Poisson kernel $K(x,y;a,R)$ is harmonic and C^∞ on x in $B(a,R)$ for all $y \in \partial B(a,R)$, while the result (ii) follows from Theorem 3.8 because, for each $x^0 \in \partial B(a,R)$ and $\delta > 0$,

$$\lim_{x \to x^0,\, x \in B(a,R)} \int_{\partial B(a,R) \setminus B(x^0,\delta)} K(x,y;a,R)\, dS_y = \int_{\partial B(a,R) \setminus B(x^0,\delta)} K(x^0,y;a,R)\, dS_y = 0.$$

□

3.3. Mean-value property

For two sets U and V in $\mathbb{R}^n$ we write $V \subset\subset U$ if $\bar{V}$ is a *compact subset* of U.

Theorem 3.11 (Mean-value property for harmonic functions)**.** *Let $u \in C^2(\Omega)$ be harmonic. Then*

$$u(x) = ⨍_{\partial B(x,r)} u(y) dS = ⨍_{B(x,r)} u(y)\, dy$$

for each ball $B(x,r) \subset\subset \Omega$.

Proof. The first equality (called the **spherical mean-value property**) has been proved above. We give a different proof. Let $B(x,r) \subset\subset \Omega$. For each $\rho \in (0,r]$, let

$$h(\rho) = ⨍_{\partial B(x,\rho)} u(y)\, dS_y = ⨍_{\partial B(0,1)} u(x+\rho z)\, dS_z.$$

Then, by Green's first identity,

$$\begin{aligned} h'(\rho) &= ⨍_{\partial B(0,1)} \nabla u(x+\rho z)\cdot z\, dS_z = ⨍_{\partial B(x,\rho)} \nabla u(y)\cdot \frac{y-x}{\rho}\, dS_y \\ &= ⨍_{\partial B(x,\rho)} \nabla u(y)\cdot \nu_y\, dS_y = ⨍_{\partial B(x,\rho)} \frac{\partial u(y)}{\partial \nu_y}\, dS_y \\ &= \frac{1}{n\alpha_n \rho^{n-1}} \int_{\partial B(x,\rho)} \frac{\partial u(y)}{\partial \nu_y}\, dS_y = \frac{1}{n\alpha_n\rho^{n-1}} \int_{B(x,\rho)} \Delta u(y)\, dy \\ &= \frac{\rho}{n} ⨍_{B(x,\rho)} \Delta u(y)\, dy. \end{aligned}$$

Hence, since $\Delta u = 0$ in Ω, it follows that $h'(\rho) = 0$ on $\rho \in (0,r]$ and so h is constant on $(0,r]$; hence,

$$h(r) = h(0^+) = \lim_{\rho\to 0^+} ⨍_{\partial B(x,\rho)} u(y)\, dS_y = u(x),$$

which proves the spherical mean-value property. From this, we have

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,\rho)} u(y)\, dS_y \right) d\rho = u(x) \int_0^r n\alpha_n \rho^{n-1}\, d\rho = u(x)\alpha_n r^n,$$

which proves the **ball mean-value property**: $u(x) = ⨍_{B(x,r)} u(y)\, dy$. □

Theorem 3.12 (Converse to mean-value property)**.** *Let $u \in C^2(\Omega)$ satisfy*

$$u(x) = ⨍_{\partial B(x,r)} u(y)\, dS_y$$

for all $x \in \Omega$ and $0 < r < r_x \le \operatorname{dist}(x,\partial\Omega)$, where $r_x > 0$ is a number depending on x. Then u is harmonic in Ω.

Proof. Suppose $\Delta u(x_0) \neq 0$ for some $x_0 \in \Omega$. WLOG, assume $\Delta u(x_0) > 0$. Then there exists a ball $B(x_0,r)$ with $0 < r < r_{x_0}$ such that $\Delta u(y) > 0$ on $\bar{B}(x_0,r)$. Consider the function

$$h(\rho) = ⨍_{\partial B(x_0,\rho)} u(y)\, dS_y \quad (0 < \rho < r_{x_0}).$$

The assumption says that h is constant on $(0, r_{x_0}]$. However, by the computation as above, $h'(r) = \frac{r}{n} ⨍_{B(x_0,r)} \Delta u(y)\, dy > 0$, giving a desired contradiction. □

This result actually holds under a much weaker assumption that u is only continuous.

Theorem 3.13. *Let $u \in C(\Omega)$ satisfy*

$$u(x) = ⨍_{\partial B(x,r)} u(y)\, dS_y$$

for all $x \in \Omega$ and $0 < r < r_x \le \operatorname{dist}(x,\partial\Omega)$, where $r_x > 0$ is a number depending on x. Then u is harmonic in Ω.

Proof. See Lemma 3.26 below. □

3.4. Maximum principles

Theorem 3.14 (Maximum principle for harmonic functions)**.** *Let Ω be bounded open in $\mathbb{R}^n$. Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω.*

(i) (**Weak maximum principle**) *We have that $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.*

(ii) (**Strong maximum principle**) *If, in addition, Ω is connected and there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$, then $u(x) \equiv u(x_0)$ for all $x \in \Omega$.*

Proof. Note that (ii) implies (i) (**Explain why?**) To prove (2), let

$$S = \{x \in \Omega \mid u(x) = u(x_0)\}.$$

This set is nonempty since $x_0 \in S$. It is relatively closed in Ω since u is continuous. We show that S is open; hence $S = \Omega$ since Ω is connected. Let $x \in S$; so $u(x) = u(x_0) = \max_{\bar{\Omega}} u$. Assume $B(x, r) \subset\subset \Omega$. By the ball mean-value property,

$$u(x) = \fint_{B(x,r)} u(y)dy \leq \fint_{B(x,r)} u(x)\, dy = u(x).$$

So the equality holds, which implies $u(y) = u(x)$ for all $y \in B(x, r)$; hence $B(x, r) \subset S$ and thus S is open. □

If u is harmonic then $-u$ is also harmonic; hence the **minimum principles** also hold. In particular, we have the following **positivity property** for harmonic functions:

Corollary 3.15. *Let Ω be connected, bounded and open in $\mathbb{R}^n$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω and $u|_{\partial\Omega} \geq 0$ but $\not\equiv 0$, then $u(x) > 0$ for all $x \in \Omega$.*

From the maximum principle, we easily have the uniqueness of Dirichlet problem.

Theorem 3.16 (Uniqueness for Dirichlet problem)**.** *Let Ω be bounded open in $\mathbb{R}^n$. Then, given f, g, the Dirichlet problem for Poisson's equation*

$$\begin{cases} -\Delta u = f & \textit{in } \Omega, \\ u = g & \textit{on } \partial\Omega \end{cases}$$

can have at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Theorem 3.17 (**C^∞-regularity of harmonic functions**)**.** *If $u \in C^2(\Omega)$ is harmonic, then $u \in C^\infty(\Omega)$.*

Proof. Let $a \in \Omega$ and $B(a, R) \subset\subset \Omega$. Then $g(y) = u(y)$ is continuous on $y \in \partial B(a, R)$. Let $\tilde{u}(x) = \int_{\partial B(a,R)} K(x, y; a, R) g(y)\, dS_y$ for $x \in B(a, R)$ be the Poisson integral of g on $B(a, R)$, and extend $\tilde{u}$ to $\partial B(a, R)$ by defining $\tilde{u}(y) = g(y) = u(y)$ for $y \in \partial B(a, R)$. Then u and $\tilde{u}$ are both solutions to Laplace's equation with the same boundary boundary data $g(y) = u(y)$ in $C^2(B(a, R)) \cap C(\bar{B}(a, R))$. Hence, by the uniqueness theorem above, $u \equiv \tilde{u}$ in $B(a, R)$. However, by Theorem 3.10, $\tilde{u} \in C^\infty(B(a, R))$; this proves that $u \in C^\infty(\Omega)$. □

3.5. Estimates of higher-order derivatives and Liouville's theorem

Theorem 3.18 (Local estimates on derivatives)**.** *Assume u is harmonic in Ω. Then*

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x,r))} \tag{3.23}$$

for each ball $B(x, r) \subset\subset \Omega$, each $k = 1, 2, \cdots$, and each multi-index α of order $|\alpha| = k$.

Proof. We prove by induction on k.

1. If $k=1$, say $\alpha=(1,0,\cdots,0)$ and so $D^\alpha u=u_{x_1}=D_1u$. Since $u\in C^\infty$ and is harmonic, D_1u is also harmonic; hence, if $B(x,r)\subset\subset\Omega$, then

$$D_1u(x)=\fint_{B(x,r/2)}D_1u(y)\,dy=\frac{2^n}{\alpha_nr^n}\int_{B(x,r/2)}D_1u(y)\,dy$$
$$=\frac{2^n}{\alpha_nr^n}\int_{\partial B(x,r/2)}u(y)\nu_1(y)\,dS_y.$$

So

$$|D_1u(x)|\le\frac{2^n}{\alpha_nr^n}\int_{\partial B(x,r/2)}|u(y)|\,dS_y\le\frac{2n}{r}\max_{\partial B(x,r/2)}|u|.$$

However, for each $y\in\partial B(x,r/2)$, one has $B(y,r/2)\subset B(x,r)\subset\subset\Omega$, and hence

$$|u(y)|=\left|\fint_{B(y,r/2)}u(z)dz\right|\le\frac{2^n}{\alpha_nr^n}\|u\|_{L^1(B(x,r))}.$$

Combining these estimates, we have

$$|D_1u(x)|\le\frac{2^{n+1}n}{\alpha_nr^{n+1}}\|u\|_{L^1(B(x,r))}.$$

This proves (3.23) when $k=1$ with constant $C_1=\frac{2^{n+1}n}{\alpha_n}$.

2. Assume now $k\ge2$ and let $|\alpha|=k$. Then for some i we have $\alpha=\beta+(0,\cdots,1,0,\cdots,0)$, where $|\beta|=k-1$ and 1 is in the i-th place. So $D^\alpha u=(D^\beta u)_{x_i}$ and hence, as in Step 1,

$$|D^\alpha u(x)|\le\frac{nk}{r}\|D^\beta u\|_{L^\infty(B(x,r/k))}.$$

If $y\in B(x,r/k)$ then $B(y,\frac{k-1}{k}r)\subset B(x,r)$; hence, by the induction assumption for $D^\beta u$ at y, we have

$$|D^\beta u(y)|\le\frac{C_{k-1}}{(\frac{k-1}{k}r)^{n+k-1}}\|u\|_{L^1(B(y,\frac{k-1}{k}r))}\le\frac{C_{k-1}(\frac{k}{k-1})^{n+k-1}}{r^{n+k-1}}\|u\|_{L^1(B(x,r))}.$$

Combining the previous estimates we derive that

$$|D^\alpha u(x)|\le\frac{C_k}{r^{n+k}}\|u\|_{L^1(B(x,r))},$$

where $C_k\ge C_{k-1}nk(\frac{k}{k-1})^{n+k-1}$. For example, we can choose

$$C_k=\frac{(2^{n+1}nk)^k}{\alpha_n}\quad\forall\;k=1,2,\cdots.$$

$\square$

Theorem 3.19 (Liouville's Theorem)**.** *Suppose u is a bounded harmonic function on whole $\mathbb{R}^n$. Then u is constant.*

Proof. Let $|u(y)|\le M$ on $y\in\mathbb{R}^n$. By (3.23) with $k=1$, for each $i=1,2,\cdots,n$,

$$|u_{x_i}(x)|\le\frac{C_1}{r^{n+1}}\|u\|_{L^1(B(x,r))}\le\frac{C_1}{r^{n+1}}M\alpha_nr^n=\frac{MC_1\alpha_n}{r}.$$

This inequality holds for all $r>0$ since $B(x,r)\subset\subset\mathbb{R}^n$. Taking $r\to\infty$ gives $u_{x_i}(x)=0$ for all $x\in\mathbb{R}^n$ and all $i=1,2,\cdots,n$. Hence $\nabla u\equiv0$ and so u is a constant on $\mathbb{R}^n$. $\square$

Theorem 3.20 (Representation formula). *Let $n \geq 3$ and $f \in C_c^\infty(\mathbb{R}^n)$. Then any bounded solution of $-\Delta u = f$ on $\mathbb{R}^n$ has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C$$

for some constant C.

Proof. Since $n \geq 3$ and thus $\Phi(y) \to 0$ as $|y| \to \infty$, it follows that the **Newton potential**

$$\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad (x \in \mathbb{R}^n)$$

is bounded on $\mathbb{R}^n$; indeed, if $\operatorname{supp} f \subset B(0,R)$ then, for all $|x| \geq R+1$ and $|y| \leq R$, we have $|x-y| \geq |x| - |y| > 1$ and hence $\Phi(x-y) \leq \frac{1}{n(n-2)\alpha_n}$; so

$$|\tilde{u}(x)| \leq \int_{B(0,R)} \Phi(x-y)|f(y)|\, dy \leq \frac{R^n}{n(n-2)} \|f\|_{L^\infty},$$

and thus $\tilde{u}$ is bounded on $\mathbb{R}^n$. This $\tilde{u}$ solves the Poisson equation $-\Delta \tilde{u} = f$ and hence $u - \tilde{u}$ is bounded harmonic on $\mathbb{R}^n$. By Liouville's theorem, $u = \tilde{u} + C$ for a constant C. $\square$

Remark 3.5. If $n = 2$, the representation formula may not hold; for example, if $\int_{\mathbb{R}^n} f(y)\, dy \neq 0$ then, as $|x| \to \infty$,

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \sim \Phi(x) \int_{\mathbb{R}^n} f(y)\, dy$$

is not bounded.

Theorem 3.21 (Compactness of sequence of harmonic functions). *Suppose $\{u^j\}$ is a sequence of harmonic functions in Ω and*

$$|u^j(x)| \leq M \quad (x \in \Omega, \ \ j = 1, 2, \cdots).$$

Let $V \subset\subset \Omega$. Then there exists a subsequence $\{u^{j_k}\}$ and a harmonic function $\bar{u}$ in V such that

$$\lim_{k \to \infty} \|u^{j_k} - \bar{u}\|_{L^\infty(V)} = 0.$$

Proof. Let $0 < r < \operatorname{dist}(V, \partial\Omega)$. Then $B(x,r) \subset\subset \Omega$ for all $x \in \bar{V}$. Applying the derivative estimates,

$$|Du^j(x)| \leq \frac{C_1 M}{r} \quad \forall\, x \in \bar{V}, \ \ j = 1, 2, \cdots.$$

Consequently, by **Arzela-Ascoli's theorem**, the family $\{u^j\}$ is uniformly bounded and equi-continuous on $\bar{V}$, and hence there exists a subsequence $\{u^{j_k}\}$ converging uniformly in $\bar{V}$ to a continuous function $\bar{u}$ on $\bar{V}$. This uniform limit $\bar{u}$ certainly satisfies the mean-value property in V and hence must be harmonic in V. $\square$

Theorem 3.22 (Harnack's Inequality). *For each subdomain $V \subset\subset \Omega$, there exists a constant $C = C(V, \Omega)$ such that the inequality*

$$\sup_V u \leq C \inf_V u$$

holds for all nonnegative harmonic *functions u in Ω.*

Proof. Let $r = \frac{1}{4}\operatorname{dist}(V, \partial\Omega)$. Let $x, y \in V$ with $|x-y| \le r$. Then $B(y,r) \subset B(x,2r) \subset\subset \Omega$; hence, for all nonnegative harmonic functions u in Ω,

$$u(x) = \frac{1}{\alpha_n (2r)^n}\int_{B(x,2r)} u(z)\,dz \ge \frac{1}{\alpha_n 2^n r^n}\int_{B(y,r)} u(z)\,dz = \frac{1}{2^n}⨍_{B(y,r)} u(z)dz = \frac{1}{2^n}u(y).$$

Therefore, $\frac{1}{2^n}u(y) \le u(x) \le 2^n u(y)$ for all $x, y \in V$ with $|x-y| \le r$.

Since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 1, 2, \cdots, N$. Then it follows that

$$u(x) \ge \frac{1}{2^{n(N+1)}}u(y) \quad \forall\, x,\ y \in V.$$

This completes the proof. $\square$

3.6. Perron's method for Dirichlet problem of Laplace's equation

(*This material is not covered in Evans's book, but can be found in other textbooks, e.g., John's book mentioned in the syllabus.*)

Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C(\partial\Omega)$. We now discuss **Perron's method** of **subharmonic functions** to solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Definition 3.6. We say a function $u \in C(\Omega)$ is **subharmonic** in Ω if for every $\xi \in \Omega$ the inequality

$$u(\xi) \le ⨍_{\partial B(\xi,\rho)} u(x)\,dS := M_u(\xi,\rho)$$

holds for all sufficiently small $\rho > 0$.

We denote by $\sigma(\Omega)$ the set of all subharmonic functions in Ω.

Lemma 3.23. *For $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$,*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. Homework. $\square$

Definition 3.7. For any $u \in C(\Omega)$ and $B(\xi,\rho) \subset\subset \Omega$, we define the **harmonic lifting** of u on $B(\xi,\rho)$ to be the function

$$u_{\xi,\rho}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus B(\xi,\rho), \\ \frac{\rho^2 - |x-\xi|^2}{n\alpha_n\rho}\int_{\partial B(\xi,\rho)} \frac{u(y)}{|y-x|^n}\,dS_y & \text{if } x \in B(\xi,\rho). \end{cases} \tag{3.24}$$

Note that in $B(\xi,\rho)$ the function $u_{\xi,\rho}$ is simply the Poisson integral of $u|_{\partial B(\xi,\rho)}$ and hence is harmonic in $B(\xi,\rho)$ and takes the boundary value $u|_{\partial B(\xi,\rho)}$ on $\partial B(\xi,\rho)$; therefore, $u_{\xi,\rho}$ is in $C(\Omega)$.

Lemma 3.24. *For $u \in \sigma(\Omega)$ and $B(\xi,\rho) \subset\subset \Omega$, we have $u_{\xi,\rho} \in \sigma(\Omega)$ and*

$$u(x) \le u_{\xi,\rho}(x) \quad \forall\, x \in \Omega. \tag{3.25}$$

Proof. We first prove (3.25). If $x \notin B(\xi,\rho)$ then $u(x) = u_{\xi,\rho}(x)$. Note that $u - u_{\xi,\rho}$ is subharmonic in $B(\xi,\rho)$ and equals zero on $\partial B(\xi,\rho)$; hence, by Lemma 3.23, $u - u_{\xi,\rho} \le 0$ in $B(\xi,\rho)$; this proves (3.25). We now prove that $u_{\xi,\rho}$ is subharmonic in Ω. We must show that for any $x \in \Omega$

$$u_{\xi,\rho}(x) \le \fint_{\partial B(x,r)} u_{\xi,\rho}(y)\, dS_y = M_{u_{\xi,\rho}}(x,r) \tag{3.26}$$

for all sufficiently small $r > 0$. We first assume $x \notin \partial B(\xi,\rho)$; then there exists a ball $B(x,r') \subset\subset \Omega$ such that either $B(x,r') \subset \Omega \setminus \bar{B}(\xi,\rho)$ or $B(x,r') \subset B(\xi,\rho)$; hence, either $u_{\xi,\rho}(y) = u(y)$ for all $y \in B(x,r')$ or $u_{\xi,\rho}(y)$ is harmonic in $y \in B(x,r')$. In the first case, (3.26) holds if $0 < r < r'$ is sufficiently small since u is subharmonic, while in the second case, (3.26) holds if $0 < r < r'$ is sufficiently small since $u_{\xi,\rho}$ is harmonic. We now prove (3.26) if $x \in \partial B(\xi,\rho)$. In this case, by (3.25),

$$u_{\xi,\rho}(x) = u(x) \le M_u(x,r) \le M_{u_{\xi,\rho}}(x,r)$$

for all sufficiently small $r > 0$. □

Lemma 3.25. *For $u \in \sigma(\Omega)$, we have $u(\xi) \le M_u(\xi,\rho)$ whenever $B(\xi,\rho) \subset\subset \Omega$.*

Proof. Let $B(\xi,\rho) \subset\subset \Omega$. Then

$$u(\xi) \le u_{\xi,\rho}(\xi) = M_{u_{\xi,\rho}}(\xi,\rho) = M_u(\xi,\rho).$$

□

Lemma 3.26. *Let $u \in C(\Omega)$. Then u is harmonic in Ω if and only if $\pm u \in \sigma(\Omega)$.*

Proof. Suppose $u, -u \in \sigma(\Omega)$. Then for all balls $B(\xi,\rho) \subset\subset \Omega$, by Lemma 3.24,

$$u \le u_{\xi,\rho}, \quad -u \le (-u)_{\xi,\rho} = -u_{\xi,\rho}.$$

This implies $u \equiv u_{\xi,\rho}$; hence u is harmonic in $B(\xi,\rho)$. Consequently u is harmonic in Ω. □

Let $g \in C(\partial\Omega)$. Define $\sigma_g(\Omega) = \{u \in C(\bar{\Omega}) \cap \sigma(\Omega) \mid u \le g \text{ on } \partial\Omega\}$, and

$$w_g(x) = \sup_{u \in \sigma_g(\Omega)} u(x) \quad (x \in \Omega). \tag{3.27}$$

Suppose $m = \min_{\partial\Omega} g$ and $M = \max_{\partial\Omega} g$. Then m, M are finite numbers and $m \in \sigma_g(\Omega)$; so $\sigma_g(\Omega)$ is nonempty. Also, by Lemma 3.23,

$$u(x) \le M \quad \forall\, u \in \sigma_g(\Omega), \;\; x \in \Omega.$$

Hence the function w_g is well-defined in Ω.

Lemma 3.27. *Let $v_1, v_2, \cdots, v_k \in \sigma_g(\Omega)$ and $v = \max\{v_1, v_2, \cdots, v_k\}$. Then $v \in \sigma_g(\Omega)$.*

Proof. Homework. □

Theorem 3.28. *The function w_g defined by (3.27) is harmonic in Ω.*

Proof. Let $\xi \in \Omega$, $B(\xi,\rho) \subset\subset \Omega$ and $0 < \rho' < \rho$ be given.

1. Assume $\{x^k\}_{k=1}^\infty$ is any sequence of points in $B(\xi,\rho')$. For each x^k, let $\{u_k^j\}_{j=1}^\infty$ be a sequence in $\sigma_g(\Omega)$ such that

$$w_g(x^k) = \lim_{j\to\infty} u_k^j(x^k) \quad (k = 1, 2, \cdots).$$

Define $u^j(x) = \max\{m, u_1^j(x), u_2^j(x), \cdots, u_j^j(x)\}$ for $x \in \bar{\Omega}$. Then $u^j \in \sigma_g(\Omega)$, $m \le u^j(x) \le w_g(x)$ and

$$\lim_{j\to\infty} u^j(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots).$$

Let $v^j = u^j_{\xi,\rho}$ be the harmonic lifting of u^j on $B(\xi, \rho)$. Then $v^j \in \sigma_g(\Omega)$ and $u^j(x) \le v^j(x) \le w_g(x)$ in Ω, and so

$$\lim_{j\to\infty} v^j(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots).$$

Since $\{v^j\}$ is a bounded sequence of harmonic functions in $B(\xi, \rho)$, by the **compactness theorem** (Theorem 3.21), there exists a subsequence $\{v^{j_m}\}$ uniformly converging to a harmonic function W on $B(\xi, \rho')$. Hence

$$W(x^k) = w_g(x^k) \quad (k = 1, 2, \cdots). \tag{3.28}$$

Note that the harmonic function W depends on the choice of points $\{x^k\}$ and the subsequence $\{v^{j_m}\}$. However the function w_g is independent of all these choices.

2. We first show that w_g is continuous in $B(\xi, \rho')$. Let $y^0 \in B(\xi, \rho')$, and let $\{y^k\}$ be any sequence in $B(\xi, \rho')$ converging to y^0. Define $x^1 = y^0$ and $x^k = y^k$ for all $k = 2, 3, \cdots$. With this sequence $\{x^k\}$ in $B(\xi, \rho')$ as in Step 1, we have, by (3.28) and the continuity of W, that

$$w_g(y^0) = W(y^0) = \lim_{k\to\infty} W(x^k) = \lim_{k\to\infty} w_g(x^k) = \lim_{k\to\infty} w_g(y^k).$$

This proves the continuity of w_g at $y^0 \in B(\xi, \rho')$.

3. We now prove that w_g is harmonic in $B(\xi, \rho')$. To show this, let $\{x^k\}$ be a **dense sequence** in $B(\xi, \rho')$. Then, by (3.28) and the continuity of w_g, it follows that $w_g \equiv W$ in $B(\xi, \rho')$. Since W is harmonic in $B(\xi, \rho')$, so is w_g in $B(\xi, \rho')$.

Finally, since $B(\xi, \rho')$ can be arbitrary, it follows that w_g is harmonic in whole Ω. $\square$

The harmonic function w_g constructed is the candidate of a solution to our Dirichlet problem. To guarantee this, we need to study the behavior of w_g near the boundary under some specific property of the boundary $\partial\Omega$.

Definition 3.8. Given a boundary point $\eta \in \partial\Omega$, a function Q_η is said to be a **Barrier function** at η if $Q_\eta \in C(\bar{\Omega}) \cap \sigma(\Omega)$ such that

$$Q_\eta(\eta) = 0, \quad Q_\eta(x) < 0 \quad (x \in \bar{\Omega} \setminus \{\eta\}).$$

In this case, we say that the point $\eta \in \partial\Omega$ is **regular** or η is a **regular boundary point** of Ω.

Theorem 3.29. *If $\eta \in \partial\Omega$ is regular, then*

$$\lim_{x\to\eta,\, x\in\Omega} w_g(x) = g(\eta).$$

Proof. 1. We first prove

$$\liminf_{x\to\eta,\, x\in\Omega} w_g(x) \ge g(\eta).$$

Let $\varepsilon > 0$, $K > 0$ be given constants and define $u(x) = g(\eta) - \varepsilon + KQ_\eta(x)$ on $\bar{\Omega}$. Then $u \in C(\bar{\Omega}) \cap \sigma(\Omega)$, $u(\eta) = g(\eta) - \varepsilon$, and $u(x) \le g(\eta) - \varepsilon$ on $\partial\Omega$. Since g is continuous, there exists a $\delta > 0$ such that $g(x) > g(\eta) - \varepsilon$ for all $x \in B(\eta, \delta) \cap \partial\Omega$. Hence $u(x) \le g(x)$ on $B(\eta, \delta) \cap \partial\Omega$. Since $Q_\eta < 0$ on the compact set $\partial\Omega \setminus B(\eta, \delta)$, it follows that $Q_\eta(x) \le -\gamma$ on $\partial\Omega \setminus B(\eta, \delta)$, where $\gamma > 0$ is a number. We now let $K = \frac{M-m}{\gamma} \ge 0$. Then

$$u(x) = g(\eta) - \varepsilon + KQ_\eta(x) \le M - K\gamma = m \le g(x) \quad (x \in \partial\Omega \setminus B(\eta, \delta)).$$

Therefore $u \le g$ on whole $\partial\Omega$. So $u \in \sigma_g(\Omega)$. Consequently, $u(x) \le w_g(x)$ for all $x \in \Omega$; so,

$$g(\eta) - \varepsilon = \lim_{x\to\eta,\, x\in\Omega} u(x) \le \liminf_{x\to\eta,\, x\in\Omega} w_g(x),$$

which completes the first step.

2. We now prove

$$\limsup_{x\to\eta,\, x\in\Omega} w_g(x) \le g(\eta).$$

We consider the function w_{-g} defined similarly with $-g$; namely,

$$w_{-g}(x) = \sup_{v\in\sigma_{-g}(\Omega)} v(x) \quad (x \in \Omega).$$

For each pair of $u \in \sigma_g(\Omega)$ and $v \in \sigma_{-g}(\Omega)$, it follows that $u + v \in \sigma(\Omega) \cap C(\bar\Omega)$ and $u+v \le g+(-g) = 0$ on $\partial\Omega$. Hence, by Lemma 3.23, $u+v \le 0$ on $\bar\Omega$; therefore, $u(x) \le -v(x)$ $(x \in \Omega)$ for all such pairs. Cosequently,

$$w_g(x) = \sup_{u\in\sigma_g(\Omega)} u(x) \le \inf_{v\in\sigma_{-g}(\Omega)} (-v(x)) = -\sup_{v\in\sigma_{-g}(\Omega)} v(x) = -w_{-g}(x);$$

that is, $w_g \le -w_{-g}$ in Ω, which is valid without the regularity of $\partial\Omega$. From this, by applying Step 1 to $-g$, we have

$$\limsup_{x\to\eta,\, x\in\Omega} w_g(x) \le \limsup_{x\to\eta,\, x\in\Omega} (-w_{-g}(x)) = -\liminf_{x\to\eta,\, x\in\Omega} w_{-g}(x) \le -(-g(\eta)) = g(\eta),$$

completing the proof. □

Theorem 3.30 (Solvability of Dirichlet problems)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded open. Then the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \textit{in } \Omega, \\ u = g & \textit{on } \partial\Omega \end{cases}$$

has a solution $u \in C^2(\Omega) \cap C(\bar\Omega)$ for every continuous boundary function $g \in C(\partial\Omega)$ if and only if every boundary point $\eta \in \partial\Omega$ is regular.

Proof. 1. Suppose every boundary point $\eta \in \partial\Omega$ is regular. Let w_g be the function defined by (3.27) and let

$$u(x) = \begin{cases} w_g(x) & (x \in \Omega), \\ g(x) & (x \in \partial\Omega). \end{cases}$$

Then $u \in C^2(\Omega) \cap C(\bar\Omega)$ is a solution to the Dirichlet problem.

2. Assume the Dirichlet problem is solvable for all continuous boundary data g. Given any $\eta \in \partial\Omega$, let $g(x) = -|x - \eta|$. Let $u = Q_\eta$ be the $C^2(\Omega) \cap C(\bar\Omega)$-solution to the Dirichlet problem with this g. Then Q_η is a barrier function at η. (Explain why?) Therefore, $\eta \in \partial\Omega$ is regular. □

Remark 3.6. A domain Ω is said to satisfy the **exterior ball property** at a boundary point $\eta \in \partial\Omega$ if there exists a closed ball $B = \bar B(x_0, \rho)$ in the exterior domain $\mathbb{R}^n \setminus \Omega$ such that $B \cap \partial\Omega = \{\eta\}$; in this case, η is regular because we can choose the barrier function Q_η to be

$$Q_\eta(x) = \Phi(x - x_0) - \phi(\rho),$$

where $\Phi(x) = \phi(|x|)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$. Domain Ω is said to satisfy the **exterior ball property** if it satisfies this property at every boundary point. For such domains the Dirichlet problem is uniquely solvable for any continuous boundary data. Note that every strictly convex domain satisfies the exterior ball property.

3.7. Maximum principles for second-order linear elliptic equations

(*This material is from Section 6.4 of the textbook.*)

3.7.1. Second-order linear elliptic PDEs. Consider the second-order linear differential operator

$$Lu(x) = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b^i(x)D_i u(x) + c(x)u(x),$$

where $D_i u = u_{x_i}$, $D_{ij}u = u_{x_i x_j}$ and $a^{ij}(x)$, $b^i(x)$, $c(x)$ are given functions in an open set Ω in $\mathbb{R}^n$ for all $i, j = 1, 2, \cdots, n$. With loss of generality, we assume $a^{ij}(x) = a^{ji}(x)$ for all i, j.

Definition 3.9. The operator L is called **elliptic** in Ω if there exists $\lambda(x) > 0$ ($x \in \Omega$) such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda(x)\sum_{i=1}^{n}\xi_i^2 \quad \forall\, x \in \Omega,\;\; \xi \in \mathbb{R}^n.$$

If $\lambda(x) \geq \lambda_0 > 0$ in Ω, we say that L is **uniformly elliptic** in Ω.

So, if L is elliptic in Ω, then for each $x \in \Omega$ the symmetry matrix $(a^{ij}(x))$ is *positive definite*, with all eigenvalues $\geq \lambda(x)$.

Lemma 3.31. *If $A = (a_{ij})$ is an $n \times n$ symmetric nonnegative definite matrix then there exists an $n \times n$ matrix $B = (b_{ij})$ such that $A = B^T B$, i.e.,*

$$a_{ij} = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n).$$

Proof. Exercise. □

3.7.2. Weak maximum principle.

Lemma 3.32. *Let L be elliptic in Ω and $u \in C^2(\Omega)$ satisfy $Lu < 0$ in Ω. If $c(x) \geq 0$, then u cannot attain a nonnegative maximum in Ω. If $c(x) \equiv 0$ then u cannot attain a maximum in Ω.*

Proof. Let $Lu < 0$ in Ω. Suppose $u(x_0)$ is maximum for some $x_0 \in \Omega$. Then, by derivative test, $D_j u(x_0) = 0$ for each $j = 1, 2, \cdots, n$ and

$$\frac{d^2 u(x_0 + t\xi)}{dt^2}\Big|_{t=0} = \sum_{i,j=1}^{n} D_{ij}u(x_0)\xi_i\xi_j \leq 0$$

for all $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$. By the lemma above, we write

$$a^{ij}(x_0) = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n),$$

where $B = (b_{ij})$ is an $n \times n$ matrix. Hence

$$\sum_{i,j=1}^{n} a^{ij}(x_0)D_{ij}u(x_0) = \sum_{k=1}^{n}\sum_{i,j=1}^{n} D_{ij}u(x_0)b_{ki}b_{kj} \leq 0,$$

which implies that $Lu(x_0) \geq c(x_0)u(x_0) \geq 0$ either when $c \geq 0$ and $u(x_0) \geq 0$ or when $c \equiv 0$. This is a contradiction. □

Theorem 3.33 (**Weak maximum principle with** $c = 0$). *Let Ω be bounded open in $\mathbb{R}^n$ and L be elliptic in Ω and*

$$|b^i(x)|/\lambda(x) \le M \quad (x \in \Omega,\ i = 1, 2, \cdots, n) \tag{3.29}$$

for some constant $M > 0$. Let $c \equiv 0$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \le 0$ in Ω. Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. Let $\alpha > 0$ and $v(x) = e^{\alpha x_1}$. Then

$$Lv(x) = (-a^{11}(x)\alpha^2 + b^1(x)\alpha)e^{\alpha x_1} = \alpha a^{11}(x)\left[-\alpha + \frac{b^1(x)}{a^{11}(x)}\right] e^{\alpha x_1} < 0$$

if $\alpha > M + 1$ because $\frac{|b^1(x)|}{a^{11}(x)} \le \frac{|b^1(x)|}{\lambda(x)} \le M$. Then consider the function $w(x) = u(x) + \varepsilon v(x)$ for $\varepsilon > 0$. Then $Lw = Lu + \varepsilon Lv < 0$ in Ω. So by Lemma 3.32, for all $x \in \bar{\Omega}$,

$$u(x) + \varepsilon v(x) \le \max_{\partial\Omega}(u + \varepsilon v) \le \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} v.$$

Letting $\varepsilon \to 0^+$ proves the theorem. $\square$

Remark 3.7. (a) The weak maximum principle still holds if $(a^{ij}(x))$ is nonnegative definite, i.e., $\lambda(x) \ge 0$ in Ω, but satisfies $\frac{|b^k(x)|}{a^{kk}(x)} \le M$ for some $k = 1, 2, \cdots, n$. (In this case use $v = e^{\alpha x_k}$.)

(b) If Ω is unbounded but bounded in a slab $|x_1| < N$, then the proof is still valid if the maximum is changed supremum.

Theorem 3.34 (**Weak maximum principle with** $c \ge 0$). *Let Ω be bounded open in $\mathbb{R}^n$ and L be elliptic in Ω satisfying* (3.29). *Let $c(x) \ge 0$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then*

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+ \quad \text{if } Lu \le 0 \text{ in } \Omega,$$

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u| \quad \text{if } Lu = 0 \text{ in } \Omega,$$

where $u^+(x) = \max\{u(x), 0\}$.

Proof. 1. Let $Lu \le 0$ in Ω. Let $\Omega^+ = \{x \in \Omega \,|\, u(x) > 0\}$. If Ω^+ is empty then the result is trivial. Assume $\Omega^+ \ne \emptyset$; then $L_0 u \equiv Lu - c(x)u(x) \le 0$ in Ω^+. Note that $\partial(\Omega^+) = [\Omega \cap \partial\Omega^+] \cup [\partial\Omega^+ \cap \partial\Omega]$, from which we easily see that $\max_{\partial(\Omega^+)} u \le \max_{\partial\Omega} u^+$; hence, by Theorem 3.33,

$$\max_{\bar{\Omega}} u = \max_{\bar{\Omega^+}} u = \max_{\partial(\Omega^+)} u \le \max_{\partial\Omega} u^+.$$

2. Let $Lu = 0$. We apply Step 1 to u and $-u$ to complete the proof. $\square$

Remark 3.8. The weak maximum principle for $Lu \le 0$ can not be replaced by $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. In fact, for any $u \in C^2(\bar{\Omega})$ satisfying

$$0 > \max_{\bar{\Omega}} u > \max_{\partial\Omega} u,$$

if we choose a constant $\theta > -\|Lu\|_{L^\infty(\Omega)}/\max_{\bar{\Omega}} u > 0$, then $\tilde{L}u = Lu + \theta u \le 0$ in Ω. But the zero-th order coefficient of $\tilde{L}$ is $c(x, t) + \theta > 0$.

The weak maximum principle easily implies the following uniqueness result for Dirichlet problems.

Theorem 3.35 (**Uniqueness of solutions).** *Let Ω be bounded open in $\mathbb{R}^n$ and the linear operator L with $c(x) \geq 0$ be elliptic in Ω and satisfy* (3.29). *Then, given any functions f and g, the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Remark 3.9. The uniqueness result fails if $c(x) < 0$ in Ω. For example, if $n = 1$, then function $u(x) = \sin x$ solves the elliptic problem $Lu \equiv -u'' - u = 0$ in $\Omega = (0, \pi)$ with $u(0) = u(\pi) = 0$; but $u \not\equiv 0$.

3.7.3. Strong maximum principle.

Theorem 3.36 (**Hopf's Lemma).** *Let L be uniformly elliptic with bounded coefficients in a ball B and let $u \in C^2(B) \cap C^1(\bar{B})$ satisfy $Lu \leq 0$ in B. Assume $x^0 \in \partial B$ such that $u(x) < u(x^0)$ for all $x \in B$.*

(a) *If $c \equiv 0$ in B, then $\frac{\partial u}{\partial \nu}(x^0) > 0$, where ν is outer unit normal to ∂B.*

(b) *If $c(x) \geq 0$ in B, then the same conclusion holds provided $u(x^0) \geq 0$.*

(c) *If $u(x^0) = 0$, then the same conclusion holds no matter what sign of $c(x)$ is.*

Proof. 1. Without loss of generality, assume $B = B(0, R)$. Consider function

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$

Let $\tilde{L}u \equiv Lu - c(x)u + c^+(x)u$, where $c^+(x) = \max\{c(x), 0\}$. This operator has the zero-th order term $c^+ \geq 0$ and hence the weak maximum principle applies to $\tilde{L}$. We compute

$$\begin{aligned} \tilde{L}v(x) &= \left[-4 \sum_{i,j=1}^n a^{ij}(x)\alpha^2 x_i x_j + 2\alpha \sum_{i=1}^n (a^{ii}(x) - b^i(x)x_i) \right] e^{-\alpha|x|^2} + c^+(x)v(x) \\ &\leq \left[-4\lambda_0 \alpha^2 |x|^2 + 2\alpha \operatorname{tr}(a^{ij}(x)) + 2\alpha |b(x)||x| + c^+(x) \right] e^{-\alpha|x|^2} < 0 \end{aligned}$$

on $\frac{R}{2} \leq |x| \leq R$ if $\alpha > 0$ is fixed and sufficiently large.

2. For any $\varepsilon > 0$, consider function $w_\varepsilon(x) = u(x) - u(x^0) + \varepsilon v(x)$. Then

$$\tilde{L}w_\varepsilon(x) = \varepsilon \tilde{L}v(x) + Lu(x) + (c^+(x) - c(x))u(x) - c^+(x)u(x^0) \leq 0$$

on $\frac{R}{2} \leq |x| \leq R$ in all cases of (a), (b) and (c).

3. By assumption, $u(x) < u(x^0)$ on $|x| = \frac{R}{2}$; hence there exists $\varepsilon > 0$ such that $w_\varepsilon(x) < 0$ on $|x| = \frac{R}{2}$. In addition, since $v|_{\partial B} = 0$, we have $w_\varepsilon(x) = u(x) - u(x^0) \leq 0$ on $|x| = R$. Hence the weak maximum principle implies that $w_\varepsilon(x) \leq 0$ for all $\frac{R}{2} \leq |x| \leq R$. But $w_\varepsilon(x^0) = 0$; this implies

$$0 \leq \frac{\partial w_\varepsilon}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \varepsilon \frac{\partial v}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - 2\varepsilon R\alpha e^{-\alpha R^2}.$$

Therefore

$$\frac{\partial u}{\partial \nu}(x^0) \geq 2\varepsilon R\alpha e^{-\alpha R^2} > 0,$$

as desired. $\square$

Theorem 3.37 (**Strong maximum principle**). *Let Ω be bounded, open and connected in $\mathbb{R}^n$ and L be uniformly elliptic with bounded coefficients in Ω and let $u \in C^2(\Omega)$ satisfy $Lu \leq 0$ in Ω.*

(a) *If $c(x) \geq 0$, then u cannot attain a nonnegative maximum in Ω unless u is constant.*

(b) *If $c \equiv 0$, then u cannot attain a maximum in Ω unless u is constant.*

Proof. Assume $c(x) \geq 0$ in Ω and u attains the maximum M at some point in Ω; also assume $M \geq 0$ if $c(x) \geq 0$. Suppose that u is not constant in Ω. Then, both of the following sets,

$$\Omega^- = \{x \in \Omega \mid u(x) < M\}; \quad \Omega_0 = \{x \in \Omega \mid u(x) = M\},$$

are nonempty, with Ω^- open and $\Omega_0 \neq \Omega$ relatively closed in Ω. Since Ω is connected, Ω_0 can not be open. Assume $x^0 \in \Omega_0$ is not an interior point of Ω_0; so, there exists a sequence $\{x^k\}$ not in Ω_0 but converging to x^0. Hence, for a ball $B(x^0, r) \subset\subset \Omega$ and an integer $N \in \mathbb{N}$, we have that $x^k \in B(x^0, r/2)$ for all $k \geq N$. Fix $k = N$ and let

$$S = \{\rho > 0 \mid B(x^N, \rho) \subset \Omega^-\}.$$

Then $S \subset \mathbb{R}$ is nonempty and bounded above by $r/2$. Let $\bar{\rho} = \sup S$; then $0 < \bar{\rho} \leq r/2$ and hence $B(x^N, \bar{\rho}) \subset B(x^0, r) \subset\subset \Omega$. So $B(x^N, \bar{\rho}) \subset \Omega^-$, and also $\Omega_0 \cap \partial B(x^N, \bar{\rho}) \neq \emptyset$. So let $y \in \Omega_0 \cap \partial B(x^N, \bar{\rho})$ and then $u(x) < u(y)$ for all $x \in B(x^N, \bar{\rho})$. Then **Hopf's Lemma** above, applied to the ball $B(x^N, \bar{\rho})$ at point $y \in \partial B(x^N, \bar{\rho})$, implies that $\frac{\partial u}{\partial \nu}(y) > 0$, where ν is the outer normal of $\partial B(x^N, \bar{\rho})$ at y. This contradicts the fact that $Du(y) = 0$, as u has a maximum at $y \in \Omega_0 \subset \Omega$. □

Finally we state without proof the following **Harnack's inequality** for nonnegative solutions of second-order elliptic PDEs, which extends the result for harmonic functions. For smooth coefficients, this result follows as a special case of **Harnack's inequality** for parabolic equations proved later.

Theorem 3.38 (**Harnack's Inequality**). *Let $V \subset\subset \Omega$ be connected and L be uniformly elliptic in Ω with bounded coefficients. Then there exists a constant $C = C(V, \Omega, L) > 0$ such that*

$$\sup_V u \leq C \inf_V u$$

for all nonnegative solutions u of $Lu = 0$ in Ω.

Chapter 4

The Heat Equation

The **heat equation**, also known as **diffusion equation**, describes in typical physical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, population, etc. Let V be any smooth subdomain, in which there is no source or sink. Then the rate of change of the total quantity within V equals the negative of the net flux $\mathbf{F}$ through ∂V:

$$\frac{d}{dt}\int_V u dx = -\int_{\partial V} \mathbf{F}\cdot \nu dS.$$

The divergence theorem tells us

$$\frac{d}{dt}\int_V u dx = -\int_V \operatorname{div}\mathbf{F} dx.$$

Since V is arbitrary, we should have

$$u_t = -\operatorname{div}\mathbf{F}.$$

For many applications $\mathbf{F}$ is proportional to the (spatial) gradient $Du = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$ of u, but points in opposite direction (flux is from regions of higher to lower concentration):

$$\mathbf{F} = -aDu \quad (a > 0).$$

Therefore we obtain the equation

$$u_t = a\operatorname{div}(Du) = a\Delta u,$$

which is called the heat equation when $a = 1$.

If there is a source in Ω, we should obtain the following nonhomogeneous equation

$$u_t - \Delta u = f(x,t) \quad x \in \Omega, \quad t \in (0,\infty).$$

4.1. Fundamental solution of heat equation

As in Laplace's equation case, we would like to find some special solutions to the heat equation. The textbook gives one way to find such a solution, and a problem in the book gives another way. Here we discuss yet another way of finding a special solution to the heat equation.

4.1.1. Fundamental solution and the heat kernel. We first make the following observations.

(1) If $u_j(x,t)$ are solution to the *one-dimensional* heat equation $u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$, $j = 1, \ldots, n$, then

$$u(x_1, \ldots, x_n, t) = u_1(x_1, t) \cdots u_n(x_n, t)$$

is a solution to the heat equation $u_t = \Delta u$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$. This simple fact is left as an exercise.

(2) If $u(x,t)$ is a solution to $u_t = u_{xx}$ with $x \in \mathbb{R}$, then so is $w(\lambda)u(\lambda x, \lambda^2 t)$ for any real λ. Especially, we expect to have a solution of the form $u(x,t) = w(t)v(\frac{x^2}{t})$. A direct computation yields

$$u_t(x,t) = w'(t)v(\frac{x^2}{t}) - w(t)v'(\frac{x^2}{t})\frac{x^2}{t^2};$$

$$u_x(x,t) = w(t)v'(\frac{x^2}{t})\frac{2x}{t};$$

$$u_{xx}(x,t) = w(t)v''(\frac{x^2}{t})\frac{4x^2}{t^2} + w(t)v'(\frac{x^2}{t})\frac{2}{t}.$$

For $u_t = u_{xx}$, we need

$$w'(t)v - \frac{w}{t}\left[v''\frac{4x^2}{t} + 2v' + v'\frac{x^2}{t}\right] = 0.$$

Separation of variables yields that

$$\frac{w'(t)t}{w(t)} = \frac{4sv''(s) + 2v'(s) + sv'(s)}{v(s)}$$

with $s = \frac{x^2}{t}$. Therefore, both sides of this equality must be constant, say, λ. So

$$s(4v'' + v') + \frac{1}{2}(4v' - 2\lambda v) = 0;$$

this equation is satisfied if we choose $\lambda = -1/2$ and $4v' + v = 0$ and hence $v(s) = e^{-\frac{s}{4}}$. In this case,

$$\frac{w'(t)t}{w(t)} = -\frac{1}{2},$$

from which we have $w(t) = t^{-\frac{1}{2}}$. Therefore

$$u = u_1(x,t) = \frac{1}{\sqrt{t}}\, e^{-\frac{x^2}{4t}}$$

is a solution of $u_t = u_{xx}$ for $x \in \mathbb{R}$. By observation (1) above, function

$$u(x_1, x_2, \cdots, x_n, t) = \frac{1}{t^{n/2}}\, e^{-\frac{|x|^2}{4t}}$$

is a solution of the heat equation $u_t = \Delta u$ for $t > 0$ and $x \in \mathbb{R}^n$.

Definition 4.1. The function

$$\Phi(x_1, \cdots, x_n, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & (t > 0), \\ 0 & (t \le 0) \end{cases}$$

is called the **fundamental solution of heat equation** $u_t = \Delta u$.

The constant $\frac{1}{(4\pi)^{n/2}}$ in the fundamental solution $\Phi(x,t)$ is due to the following

Lemma 4.1. *For each $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x_1, \ldots, x_n, t) dx_1 \cdots dx_n = 1.$$

Proof. This is a straight forward computation using the fact

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

$\square$

Note that, unlike the fundamental solution of Laplace's equation, the fundamental solution $\Phi(x, t)$ of the heat equation is C^∞ in $x \in \mathbb{R}^n$ for each $t > 0$. Furthermore, all *space-time* derivatives $D^\alpha \Phi$ are integrable on $\mathbb{R}^n$ for each $t > 0$. Also, as $t \to 0^+$, $\Phi(x, t) \to 0$ ($x \neq 0$) and $\Phi(0, t) \to \infty$. In fact, below we show that $\Phi(\cdot, t) \to \delta_0$ in distribution on $\mathbb{R}^n$ as $t \to 0^+$.

Definition 4.2. We call the function

$$K(x, y, t) = \Phi(x - y, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \quad (x,\, y \in \mathbb{R}^n,\ t > 0)$$

the **heat kernel** in $\mathbb{R}^n$.

4.1.2. Initial-value problem. Consider the **initial-value problem** or **Cauchy problem** of the heat equation

$$\begin{cases} u_t = \Delta u, & x \in \mathbb{R}^n,\ t \in (0, \infty), \\ u(x, 0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

Define

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy = \int_{\mathbb{R}^n} K(x, y, t) g(y)\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t > 0). \end{aligned} \tag{4.2}$$

Theorem 4.2. *Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and u be defined by (4.2). Then*

(i) $u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap L^\infty(\mathbb{R}^n \times (0, \infty))$,

(ii) $u_t = \Delta u$ *on* $\mathbb{R}^n \times (0, \infty)$,

(iii) for each $x_0 \in \mathbb{R}^n$,

$$\lim_{x \to x_0,\, t \to 0^+} \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy = g(x_0).$$

Proof. 1. Clearly, $K(x, y, t) > 0$ and $\int_{\mathbb{R}^n} K(x, y, t)\, dy = 1$ for all $x \in \mathbb{R}^n$, $t > 0$. Hence

$$|u(x, t)| \leq \|g\|_{L^\infty} \int_{\mathbb{R}^n} K(x, y, t)\, dy = \|g\|_{L^\infty} \quad (x \in \mathbb{R}^n,\ t > 0).$$

Furthermore, since $K(x, y, t)$ and along all derivatives are uniformly bounded on $\mathbb{R}^n \times \mathbb{R}^n \times [\delta, \infty)$, we see that $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$ and

$$u_t(x, t) - \Delta u(x, t) = \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x - y, t)] g(y)\, dy = 0 \quad (x \in \mathbb{R}^n,\ t > 0).$$

2. Fix $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(x_0)| < \varepsilon \quad \forall\, |y - x_0| < \delta,\ y \in \mathbb{R}^n.$$

Then if $|x - x_0| < \delta/2$, we have

$$|u(x,t) - g(x_0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x_0)]\, dy \right|$$

$$\leq \int_{B(x_0,\delta)} \Phi(x-y,t)|g(y) - g(x_0)|\, dy + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)|g(y) - g(x)| dy$$

$$:= I + J.$$

Now $I \leq \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\, dy = \varepsilon$. Furthermore, if $|x - x_0| < \delta/2$ and $|y - x_0| \geq \delta$, then

$$|y - x_0| \leq |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} \leq |x - y| + \frac{1}{2}|y - x_0|.$$

Thus $|y - x| \geq \frac{1}{2}|y - x_0|$. Consequently,

$$\begin{aligned} J \leq & 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\, dy = \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x-y|^2}{4t}}\, dy \\ \leq & \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|y-x_0|^2}{16t}}\, dy = C \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}}\, dz \to 0, \end{aligned}$$

as $t \to 0^+$. Hence if $|x - x_0| < \delta/2$ and $t > 0$ is sufficiently small, $|u(x,t) - g(x_0)| < 2\varepsilon$. $\square$

Remark 4.1. (a) Solution $u(x,t)$ defined by (4.2) depends on the values of $g(y)$ at all points $y \in \mathbb{R}^n$ no matter how far y and x are away. Even the initial datum g is compactly supported, its **domain of influence** on the solution $u(x,t)$ is still all of $x \in \mathbb{R}^n$ as long as $t > 0$. This phenomenon is known as the **infinite speed of propagation** of disturbances of the heat equation.

(b) The solution u defined by (4.2) is in $C^\infty(\mathbb{R}^n \times (0,\infty))$ even if g is not continuous; actually u is real analytic in $\mathbb{R}^n \times (0,\infty)$. This phenomenon is known as the **smoothing effect** of the heat kernel.

4.1.3. Nonhomogeneous initial value problems. We consider the initial-value problem for heat equation with source:

$$\begin{cases} u_t - \Delta u = f(x,t) & (x \in \mathbb{R}^n,\ t > 0), \\ u(x,0) = 0 & (x \in \mathbb{R}^n). \end{cases}$$

A general method for solving nonhomogeneous problems of general linear evolution equations using the solutions of homogeneous problem with variable initial data is known as **Duhamel's principle**. We use the idea of this method to solve the above nonhomogeneous heat equation.

Given $s > 0$, we solve the following homogeneous problem

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } \mathbb{R}^n \times (s,\infty), \\ \tilde{u}(x,s) = f(x,s) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{4.3}$$

to obtain a solution $\tilde{u}(x,t) = U(x,t;s)$, indicating the dependence on $s > 0$. By letting $\tilde{u}(x,t) = v(x,t-s)$ for a function v on $\mathbb{R}^n \times (0,\infty)$; then v solves the heat equation with initial data $v(x,0) = f(x,s)$. We use the heat kernel to have such a solution v as

$$v(x,t) = \int_{\mathbb{R}^n} K(x,y,t) f(y,s)\, dy \quad (x \in \mathbb{R}^n,\ t > 0).$$

In this way we obtain a solution $\tilde{u}$ to (4.3) as

$$\tilde{u}(x,t) = U(x,t;s) = \int_{\mathbb{R}^n} K(x,y,t-s)f(y,s)\,dy \quad (x \in \mathbb{R}^n,\ t > s > 0).$$

Then **Duhamel's principle** asserts that the function

$$u(x,t) = \int_0^t U(x,t;s)\,ds \quad (x \in \mathbb{R}^n,\ t > 0)$$

would be a solution to the original nonhomogeneous problem. Formally, $u(x,0) = 0$ and

$$u_t(x,t) = U(x,t;t) + \int_0^t U_t(x,t;s)\,ds = f(x,t) + \int_0^t \Delta U(x,t;s)\,ds = f(x,t) + \Delta u(x,t).$$

However, we have to justify the differentiation under the integral.

Rewriting,

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dyds \quad (x \in \mathbb{R}^n,\ t > 0). \end{aligned} \tag{4.4}$$

In the following, we use $C_1^2(\Omega \times I)$ to denote the space of functions $u(x,t)$ on $\Omega \times I$ such that $u, u_t, u_{x_i}, u_{x_ix_j}$ are in $C(\Omega \times I)$.

Theorem 4.3. *Assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ is such that f, f_t, f_{x_i} and $f_{x_ix_j}$ are bounded on $\mathbb{R}^n \times [0,\infty)$. Define $u(x,t)$ by (4.4). Then $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$ and satisfies*

(i) $u_t(x,t) - \Delta u(x,t) = f(x,t)$ $(x \in \mathbb{R}^n,\ t > 0)$,

(ii) for each $x_0 \in \mathbb{R}^n$,

$$\lim_{x \to x_0,\, t \to 0^+} u(x,t) = 0.$$

Proof. By the change of variables, we have that

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f(x-y,t-s)\,dyds \quad (x \in \mathbb{R}^n,\ t > 0).$$

As f, f_t, f_{x_i} and $f_{x_ix_j}$ are all bounded on $\mathbb{R}^n \times [0,\infty)$, it follows that, for all $x \in \mathbb{R}^n$ and $t > 0$,

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f_t(x-y,t-s)\,dyds + \int_{\mathbb{R}^n} \Phi(y,t)f(x-y,0)\,dy$$

and

$$u_{x_ix_j} = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f_{x_ix_j}(x-y,t-s)\,dyds \quad (i,j = 1,2,\cdots,n).$$

This proves that $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$. So

$$\begin{aligned}
u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)[(\partial_t - \Delta_x) f(x-y,t-s)]\, dyds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\
&= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[(-\partial_s - \Delta_y) f(x-y,t-s)]\, dyds \\
&\quad + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)[(-\partial_s - \Delta_y) f(x-y,t-s)]\, dyds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\
&:= I_\varepsilon + J_\varepsilon + N.
\end{aligned}$$

Now $|J_\varepsilon| \le C \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) dyds \le C\varepsilon$. By integration by parts, we have

$$\begin{aligned}
I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} [(\partial_s - \Delta_y)\Phi(y,s)] f(x-y,t-s)\, dyds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\
&= \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t)\, dy + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)[f(x-y,t-\varepsilon) - f(x-y,t)]\, dy \\
&\quad - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy,
\end{aligned}$$

using $(\partial_s - \Delta_y)\Phi(y,s) = 0$. Therefore,

$$u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon \to 0^+} (I_\varepsilon + J_\varepsilon + N)$$

$$\begin{aligned}
&= \lim_{\varepsilon \to 0^+} \left[\int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t)\, dy + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)[f(x-y,t-\varepsilon) - f(x-y,t)]\, dy \right] \\
&= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t)\, dy = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \Phi(x-y,\varepsilon) f(y,t)\, dy = f(x,t),
\end{aligned}$$

by (iii) of Theorem 4.2. Finally, we also have $\|u(\cdot,t)\|_{L^\infty} \le t\|f\|_{L^\infty} \to 0$ as $t \to 0^+$. This completes the proof. □

4.1.4. Nonuniqueness of the heat equation. The Cauchy problem (4.1) of the heat equation does not have unique solution. In fact we have the following result.

Theorem 4.4 (Tychonoff's solution)**.** *There are infinitely many solutions to Problem* (4.1).

Proof. We only need to construct infinitely many nonzero solutions to the one-dimensional heat equation with 0 initial condition.

We first solve the following Cauchy problem

$$\begin{cases} u_t = u_{xx}, \quad x \in \mathbb{R},\ t \in (-\infty,\infty), \\ u(0,t) = g(t), \quad u_x(0,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{4.5}$$

by formally expanding u as a Taylor series of x,

$$u(x,t) = \sum_{j=0}^\infty g_j(t) x^j.$$

A formal computation from $u_t = u_{xx}$ requires

$$g_0(t) = g(t),\ g_1(t) = 0,\ g_j'(t) = (j+2)(j+1)g_{j+2},\ j = 0, 1, 2, \dots;$$

therefore

$$g_{2k}(t) = \frac{1}{(2k)!}g^{(k)}(t), \quad g_{2k+1}(t) = 0, \quad k = 0, 1, \dots,$$

which leads to

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!}x^{2k}. \tag{4.6}$$

We then would like to choose a function g so that this series defines a true solution of (4.5). For this purpose, let $g(t)$ be defined by

$$g(t) = \begin{cases} e^{-t^{-\alpha}}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

where $\alpha > 1$ is a constant. Then it is a good exercise to show that there exists a number $\theta = \theta(\alpha) > 0$ with

$$|g^k(t)| \le \frac{k!}{(\theta t)^k} e^{-\frac{1}{2}t^{-\alpha}} \quad (k = 0, 1, 2, \dots,\ \ t > 0),$$

and hence the function u defined above is a honest solution of (4.5) that also satisfies $u(x, 0) = 0$; such a solution is called a **Tychonoff solution** to the heat equation with zero initial datum, and it is not identically zero as $u(0,t) = g(t)$ for all $t > 0$.

Actually, the Tychonoff solution $u(x,t)$ is an entire function of x for any real t, but is not analytic in t. Of course, for different $\alpha > 1$ we have the different Tychonoff solutions. □

4.2. Weak maximum principle and uniqueness

More practical question is the existence and uniqueness of the the **mixed-value problem** of the heat equation in a bounded open set. We handle the uniqueness by **maximum principles** of the heat equation similar to that of Laplace's equation.

4.2.1. Parabolic cylinder and weak maximum principle for heat equation. We assume Ω is a bounded open set in $\mathbb{R}^n$, $T > 0$. Consider the **parabolic cylinder**

$$\Omega_T = \Omega \times (0, T] = \{(x,t) \mid x \in \Omega,\ t \in (0,T]\}.$$

We define the **parabolic boundary** of Ω_T to be

$$\Gamma_T = \partial'\Omega_T := \overline{\Omega_T} \setminus \Omega_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t = 0\}). \tag{4.7}$$

Theorem 4.5 (Weak maximum principle). *Let u be continuous in $\bar{\Omega}_T$ and u_t, $D_i u$, $D_{ij}u$ exist and be continuous in Ω_T* (that is, $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$) *and satisfy $u_t - \Delta u \le 0$ in Ω_T* (that is, u is a **subsolution of the heat equation**). *Then*

$$\max_{\overline{\Omega_T}} u = \max_{\partial'\Omega_T} u.$$

Proof. Consider $v = u - \varepsilon t$ for any $\varepsilon > 0$. Then

$$v_t - \Delta v = u_t - \Delta u - \varepsilon \le -\varepsilon < 0 \quad \text{in } \Omega_T.$$

Let $v(x_0, t_0) = \max_{\overline{\Omega_T}} v$ for some $(x_0, t_0) \in \overline{\Omega_T}$. We claim $(x_0, t_0) \in \partial'\Omega_T$. Otherwise, assume $(x_0, t_0) \in \Omega_T$. Then $x_0 \in \Omega$ and $0 < t_0 \le T$. So at this maximum point, we have

$\Delta v(x_0,t_0) \le 0$ and $v_t(x_0,t_0) \ge 0$, which contradicts $v_t - \Delta v \le -\varepsilon$. Therefore $(x_0,t_0) \in \partial'\Omega_T$ and so

$$\max_{\overline{\Omega_T}} v = \max_{\partial'\Omega_T} v \le \max_{\partial'\Omega_T} u,$$

which implies

$$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}} (v+\varepsilon t) \le \max_{\overline{\Omega_T}} v + \varepsilon T = \max_{\partial'\Omega_T} v + \varepsilon T \le \max_{\partial'\Omega_T} u + \varepsilon T.$$

Finally, letting $\varepsilon \to 0^+$ proves

$$\max_{\overline{\Omega_T}} u \le \max_{\partial'\Omega_T} u.$$

The opposite inequality is obvious. □

Similarly, **weak minimum principle** holds for **supersolutions** u of the heat equation $u_t - \Delta u \ge 0$.

4.2.2. Uniqueness of mixed-value problems. From the weak maximum (minimum) principle, we easily obtain the following uniqueness result.

Theorem 4.6 (Uniqueness of mixed-value problem)**.** *Given any functions f, h and g, the mixed boundary value problem*

$$\begin{cases} u_t - \Delta u = f(x,t) & (x,t) \in \Omega_T, \\ u(x,t) = h(x,t), & x \in \partial\Omega,\ t \in [0,T], \\ u(x,0) = g(x), & x \in \Omega \end{cases}$$

can have at most one solution u in $C^2_1(\Omega_T) \cap C(\overline{\Omega_T})$.

4.2.3. Uniqueness by the energy method. Let us study another method for the uniqueness, but only for more smooth solutions on smooth domains.

Assume Ω is a bounded smooth domain. Let $u \in C^2_1(\overline{\Omega_T})$ be a solution to the homogeneous mixed-value problem

$$u_t = \Delta u \ \text{ in } \Omega_T, \quad u = 0 \ \text{ on } \partial'\Omega_T.$$

Of course, by the previous theorem, we know that $u \equiv 0$. However, we give another proof based on integration by parts; this is known as the **energy method**. Let $e(t)$ be the "energy" defined by

$$e(t) = \int_\Omega u(x,t)^2\,dx \quad (0 \le t \le T).$$

Then $e(0) = 0$, $e(t) \ge 0$ and is differentiable in $(0,T)$ and, by Green's identity,

$$e'(t) = 2\int_\Omega u u_t\,dx = 2\int_\Omega u\Delta u\,dx = -2\int_\Omega |Du|^2\,dx \le 0,$$

which implies that $e(t)$ is non-increasing in $(0,T)$. Hence $e(t) \le e(0) = 0$; so $e(t) \equiv 0$. This proves that $u \equiv 0$ in Ω_T.

The energy method can also be used to obtain other types of uniqueness result.

Theorem 4.7 (Uniqueness of mixed-Neumann value problem)**.** *Given any functions f, h and g, the* **mixed-Neumann value problem**

$$\begin{cases} u_t - \Delta u = f(x,t) & (x,t) \in \Omega_T, \\ \frac{\partial u}{\partial \nu}(x,t) = h(x,t), & x \in \partial\Omega,\ t \in [0,T], \\ u(x,0) = g(x), & x \in \Omega \end{cases}$$

can have at most one solution u in $C_1^2(\bar{\Omega} \times [0,T])$.

Proof. The proof is standard by showing the difference of any two solutions must be zero, which can be proved by a similar energy method as above. □

Theorem 4.8 (Backward uniqueness of the heat equation)**.** *Let $u \in C^2(\overline{\Omega_T})$ solve*

$$u_t = \Delta u \;\; in\ \Omega_T, \quad u(x,t) = 0 \;\; on\ \partial\Omega \times [0,T].$$

If $u(x,T) = 0$, then $u \equiv 0$ in Ω_T.

Proof. 1. Let $e(t) = \int_\Omega u(x,t)^2\,dx$ for $0 \le t \le T$. As above $e'(t) = -2\int_\Omega |Du|^2\,dx$ and hence

$$e''(t) = -4\int_\Omega Du\cdot Du_t = 4\int_\Omega (\Delta u)^2\,dx.$$

Now, by Hölder's inequality (or the Cauchy-Schwartz inequality)

$$\int_\Omega |Du|^2\,dx = -\int_\Omega u\Delta u\,dx \le \left(\int_\Omega u^2\,dx\right)^{1/2}\left(\int_\Omega (\Delta u)^2\,dx\right)^{1/2}$$

and so

$$(e'(t))^2 = 4\left(\int_\Omega |Du|^2\,dx\right)^2 \le e(t)e''(t).$$

2. If $e(t) = 0$ for all $0 \le t \le T$, then we are done. Otherwise, suppose $e(t) \not\equiv 0$ on $[0,T]$. Since $e(T) = 0$, we can find $0 \le t_1 < t_2 \le T$ such that

$$e(t) > 0 \;\text{ on } t \in [t_1,t_2), \quad e(t_2) = 0.$$

Set $f(t) = \ln e(t)$ for $t \in [t_1,t_2)$. Then

$$f''(t) = \frac{e''(t)e(t) - e'(t)^2}{e(t)^2} \ge 0 \quad (t_1 \le t < t_2).$$

This shows that f is convex on $[t_1,t_2)$; hence $f((1-\lambda)t_1 + \lambda t) \le (1-\lambda)f(t_1) + \lambda f(t)$ and thus $e((1-\lambda)t_1 + \lambda t) \le e(t_1)^{1-\lambda}e(t)^\lambda$ for all $0 < \lambda < 1$ and $t_1 < t < t_2$. Letting $t \to t_2^-$ and in view of $e(t_2) = 0$ we have

$$0 \le e((1-\lambda)t_1 + \lambda t_2) \le e(t_1)^{1-\lambda}e(t_2)^\lambda = 0 \quad \forall\, 0 < \lambda < 1,$$

which is a contradiction to that assumption $e(t) > 0$ on $[t_1,t_2)$. □

Remark 4.2. The **backward uniqueness theorem** for the heat equation asserts that if two temperature distributions on Ω agree at some time $T > 0$ and have the same boundary values at all earlier times $0 \le t \le T$ then these temperature distributions must be identical within Ω at all earlier times.

4.2.4. Maximum principle for Cauchy problems.

We now extend the maximum principle and uniqueness theorem to the region $\mathbb{R}^n \times (0,T]$.

Theorem 4.9 (Weak maximum principle for Cauchy problem)**.** *Let $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ satisfy*

$$\begin{cases} u_t - \Delta u \le 0, & in\ \mathbb{R}^n \times (0,T), \\ u(x,t) \le Me^{a|x|^2} & on\ \mathbb{R}^n \times [0,T], \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases}$$

for some constants M, $a > 0$. Then

$$u(x,t) \le \sup_{z\in\mathbb{R}^n} g(z) \quad \forall\, x \in \mathbb{R}^n,\; 0 \le t \le T.$$

Proof. 1. Without loss of generality, assume that $\sup_{\mathbb{R}^n} g < \infty$. We first assume that $4aT < 1$. Let $\epsilon > 0$ be such that $4a(T+\epsilon) < 1$. Fix $\mu > 0$ and consider the function

$$v(x,t) = u(x,t) - \mu w(x,t), \quad (x \in \mathbb{R}^n,\ 0 \le t \le T),$$

where $w(x,t)$ is the function defined by

$$w(x,t) = \frac{1}{(T+\epsilon-t)^{n/2}} e^{\frac{|x|^2}{4(T+\epsilon-t)}}$$

A direct calculation shows that $w(x,t)$ satisfies the heat equation $w_t = \Delta w$ on $\mathbb{R}^n \times [0,T]$, and hence

$$v_t - \Delta v \le 0 \quad \text{in } \mathbb{R}^n \times (0,T].$$

2. Clearly,

$$v(x,0) < u(x,0) = g(x) \le \sup_{\mathbb{R}^n} g \quad \forall\, x \in \mathbb{R}^n.$$

For $r > 0$, if $|x| = r$ and $0 \le t \le T$, then

$$\begin{aligned} v(x,t) = u(x,t) - \mu w(x,t) = u(x,t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ \le Me^{ar^2} - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \le Me^{ar^2} - \frac{\mu}{(T+\epsilon)^{n/2}} e^{\frac{r^2}{4(T+\epsilon)}} \\ := Me^{ar^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2}, \end{aligned}$$

where $a + \gamma = \frac{1}{4(T+\epsilon)}$ for some number $\gamma > 0$. We choose an $R > 0$ sufficiently large (depending on μ) so that

$$Me^{ar^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \le \sup_{\mathbb{R}^n} g \quad \forall\, r \ge R.$$

Therefore, we have that

$$v(x,t) \le \sup_{\mathbb{R}^n} g \quad \forall\, |x| \ge R,\ 0 \le t \le T, \tag{4.8}$$

and thus that $v(x,t) \le \sup_{\mathbb{R}^n} g$ for all $(x,t) \in \partial' \Omega^R_T$, where $\Omega^R = \{x \in \mathbb{R}^n : |x| < R\}$. Hence, applying the weak maximum principle to the circular cylinder $\bar{\Omega}^R_T$, we have

$$v(x,t) \le \sup_{\mathbb{R}^n} g \quad \forall\, |x| \le R,\ 0 \le t \le T. \tag{4.9}$$

Consequently, combining (4.8) and (4.9), we have

$$v(x,t) \le \sup_{\mathbb{R}^n} g \quad \forall\, x \in \mathbb{R}^n, \ \ 0 \le t \le T;$$

this implies that

$$u(x,t) \le \sup_{\mathbb{R}^n} g + \mu w(x,t) \quad \forall\, x \in \mathbb{R}^n,\ 0 \le t \le T.$$

Letting $\mu \to 0^+$, we obtain that $u(x,t) \le \sup_{\mathbb{R}^n} g$ for all $x \in \mathbb{R}^n$ and $0 \le t \le T$.

3. Finally, if $4aT \ge 1$, we repeatedly apply the result on intervals $[0,T_1]$, $[T_1, 2T_1], \cdots$, where $T_1 = \frac{1}{8a}$, until we reach to time T. □

Theorem 4.10 (Uniqueness under growth condition)**.** *Given any f, g, there exists at most one solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ to the Cauchy problem*

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

that satisfies the growth condition $|u(x,t)| \leq Me^{a|x|^2}$ *in* $\mathbb{R}^n \times (0,T)$ *for some constants* M, $a > 0$.

4.3. Nonnegative Solutions

From the example of Tychonoff's solutions above, we know that the initial data can not determine the solution uniquely and some additional information is needed for uniqueness; for example, the suitable growth condition at infinity or the boundary conditions when the domain Ω is bounded.

This section we discuss another important uniqueness result due to D.V. Widder concerning for *nonnegative solutions* of the heat equation.

Theorem 4.11 (Widder's theorem). *Let u be continuous for $x \in \mathbb{R}$, $0 \leq t < T$, and let u_t, u_x and u_{xx} exist and be continuous for $x \in \mathbb{R}$, $0 < t < T$. Assume that*

$$u_t = u_{xx}, \quad u(x,0) = g(x), \quad u(x,t) \geq 0.$$

Then u is determined uniquely and represented by

$$u(x,t) = \int_{\mathbb{R}} K(x,y,t)g(y)dy.$$

Proof. The idea is to show that (i) $u(x,t) \geq \int_{\mathbb{R}} K(x,y,t)g(y)dy$ (the representation formula gives the smallest nonnegative solution), and (ii) $w(x,t) = u(x,t) - \int_{\mathbb{R}} K(x,y,t)g(y)dy$ must be identically zero.

1. For $a > 1$ define function $\zeta^a(x)$ by $\zeta^a(x) = 1$ for $|x| \leq a-1$; $\zeta^a(x) = 0$ for $|x| \geq a$; $\zeta^a(x) = a - |x|$ for $a-1 < |x| < a$. Consider the expression

$$v^a(x,t) = \int_{\mathbb{R}} K(x,y,t)\zeta^a(y)g(y)dy.$$

Since $\zeta^a(y)g(y)$ is bounded and continuous on $\mathbb{R}$, we know that

$$v^a_t - v^a_{xx} = 0 \ \text{ for } x \in \mathbb{R},\ 0 < t < T, \quad v^a(x,0) = \zeta^a(x)g(x).$$

Let M_a be the maximum of $g(x)$ on $|x| \leq a$. Using $K(x,y,t) \leq \frac{1}{\sqrt{2\pi e}|x-y|}$, we have that, for $|x| > a$,

$$0 \leq v^a(x,t) \leq M_a \int_{-a}^{a} K(x,y,t)dy \leq \frac{2M_a a}{\sqrt{2\pi e}} \frac{1}{|x|-a}.$$

Let $\epsilon > 0$ and let $\rho > a + \frac{2M_a a}{\epsilon\sqrt{2\pi e}}$. Then

$$\begin{cases} v^a(x,t) \leq \epsilon \leq \epsilon + u(x,t), & |x| = \rho,\ 0 < t < T, \\ v^a(x,0) \leq g(x) \leq \epsilon + u(x,0), & |x| \leq \rho. \end{cases}$$

By the weak maximum principle,

$$v^a(x,t) \leq \epsilon + u(x,t) \quad \text{for } |x| \leq \rho,\ 0 \leq t < T.$$

Let $\rho \to \infty$ and we find the same inequality for all $x \in \mathbb{R}$, $0 \leq t < T$. Letting $\epsilon \to 0^+$, it follows that

$$v^a(x,t) \leq u(x,t), \quad \forall\, x \in \mathbb{R},\ 0 \leq t < T.$$

Since ζ^a is a non-decreasing bounded functions of a we find that

$$v(x,t) = \lim_{a\to\infty} v^a(x,t) = \int_{\mathbb{R}} K(x,y,t)g(y)dy$$

exists for $x \in \mathbb{R},\ 0 \le t < T$ and $0 \le v(x,t) \le u(x,t)$. The analytic regularity of $v(x,t)$ in $x \in \mathbb{R}$, $0 < t < T$ can be obtained from the analyticity of $v^a(x,t)$. Furthermore, as $v^a(x,t) \le v(x,t) \le u(x,t)$ for $x \in \mathbb{R}$, $0 \le t < T$ it also follows that $v(x,t)$ is continuous on $\mathbb{R} \times [0,T)$ with $v(x,0) = g(x)$.

2. Let $w = u - v$, which is continuous for $x \in \mathbb{R},\ 0 \le t < T$, $w_t = w_{xx},\ w(x,t) \ge 0$ and $w(x,0) = 0$. It remains to show that $w \equiv 0$. (That is, we reduced the problem to the case $g = 0$.) We introduce a new function

$$W(x,t) = \int_0^t w(x,s)ds.$$

Clearly $W_t = w$ and W are nonnegative continuous on $\mathbb{R} \times [0,T)$ with $W(x,0) = 0$. The existence of W_x, W_{xx} is not obvious since $w_x(x,s)$ may not exist for $s = 0$. To prove the existence of W_x, W_{xx}, we use the difference quotient operator

$$\delta_h f(x,t) = \frac{f(x+h,t) - f(x,t)}{h} \quad (h \ne 0).$$

Note that if f_x exists then $\delta_h f(x,t) = \int_0^1 f_x(x+zh,t)\,dz$. Hence

$$\begin{aligned}\delta_H w(x,t) - \delta_h w(x,t) &= \int_0^1 (w_x(x+zH,t) - w_x(x+zh,t))\,dz \\ &= \int_0^1 \int_h^H w_{xx}(x+zp,t) z\,dpdz = \int_0^1 \int_h^H w_t(x+zp,t) z\,dpdz,\end{aligned}$$

and so by integration with respect to t it follows that for any $\varepsilon \in (0,t)$

$$\begin{aligned}\delta_H W(x,t) - \delta_h W(x,t) &= \delta_H W(x,\varepsilon) - \delta_h W(x,\varepsilon) \\ &\quad + \int_0^1 \int_h^H (w(x+zp,t) - w(x+zp,\varepsilon)) z\,dpdz.\end{aligned}$$

Letting $\varepsilon \to 0^+$, since W and w are continuous at $t = 0$, gives

$$\delta_H W(x,t) - \delta_h W(x,t) = \int_0^1 \int_h^H w(x+zp,t) z\,dpdz. \tag{4.10}$$

This proves that $W_x(x,t) = \lim_{h\to 0} \delta_h W(x,t)$ exists. Letting $h \to 0^+$ in (4.10) and rewriting, we obtain

$$W(x+H,t) = W(x,t) + HW_x(x,t) + \int_0^1 \int_0^H Hw(x+zp,t) z\,dpdz,$$

and thus $W(x+H,t)$ is twice differentiable with respect to H and hence is also twice differentiable with respect to x; moreover,

$$W_{xx}(x+H,t) = W_{HH}(x+H,t) = \int_0^1 (2zw(x+zH,t) + Hz^2 w_x(x+zH,t))\,dz.$$

For $H \to 0^+$ we find that $W_{xx}(x,t) = w(x,t) = W_t(x,t) \ge 0$. So $W(x,t)$ is convex in x; hence

$$2W(x,t) \le W(x+H,t) + W(x-H,t)$$

for any $H > 0$. Integrating the inequality with respect to H from 0 to $x > 0$ we find

$$2xW(x,t) \le \int_0^x W(x+H,t)dH + \int_0^x W(x-H,t)dH = \int_0^{2x} W(y,t)dy.$$

From Step 1, for all $t > s,\ x > 0$,

$$W(0,t) \geq \int_{\mathbb{R}} K(0,y,t-s)W(y,s)dy \geq \int_0^{2x} K(0,y,t-s)W(y,s)dy$$

$$\geq \frac{e^{-x^2/(t-s)}}{\sqrt{4\pi(t-s)}} \int_0^{2x} W(y,s)dy.$$

The similar argument can be carried out to the case $x < 0$ (or applied to $W(-x,t)$). Therefore we deduce that

$$W(x,s) \leq \sqrt{\frac{\pi(t-s)}{x^2}}\, e^{x^2/(t-s)} W(0,t) \quad (x \in \mathbb{R},\ t > s \geq 0). \tag{4.11}$$

3. Now for $T > \epsilon > 0$, consider $W(x,s)$ for $x \in \mathbb{R}$ and $0 \leq s \leq T - 2\epsilon$. Then $W(x,s)$ is bounded for $|x| \leq \sqrt{\pi T}$ and $0 \leq s \leq T - 2\epsilon$. Using (4.11) with $t = T - \epsilon$, we obtain, for $|x| \geq \sqrt{\pi T}$ and $0 \leq s \leq T - 2\epsilon$,

$$W(x,s) \leq e^{x^2/\epsilon} W(0, T-\epsilon).$$

Hence W satisfies the assumption of Theorem (4.9). It follows that $W(x,s) = 0$ for $s \in [0, T-2\epsilon]$. Since $\epsilon > 0$ is arbitrary, $W(x,s) = 0$ for $s \in [0,T)$; that is $u(x,t) = v(x,t)$. $\square$

4.4. Regularity of Solutions

We now establish the regularity of solution for the heat equation.

Theorem 4.12 (Smoothness). *Let $u \in C_1^2(\Omega_T)$ solve the heat equation in Ω_T. Then $u \in C^\infty(\Omega_T)$.*

Proof. 1. Consider the closed circular cylinder

$$C(x,t;r) = \{(y,s) \mid |y-x| \leq r,\ t - r^2 \leq s \leq t\}.$$

Fix $(x_0,t_0) \in \Omega_T$. Choose $r > 0$ small enough so that $C = C(x_0,t_0;r) \subset \Omega_T$. Consider two smaller cylinders

$$C' = C(x_0,t_0;\frac{3}{4}r), \quad C'' = C(x_0,t_0;\frac{1}{2}r).$$

Let $\zeta(x,t) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$ such that $\zeta = 0$ outside of C, $\zeta = 1$ on C' and $0 \leq \zeta \leq 1$.

2. We temporarily assume that $u \in C^\infty(C)$. This seems contradicting to what we are going to prove, but the following arguments aim to establishing an identity that is valid for all C_1^2-solutions. Let $v = \zeta u$ on $\mathbb{R}^n \times [0,t_0]$. Then $v \in C^\infty(\mathbb{R}^n \times (0,t_0])$ and $v = 0$ on $\mathbb{R}^n \times \{t = 0\}$. Furthermore,

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta := \tilde{f} \quad \text{in } \mathbb{R}^n \times (0,t_0).$$

Note that $\tilde{f}$ is C^∞ and has compact in $\mathbb{R}^n \times [0,\infty)$; moreover, v is bounded on $\mathbb{R}^n \times [0,T]$. Hence, by the uniqueness of bounded solution, we have

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s)\,dyds \quad (x \in \mathbb{R}^n,\ t \in (0,t_0)).$$

Let $(x,t)\in C''$. Then

$$\begin{aligned} u(x,t) &= \iint_C \Phi(x-y,t-s)\tilde f(y,s)\,dyds \\ &= \iint_C \Phi(x-y,t-s)[(\zeta_s-\Delta\zeta)u(y,s)-2D\zeta\cdot Du]\,dyds \\ &= \iint_C [\Phi(x-y,t-s)(\zeta_s+\Delta\zeta)+2D_y\Phi(x-y,t-s)\cdot D\zeta]u(y,s)\,dyds \\ &= \iint_{C\setminus C'} [\Phi(x-y,t-s)(\zeta_s+\Delta\zeta)+2D_y\Phi(x-y,t-s)\cdot D\zeta]u(y,s)\,dyds. \end{aligned}$$

Let $\Gamma(x,y,t,s)=\Phi(x-y,t-s)(\zeta_s+\Delta\zeta)(y,s)+2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)$; then

$$u(x,t)=\iint_{C\setminus C'}\Gamma(x,y,t,s)u(y,s)\,dyds \quad \forall\,(x,t)\in C''. \tag{4.12}$$

3. We have derived (4.12) assuming u is C^∞-solution of the heat equation in Ω_T. Using the standard mollifying technique, this formula is valid for all $C^2_1(\Omega_T)$-solution of the heat equation. From this formula, since $\Gamma(x,y,t,s)$ is C^∞ in $(x,t)\in C''$ and $(y,s)\in C\setminus C'$, we deduce that $u\in C^\infty(C'')$. □

Theorem 4.13 (Estimates on derivatives)**.** *There exist constants $C_{k,l}$ for $k,l=0,1,2,\cdots$ such that*

$$\max_{C(x,t;\frac{r}{2})}|D_x^\alpha D_t^l u|\le \frac{C_{k,l}}{r^{k+2l+n+2}}\|u\|_{L^1(C(x,t;r))}\quad (|\alpha|=k)$$

for all cylinders $C(x,t;r)\subset\Omega_T$ and all $C^2_1(\Omega_T)$-solutions u of the heat equation in Ω_T.

Proof. 1. Fix some point $(x_0,t_0)\in\Omega_T$. Upon shifting the coordinates we may assume the point is $(0,0)$. Suppose first that the cylinder $C(1)=C(0,0;1)\subset\Omega_T$; then by (4.12) above,

$$u(x,t)=\iint_{C(1)\setminus C(\frac34)}\Gamma(x,y,t,s)u(y,s)\,dyds\quad\forall\,(x,t)\in C(\frac12).$$

Consequently, for all $k,l=0,1,2,\cdots$ and all α with $|\alpha|=k$,

$$|D_x^\alpha D_t^l u(x,t)|\le\iint_{C(1)\setminus C(\frac34)}|D_x^\alpha D_t^l\Gamma(x,y,t,s)|u(y,s)\,dyds\le C_{k,l}\|u\|_{L^1(C(1))}, \tag{4.13}$$

for all $(x,t)\in C(\frac12)$, where $C_{k,l}$ are some constants.

2. Now suppose $C(r)=C(0,0;r)\subset\Omega_T$. We rescale u by defining

$$v(x,t)=u(rx,r^2t)\quad\forall\,(x,t)\in C(1).$$

Then $v_t-\Delta v=0$ in $C(1)$. Note that for all $|\alpha|=k$,

$$D_x^\alpha D_t^l v(x,t)=r^{2l+k}D_y^\alpha D_s^l u(rx,r^2t)$$

and $\|v\|_{L^1(C(1))}=\frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$. Then, with (4.13) applied to v, we deduce

$$|D_y^\alpha D_s^l u(y,s)|\le\frac{C_{k,l}}{r^{k+2l+n+2}}\|u\|_{L^1(C(r))}$$

for all $(y,s)\in C(\frac r2)$. This completes the proof. □

4.5. Mean value property and the strong maximum principle

In this section we derive for the heat equation some kind of analogue of the mean value property of harmonic functions.

Definition 4.3. Fix $x \in \mathbb{R}^n,\ t \in \mathbb{R},\ r > 0$. We define the **heat ball** to be

$$E(x,t,r) = \{(y,s) \mid \Phi(x-y,t-s) \geq \frac{1}{r^n}\}.$$

This is a region in space-time, whose boundary is a level set of $\Phi(x-y,t-s)$.

4.5.1. Mean-value property for the heat equation.

Theorem 4.14. *Let $u \in C_1^2(\Omega_T)$ satisfy $u_t - \Delta u \leq 0$ in Ω_T. Then*

$$u(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$$

for all $E(x,t,r) \subset \Omega_T$.

Proof. 1. We may assume that $x = 0, t = 0$ and write $E(r) = E(0,0,r)$. Set

$$\phi(r) = \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds = \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds.$$

Let $v(y,s) = u(ry, r^2 s)$; then $v_s(y,s) \leq \Delta v(y,s)$ in $E(1)$. Let

$$H(y,s) = \Phi(y,-s) = \frac{1}{(-4\pi s)^{n/2}} e^{\frac{|y|^2}{4s}} \quad (y \in \mathbb{R}^n,\ s < 0).$$

Then $H(y,s) = 1$ for $(y,s) \in \partial E(1)$; moreover,

$$\begin{aligned} \ln H(y,s) &= -\frac{n}{2} \ln(-4\pi s) + \frac{|y|^2}{4s} \geq 0 \quad \forall\, (y,s) \in E(1), \\ 4(\ln H)_s + \frac{|y|^2}{s^2} &= -\frac{2n}{s}, \quad D(\ln H) = D_y(\ln H) = \frac{y}{2s}. \end{aligned} \tag{4.14}$$

2. Note also that $D_x u(ry, r^2 s) = \frac{1}{r} Dv(y,s)$ and $u_t(ry, r^2 s) = \frac{1}{r^2} v_s(y,s)$. We calculate $\phi'(r)$ as follows.

$$\begin{aligned}
\phi'(r) =& \iint_{E(1)} \left[D_x u(ry, r^2 s) \cdot y + 2rs u_t(ry, r^2 s) \right] \frac{|y|^2}{s^2}\, dyds \\
=& \frac{1}{r} \iint_{E(1)} \left[y \cdot Dv(y,s) + 2s v_s(y,s) \right] \frac{|y|^2}{s^2}\, dyds \\
=& \frac{1}{r} \iint_{E(1)} \left[y \cdot Dv \frac{|y|^2}{s^2} + 4 v_s\, y \cdot D(\ln H) \right] dyds \\
=& \frac{1}{r} \iint_{E(1)} y \cdot Dv \frac{|y|^2}{s^2} dyds - \frac{4}{r} \iint_{E(1)} (n v_s + y \cdot (Dv)_s) \ln H\, dyds \\
=& \frac{1}{r} \iint_{E(1)} y \cdot Dv \frac{|y|^2}{s^2} dyds - \frac{4}{r} \iint_{E(1)} [n v_s \, \ln H - y \cdot Dv\, (\ln H)_s]\, dyds \\
=& -\frac{1}{r} \left[\iint_{E(1)} y \cdot Dv \frac{2n}{s} dyds + 4n \iint_{E(1)} v_s \, \ln H dyds \right] \\
\geq& -\frac{1}{r} \left[\int_{E(1)} y \cdot Dv \frac{2n}{s} dyds + 4n \iint_{E(1)} \Delta v \, \ln H dyds \right] \\
=& -\frac{2n}{r} \iint_{E(1)} \left[\frac{y}{s} \cdot Dv - 2Dv \cdot D(\ln H) \right] dyds = 0.
\end{aligned}$$

Consequently, $\phi(r)$ is nondecreasing for $r > 0$; hence

$$\phi(r) \geq \phi(0^+) = u(0,0) \iint_{E(1)} \frac{|y|^2}{s^2} dyds.$$

3. It remains to show that $\iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4$. Note that

$$E(1) = \{(y,s) \mid -\frac{1}{4\pi} \leq s < 0,\ |y|^2 \leq 2ns \ln(-4\pi s)\}.$$

Using the change of variables $(y,s) \mapsto (y, -\tau)$, where $\tau = -\ln(-4\pi s)$ (and hence $s = -\frac{e^{-\tau}}{4\pi}$), the set $E(1)$ is mapped one-to-one and onto the set $\tilde{E}(1) = \{(y,\tau) \mid \tau \in (0,\infty),\ |y|^2 \leq \frac{n}{2\pi} \tau e^{-\tau}\}$. Therefore, using $ds = \frac{1}{4\pi} e^{-\tau} d\tau$,

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{s^2} dyds &= \int_0^\infty \int_{|y|^2 \leq \frac{n}{2\pi}\tau e^{-\tau}} 4\pi |y|^2 e^{\tau} dy d\tau \\
&= 4\pi n \alpha_n \int_0^\infty \int_0^{\sqrt{\frac{n}{2\pi}\tau e^{-\tau}}} r^{n+1} e^{\tau} dr d\tau \\
&= \frac{4\pi n \alpha_n}{n+2} (\frac{n}{2\pi})^{\frac{n+2}{2}} \int_0^\infty \tau^{\frac{n+2}{2}} e^{-\frac{n}{2}\tau} d\tau \\
&= \frac{4\pi n \alpha_n}{n+2} (\frac{n}{2\pi})^{\frac{n+2}{2}} (\frac{n}{2})^{-\frac{n}{2}-2} \int_0^\infty t^{\frac{n+2}{2}} e^{-t} dt \quad (\text{using } \tau = \tfrac{2}{n} t) \\
&= \frac{4n\alpha_n}{n+2} \frac{2}{n} (\frac{1}{\pi})^{n/2} \Gamma(\frac{n}{2} + 2) \\
&= 2n\alpha_n (\frac{1}{\pi})^{n/2} \Gamma(\frac{n}{2}) = 4,
\end{aligned}$$

noting that $\alpha_n = \frac{2}{n\Gamma(\frac{n}{2})}\pi^{n/2}$. $\square$

4.5.2. Strong maximum principle.

Theorem 4.15 (Strong Maximum Principle). *Assume that $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $u_t - \Delta u \le 0$ in Ω_T.*

(i) *We have that $\max_{\overline{\Omega_T}} u = \max_{\partial'\Omega_T} u$.*

(ii) *If Ω is connected and there exists a point $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) = \max_{\overline{\Omega_T}} u$, then u is constant in $\overline{\Omega_{t_0}}$.*

Proof. 1. The part (i) is just the weak maximum principle we have proved earlier; it also follows from part (ii). (Explain why?) So we only prove (ii).

2. Let $M = u(x_0, t_0) = \max_{\overline{\Omega_T}} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0, r) \subset \Omega_T$, and we employ the mean value theorem to get

$$M = u(x_0, t_0) = \frac{1}{4r^n}\int_{E(x_0,t_0,r)} u(y,s)\frac{|x_0 - y|^2}{(t_0 - s)^2}dyds \le \frac{M}{4r^n}\int_{E(x_0,t_0,r)} \frac{|x_0 - y|^2}{(t_0 - s)^2}dyds = M.$$

Therefore, $u(y, s) \equiv M$ on $E(x_0, t_0, r)$. Draw any line segment L in Ω_T connecting (x_0, t_0) to some point $(y_0, s_0) \in \Omega_T$ with $s_0 < t_0$. Consider

$$r_0 = \inf\{s \in [s_0, t_0] \mid u(x,t) = M \quad \forall\,(x,t) \in L,\ s \le t \le t_0\}.$$

Since u is continuous, the infimum is attained. We claim $r_0 = s_0$ and hence $u \equiv M$ on L. Suppose, for the contrary, that $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point $(z_0, r_0) \in L$. From the previous argument, $u(x, t) = M$ on $E(z_0, r_0, r)$ for all sufficiently small r. Note that $E(z_0, r_0, r)$ contains $L \cap \{r_0 - \sigma < t \le r_0\}$ for some $\sigma > 0$. We obtain a contradiction.

3. Now fix any $x \in \Omega$ and $t \in (0, t_0)$. We show $u(x, t) = M$. Since Ω is connected, so is Ω_T; hence, there exists a piece-wise continuous line segments $\{L_i\}$ connecting points (x_0, t_0) to (x, t) in such a way that the t-coordinates of the endpoints of L_i are decreasing. By Step 2, we know $u \equiv M$ on each L_i and hence $u(x, t) = M$. Consequently, $u \equiv M$ in Ω_{t_0}. $\square$

Remark 4.3. (a) If a solution u to the heat equation attains its maximum (or minimum) at an interior point (x_0, t_0) then u is constant at all *earlier times* $t \le t_0$. However, the solution may change values at later times $t > t_0$ if, for example, the boundary conditions alter after t_0.

(b) Suppose that u solves the heat equation in Ω_T and equals zero on $\partial\Omega \times [0, T]$. If the initial data $u(x, 0) = g(x)$ is nonnegative and is positive somewhere, then u is positive everywhere within Ω_T. This is another illustration of **infinite speed of propagation** of disturbances for the heat equation.

4.6. Maximum principles for second-order linear parabolic equations

We consider the second-order linear differential operator of the form

$$(\partial_t + L)u = u_t + Lu = u_t - \sum_{ij=1}^{n} a^{ij}(x,t)D_{ij}u + \sum_{i=1}^{n} b^i(x,t)D_i u + c(x,t)u,$$

where a^{ij}, b^i and c are given functions in Ω_T. Without loss of generality, we assume $a^{ij} = a^{ji}$.

Definition 4.4. The operator $\partial_t + L$ is called **parabolic** in Ω_T if there exists $\lambda(x,t) > 0$ in Ω_T such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \lambda(x,t)|\xi|^2 \quad \forall\,(x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n.$$

If $\lambda(x,t) \geq \lambda_0 > 0$ in Ω_T for some constant $\lambda_0 > 0$, we say $\partial_t + L$ is **uniformly parabolic** in Ω_T.

4.6.1. Weak maximum principle.

Theorem 4.16 (Weak maximum principle)**.** *Let Ω be bounded open in $\mathbb{R}^n$. Assume $\partial_t + L$ is parabolic in Ω_T with bounded coefficients in Ω_T. Let $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $u_t + Lu \leq 0$ in Ω_T* (that is, u is a **subsolution of** $\partial_t + L$). *Then*

$$(a) \qquad \max_{\overline{\Omega_T}} u = \max_{\partial'\Omega_T} u \quad \text{if } c(x,t) = 0 \text{ in } \Omega_T;$$

$$(b) \qquad \max_{\overline{\Omega_T}} u \leq \max_{\partial'\Omega_T} u^+ \quad \text{if } c(x,t) \geq 0 \text{ in } \Omega_T.$$

Proof. Fix $\varepsilon > 0$ and let $v = u - \varepsilon t$. Then

$$v_t + Lv = u_t + Lu - [\varepsilon + c(x,t)\varepsilon t] \leq -\varepsilon < 0 \quad \forall\,(x,t) \in \Omega_T$$

in both cases of (a) and (b) above. We claim

$$(4.15) \qquad (a)\ \max_{\overline{\Omega_T}} v = \max_{\partial'\Omega_T} v \ \text{ if } c = 0 \text{ in } \Omega_T; \quad (b)\ \max_{\overline{\Omega_T}} v \leq \max_{\partial'\Omega_T} v^+ \ \text{ if } c \geq 0 \text{ in } \Omega_T.$$

Let $v(x_0,t_0) = \max_{\overline{\Omega_T}} v$ for some $(x_0,t_0) \in \overline{\Omega_T}$. If $(x_0,t_0) \in \Omega_T$, then $Dv(x_0,t_0) = 0$, $v_t(x_0,t_0) \geq 0$ and $(D_{ij}v(x_0,t_0)) \leq 0$ (nonnegative definite in the matrix sense). Hence

$$(4.16) \qquad v_t + Lv = v_t - \sum_{i,j=1}^{n} a^{ij} D_{ij} v + cv \quad \text{at } (x_0,t_0).$$

In case (b) we assume $v(x_0,t_0) = \max_{\overline{\Omega_T}} v > 0$ since otherwise (4.15)(b) is obvious. Therefore, by (4.16), in both cases of (a) and (b) of (4.15), we have $v_t + Lv \geq 0$ at (x_0,t_0), which is contradiction to $v_t + Lv \leq -\varepsilon < 0$. Consequently, we must have $(x_0,t_0) \in \partial'\Omega_T$, which proves (4.15). Finally, by taking $\varepsilon \to 0^+$, the results for u follow. $\square$

Theorem 4.17 (Estimate for general $c(x,t)$)**.** *Let Ω be bounded open in $\mathbb{R}^n$. Assume $\partial_t + L$ is parabolic in Ω_T with bounded coefficients in Ω_T. Let $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $u_t + Lu \leq 0$ in Ω_T. Then*

$$\max_{\overline{\Omega_T}} u \leq e^{CT} \max_{\partial'\Omega_T} u^+,$$

where C is any constant satisfying $C + c(x,t) \geq 0$ in Ω_T.

Proof. The inequality is obvious if $\max_{\overline{\Omega_T}} u \leq 0$. So we assume $\max_{\overline{\Omega_T}} u > 0$. Consider $w(x,t) = e^{-Ct}u(x,t)$. Then

$$w_t + Lw = e^{-Ct}(u_t + Lu) - Ce^{-Ct}u = e^{-Ct}(u_t + Lu) - Cw$$

and hence $w_t + (Lw + Cw) = e^{-Ct}(u_t + Lu) \leq 0$ in Ω_T. Observe that the operator $\partial_t + \tilde{L}$, where $\tilde{L}w = Lw + Cw$, is parabolic and has the zeroth order coefficient $\tilde{c}(x,t) = c(x,t) + C \geq 0$ in Ω_T. Hence, by the weak maximum principle,

$$\max_{\overline{\Omega_T}} w \leq \max_{\partial'\Omega_T} w^+,$$

which implies

$$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}} (e^{Ct} w) \le e^{CT} \max_{\overline{\Omega_T}} w \le e^{CT} \max_{\partial'\Omega_T} w^+ \le e^{CT} \max_{\partial'\Omega_T} u^+,$$

completing the proof. □

Similarly, the **weak minimum principle** with $c(x,t) \ge 0$ and the estimate of minimum with general $c(x,t)$ also hold for a **supersolution** u; namely, $u_t + Lu \ge 0$.

4.6.2. Uniqueness of mixed-value problem. The estimate for general $c(x,t)$ implies the uniqueness of **mixed-value problem** for parabolic equations *regardless the sign of* $c(x,t)$. This is different from the elliptic equations.

Theorem 4.18. *Let Ω be bounded open in $\mathbb{R}^n$. Assume $\partial_t + L$ is parabolic in Ω_T with bounded coefficients in Ω_T. Then, given any f, h and g,* the mixed-value problem

$$\begin{cases} u_t + Lu = f(x,t) & (x,t) \in \Omega_T, \\ u(x,t) = h(x,t) & x \in \partial\Omega,\ t \in [0,T], \\ u(x,0) = g(x) & x \in \Omega \end{cases}$$

can have at most one solution u in $C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$.

4.7. Harnack's inequality

Assume Ω is a bounded open set in $\mathbb{R}^n$. Let

$$(\partial_t + L)u = u_t + Lu = u_t - \sum_{i,j=1}^n a^{ij}(x,t) D_{ij} u + \sum_{i=1}^n b^i(x,t) D_i u + c(x,t) u, \tag{4.17}$$

where a^{ij}, b^i and c are smooth functions on $\overline{\Omega_T}$, with $a^{ij} = a^{ji}$ satisfying, for a constant $\theta > 0$,

$$\sum_{i,j=1}^n a^{ij}(x,t) \xi_i \xi_j \ge \theta |\xi|^2 \quad \forall\ (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n.$$

4.7.1. Harnack's inequality.

Theorem 4.19. *Let $V \subset\subset \Omega$ be connected and $0 < t_1 < t_2 \le T$. Then, there is a constant C depending only on V, t_1, t_2 and the coefficients of L such that*

$$\sup_V u(., t_1) \le C \inf_V u(., t_2)$$

for all nonnegative smooth solutions u of $u_t + Lu = 0$ in Ω_T.

Proof. Without loss of generality, assume V is a ball since in general V can be covered by finitely many balls. Let $\varepsilon > 0$; the idea is to show that there is some $C > 0$, depending only on V, t_1, t_2 and the coefficients of L but independent of ε, such that

$$v(x_2, t_2) - v(x_1, t_1) \ge -C \quad (\forall\, x_1,\, x_2 \in V), \tag{4.18}$$

where $v = \ln(u + \varepsilon)$ and u is any nonnegative smooth solution of $u_t + Lu = 0$ in Ω_T. Once this is proved, we have $u(x_1, t_1) + \varepsilon \le e^C (u(x_2, t_2) + \varepsilon)$ and so letting $\varepsilon \to 0^+$ gives the desired inequality

$$\sup_V u(., t_1) \le e^C \inf_V u(., t_2).$$

Now that

$$v(x_2,t_2)-v(x_1,t_1)=\int_0^1 \frac{d}{ds}v(sx_2+(1-s)x_1,st_2+(1-s)t_1)ds$$
$$=\int_0^1 [(x_2-x_1)\cdot Dv+(t_2-t_1)v_t]ds,$$

and so, to show (4.18), it is enough to show that

$$v_t \ge \nu|Dv|^2 - C \quad \text{in } V\times[t_1,t_2], \tag{4.19}$$

for some constants $\nu>0$ and $C>0$. A direct computation shows that

$$D_jv=\frac{D_ju}{u+\varepsilon},\quad D_{ij}v=\frac{D_{ij}u}{u+\varepsilon}-D_ivD_jv,$$

and so, using $u_t=\sum_{ij}a^{ij}D_{ij}u-\sum_i b^iD_iu-cu$, we have

$$v_t=\frac{u_t}{u+\varepsilon}=\sum_{ij}a^{ij}D_{ij}v+\sum_{ij}a^{ij}D_ivD_jv-\sum_i b^iD_iv-c\frac{u}{u+\varepsilon}$$
$$:=\alpha+\beta+\gamma+f,$$

where

$$\alpha=\sum_{ij}a^{ij}D_{ij}v,\ \ \beta=\sum_{ij}a^{ij}D_ivD_jv,\ \ \gamma=-\sum_i b^iD_iv,\ \ f=-\frac{cu}{u+\varepsilon}.$$

Note that $\beta\ge\theta|Dv|^2$ and

$$|f|\le C,\quad |Df|\le C(1+|Dv|),\quad |D^2f|\le C(1+|D^2v|+|Dv|^2),$$

where constant $C>0$ depends only on $c(x,t)$. Therefore, to establish (4.19), it suffices to show that

$$w:=\alpha+\kappa\beta+\gamma\ge -C \quad \text{on } V\times[t_1,t_2] \tag{4.20}$$

for some constants $C>0$ and $\kappa\in(0,1/2)$ depending only on V,t_1,t_2 and the coefficients of L. To this end, we calculate

$$D_k\alpha=\sum_{ij}a^{ij}D_{ijk}v+\sum_{ij}D_ka^{ij}D_{ij}v:=\sum_{ij}a^{ij}D_{ijk}v+R_1,$$
$$D_k\beta=2\sum_{ij}a^{ij}D_{ik}vD_jv+\sum_{ij}D_ka^{ij}D_ivD_jv:=2\sum_{ij}a^{ij}D_{ik}vD_jv+R_2,$$
$$D_k\gamma=-\sum_i b^iD_{ik}v-\sum_i D_kb^iD_iv:=-\sum_i b^iD_{ik}v+R_3,$$

where $|R_1|\le C|D^2v|,\ |R_2|\le C|Dv|^2,\ |R_3|\le C|Dv|$ with some constant $C>0$ depending only on a^{ij} and b^i. In the following, if not specifically stated, R_k or $\tilde R_k$ will always denote a term satisfying

$$|R_k|,\ |\tilde R_k|\le C(|D^2v||Dv|+|D^2v|+|Dv|^2+|Dv|+1)$$

with a constant $C>0$ depending only on the coefficients of L. Below we will also use C or C_k to denote a constant that depends only on the coefficients of L and $\theta>0$ but could be different in different estimates. Note that

$$D_{kl}\beta=2\sum_{ij}a^{ij}D_{ikl}vD_jv+2\sum_{ij}a^{ij}D_{ik}vD_{jl}v+R_4,$$

and hence

$$
\begin{aligned}
\sum_{kl} a^{kl} D_{kl}\beta &= 2\sum_{ijkl} a^{ij}a^{kl}D_{ikl}vD_jv + 2\sum_{ijkl} a^{ij}a^{kl}D_{ik}vD_{jl}v + \tilde{R}_4 \\
&= 2\sum_{ij} a^{ij}D_jv(D_i\alpha - R_1) + 2\sum_{ijkl} a^{ij}a^{kl}D_{ik}vD_{jl}v + \tilde{R}_4 \\
&= \sum_i \tilde{b}^i D_i\alpha + 2\sum_{ijkl} a^{ij}a^{kl}D_{ik}vD_{jl}v + \tilde{R}_5 \\
&\geq \sum_i \tilde{b}^i D_i\alpha + 2\theta^2|D^2v|^2 + R_5,
\end{aligned}
\tag{4.21}
$$

where $\tilde{b}^i = 2\sum_j a^{ij}D_jv$, using that, for all $n \times n$-symmetric matrices A, B,

$$
\operatorname{tr}(ABAB) \geq \theta^2|B|^2 \quad \text{if } A \geq \theta I_n.
$$

We now calculate, using $v_t = \alpha + \beta + \gamma + f$, that

$$
\begin{aligned}
\alpha_t &= \sum_{kl} a^{kl}D_{kl}v_t + \sum_{kl} a_t^{kl}D_{kl}v \\
&= \sum_{kl} a^{kl}D_{kl}(\alpha+\gamma) + \sum_{kl} a^{kl}D_{kl}\beta + \sum_{kl} a^{kl}D_{kl}f + \sum_{kl} a_t^{kl}D_{kl}v \\
&= \sum_{kl} a^{kl}D_{kl}(\alpha+\gamma) + \sum_i \tilde{b}^i D_i\alpha + 2\sum_{ijkl} a^{ij}a^{kl}D_{ik}vD_{jl}v + R_6 \\
&\geq \sum_{kl} a^{kl}D_{kl}(\alpha+\gamma) + \sum_i \tilde{b}^i D_i\alpha + 2\theta^2|D^2v|^2 + R_6 \\
&\geq \sum_{kl} a^{kl}D_{kl}(\alpha+\gamma) + \sum_i \tilde{b}^i D_i\alpha + \theta^2|D^2v|^2 - C|Dv|^2 - C.
\end{aligned}
\tag{4.22}
$$

Similarly, we compute

$$
\begin{aligned}
\beta_t &= 2\sum_{kl} a^{kl}D_kvD_lv_t + \sum_{kl} a_t^{kl}D_kvD_lv \\
&= 2\sum_{kl} a^{kl}D_kvD_l(\alpha+\beta+\gamma+f) + \sum_{kl} a_t^{kl}D_kvD_lv \\
&= \sum_l \tilde{b}^l D_l\alpha + \sum_l \tilde{b}^l D_l\beta + 2\sum_{kl} a^{kl}D_kvD_l(\gamma+f) + \sum_{kl} a_t^{kl}D_kvD_lv \\
&= \sum_{kl} a^{kl}D_{kl}\beta - 2\sum_{ijkl} a^{ij}a^{kl}D_{ik}vD_{jl}v - \tilde{R}_5 + \sum_l \tilde{b}^l D_l\beta + R_7 \\
&\geq \sum_{kl} a^{kl}D_{kl}\beta + \sum_l \tilde{b}^l D_l\beta - C(|D^2v|^2 + |Dv|^2 + 1),
\end{aligned}
\tag{4.23}
$$

and

$$
\begin{aligned}
\gamma_t &= -\sum_k b^kD_kv_t - \sum_k b_t^kD_kv = -\sum_k b^kD_k(\alpha+\beta+\gamma+f) - \sum_k b_t^kD_kv \\
&= -\sum_k b^kD_k\alpha + R_8,
\end{aligned}
$$

$$
|\sum_l b^lD_l\beta| + |\sum_l b^lD_l\gamma| + |\sum_l \tilde{b}^lD_l\gamma| \leq C(|D^2v||Dv| + |Dv|^2 + 1).
$$

Let $B^k = \tilde{b}^k - b^k$; then, for $w = \alpha + \kappa\beta + \gamma$, with $\kappa \in (0, 1/2)$ sufficiently small, we have

$$w_t - \sum_{kl} a^{kl} D_{kl} w - \sum_k B^k D_k w \geq \frac{\theta^2}{2}|D^2 v|^2 - C|Dv|^2 - C. \tag{4.24}$$

Let $\zeta \in C^\infty(\Omega_T)$ be a cutoff function such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on $V \times [t_1, t_2]$ and $\zeta = 0$ on $\partial'\Omega_T$.

Lemma 4.20. *There is a number $\mu > 0$ depending only on V, t_1, t_2 and the coefficients of L such that $H(x,t) = \zeta^4 w + \mu t \geq 0$ in Ω_T.*

Proof. Suppose that H is negative somewhere on Ω_T. Then, since $H = 0$ on $\partial'\Omega_T$, the negative minimum of H on $\overline{\Omega_T}$ must be attained at some point $(x_0, t_0) \in \Omega_T$. At this point (x_0, t_0), we must have that $\zeta > 0$ and $D_k H = \zeta^3(4wD_k\zeta + \zeta D_k w) = 0$, and so $\zeta D_k w = -4wD_k\zeta$; moreover, at (x_0, t_0),

$$0 \geq H_t - \sum_{kl} a^{kl} D_{kl} H - \sum_k B^k D_k H,$$

which, from $D_{kl}(\zeta^4 w) = \zeta^4 D_{kl} w + D_l(\zeta^4) D_k w + D_k(\zeta^4) D_l w + w D_{kl}(\zeta^4)$, implies

$$\begin{aligned} 0 \geq & \mu + \zeta^4 \Big(w_t - \sum_{kl} a^{kl} D_{kl} w - \sum_k B^k D_k w \Big) \\ & - \sum_k B^k w D_k(\zeta^4) - 2\sum_{kl} a^{kl} D_k w D_l(\zeta^4) - \sum_{kl} a^{kl} w D_{kl}(\zeta^4) \\ \geq & \mu + \zeta^4 \Big(\frac{\theta^2}{2}|D^2 v|^2 - C|Dv|^2 - C \Big) + R_9, \end{aligned} \tag{4.25}$$

in views of (4.24) and that $\zeta D_k w = -4wD_k\zeta$ at (x_0, t_0), where

$$|R_9| \leq C(\zeta^2|w| + \zeta^3|w||Dv|).$$

At (x_0, t_0), since $H = \zeta^4 w + \mu t < 0$, we have $w = \alpha + \kappa\beta + \gamma < 0$ and thus $\kappa\beta \leq -\alpha - \gamma$. Since $\beta \geq \theta|Dv|^2$, we have, at (x_0, t_0), that $|Dv|^2 \leq C(|D^2 v| + |Dv|)$; so the inequalities

$$\begin{aligned} & |Dv|^2 \leq C(1 + |D^2 v|), \quad |w| \leq C(1 + |D^2 v|), \\ & |w||Dv| \leq C(1 + |D^2 v|)^{3/2} \leq C[2^{3/2}|D^2 v|^{3/2} + 2^{3/2}] \end{aligned} \tag{4.26}$$

hold at (x_0, t_0). Consequently, at (x_0, t_0),

$$|R_9| \leq C(\zeta^2|D^2 v| + \zeta^3|D^2 v|^{3/2} + 1) \leq \epsilon\zeta^4|D^2 v|^2 + C(\epsilon),$$

for each $\epsilon > 0$; here we have used **Young's inequality** with $\epsilon > 0$. Now let $\epsilon = \frac{\theta^2}{4}$; then by (4.25) and (4.26) we have

$$\begin{aligned} 0 & \geq \mu + \zeta^4 \Big(\frac{\theta^2}{4}|D^2 v|^2 - C|Dv|^2 - C \Big) - C_1 \\ & \geq \mu + \zeta^4 \Big(\frac{\theta^2}{4}|D^2 v|^2 - C_2|D^2 v| - C_3 \Big) - C_1 \\ & = \mu + \zeta^4 \Big(\frac{\theta}{2}|D^2 v| - \frac{C_2}{\theta} \Big)^2 - \zeta^4 \Big(\frac{C_2^2}{\theta^2} + C_3 \Big) - C_1 \end{aligned}$$

at (x_0, t_0), which implies

$$\mu \leq \zeta^4 \Big(\frac{C_2^2}{\theta^2} + C_3 \Big) + C_1 \leq \frac{C_2^2}{\theta^2} + C_3 + C_1.$$

Therefore, if $\mu = \frac{C_2^2}{\theta^2} + C_3 + C_1 + 1$, then $H(x,t) = \zeta^4 w + \mu t \geq 0$ in Ω_T. □

To complete the proof of our main theorem, let $\mu > 0$ be a constant determined in the lemma above. Since $H = w + \mu t \geq 0$ in $V \times [t_1, t_2]$, we have that $w \geq -\mu t_2 = -C$ in $V \times [t_1, t_2]$. This proves (4.20) and hence completes the proof. $\square$

4.7.2. Strong maximum principle.

Theorem 4.21 (**Strong maximum principle**). *Let $\partial_t + L$ be as given in* (4.17) *and Ω be open, bounded and connected. Assume $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega_T})$, $u_t + Lu \leq 0$ in Ω_T and $u(x_0, t_0) = M = \max_{\overline{\Omega_T}} u$ at some point $(x_0, t_0) \in \Omega_T$. Then*

(i) *$u \equiv M$ in Ω_{t_0} if $c \equiv 0$.*

(ii) *$u \equiv 0$ in Ω_{t_0} if $M = 0$.*

(iii) *$u \equiv M$ in Ω_{t_0} if $c \geq 0$ and $M \geq 0$.*

Proof. 1. Let $c \equiv 0$. Select any smooth open set $W \subset\subset \Omega$ with $x_0 \in W$. Let v be the solution of $v_t + Lv = 0$ in W_T and $v|_{\partial' W_T} = u|_{\partial' W_T}$. (Here and below we assume the existence of such a solution.) Then by the weak maximum principle, we have $u(x,t) \leq v(x,t) \leq M$ for all $(x,t) \in W_T$. From this we deduce that $v(x_0,t_0) = u(x_0,t_0) = M$ is maximum of v. Let $w = M - v(x,t)$. Then $w_t + Lw = 0$ and $w(x,t) \geq 0$ in W_T. Choose any connected set $V \subset\subset W$ with $x_0 \in V$. Let $0 < t < t_0$. Using Harnack's inequality we have a constant $C = C(L, V, t, t_0)$ such that $0 \leq w(x,t) \leq C \inf_{y \in V} w(y, t_0) = Cw(x_0, t_0) = 0$, which implies that $w \equiv 0$ on V_{t_0}. Since V is arbitrary, this implies $w \equiv 0$ in W_{t_0}; but then $v \equiv M$ in W_{t_0}. Since $v = u$ on $\partial' W_T$, we have $u \equiv M$ on $\partial' W_{t_0}$. Since $W \subset\subset \Omega$ is arbitrary, we have $u \equiv M$ in Ω_{t_0}.

2. Let $M = 0$. As in Step 1, select any smooth open set $W \subset\subset \Omega$ with $x_0 \in W$. Let v be the solution of $v_t + Lv = 0$ in W_T and $v|_{\partial' W_T} = u|_{\partial' W_T}$. Then, based on Theorem 4.17, we have $u(x,t) \leq v(x,t) \leq 0$ for all $(x,t) \in W_T$. So $v(x_0,t_0) = u(x_0,t_0) = 0$ is maximum of v. Let $w = -v(x,t)$. Then $w_t + Lw = 0$ and $w(x,t) \geq 0$ in W_T. As in Step 1, Harnack's inequality implies that $w = -v \equiv 0$ in W_{t_0}; but then $u \equiv 0$ on $\partial' W_{t_0}$. Since $W \subset\subset \Omega$ is arbitrary, we have $u \equiv 0$ in Ω_{t_0}.

3. Let $c \geq 0$ and $M > 0$. Let $\tilde{L}$ be the operator obtained from L by dropping the zeroth term. Let v be the solution of $v_t + \tilde{L}v = 0$ in W_T and $v|_{\partial' W_T} = u^+|_{\partial' W_T}$. Then $0 \leq v(x,t) \leq M$. Consider the set $U = \{(x,t) \in W_{t_0} \,|\, u(x,t) > 0\}$ (which may not be a parabolic cylinder). Then $\tilde{L}(u - v) \leq 0$ in the interior of U and $u - v \leq 0$ on $\partial' W_{t_0} \cap \partial U$. As in the proof of weak maximum principle for cylindrical domains, we can prove that $u - v \leq 0$ in U. This implies $v(x_0,t_0) = M$. Let $w = M - v(x,t)$. Then $w_t + \tilde{L}w = 0$ and $w(x,t) \geq 0$ in W_{t_0}. As in Step 1, Harnack's inequality implies that $w \equiv 0$ in W_{t_0}; but then $v \equiv M$ in W_{t_0}. Since $v = u^+$ on $\partial' W_T$, we have $u^+ \equiv M$ on $\partial' W_{t_0}$. Since $W \subset\subset \Omega$ is arbitrary, we have $u^+ \equiv M$ in Ω_{t_0}; hence $u \equiv M$ in Ω_{t_0}. $\square$

Remark 4.4. Assume $u \in C^2_1(\Omega_T) \cap C^1(\overline{\Omega_T})$ and for some $x_0 \in \partial\Omega$ and $0 < t_0 < T$,

$$u(x,t) < u(x_0,t_0) = M \quad \forall\, x \in \Omega,\ 0 < t < t_0.$$

Assume one of the following conditions holds:

$$\text{(a) } c(x,t) \equiv 0; \quad \text{(b) } M = 0; \quad \text{(c) } c(x,t) \geq 0 \text{ and } M > 0.$$

Then the following version of **parabolic Hopf's inequality** holds:

$$\frac{\partial u}{\partial \nu}(x_0,t_0) > 0 \tag{4.27}$$

provided the exterior normal derivative exists at (x_0, t_0) and $\partial\Omega$ satisfies the **interior ball condition** at x_0.

4.7.3. Elliptic Harnack's Inequality. Using Harnack's inequality for the parabolic equation we can derive Harnack's inequality for elliptic equations since when the coefficients of L depend only on x, a solution to elliptic equation $Lu = 0$ can be thought as a steady-state solution of the parabolic equation $u_t + Lu = 0$.

Corollary 4.22 (**Harnack's inequality for elliptic equations**). *Given any connected open bounded subset $V \subset\subset \Omega$, there exists a constant $C(L, V) > 0$ such that*

$$\sup_{x\in V} u(x) \leq C(L, V) \inf_{x\in V} u(x)$$

for all nonnegative solutions $u \in C^2(\Omega)$ to a uniformly elliptic equation $Lu = 0$ in Ω.

Remark 4.5. In the similar way, the elliptic strong maximum principle proved before using **Hopf's lemma** also follows from the parabolic strong maximum principle (Theorem 4.21) or can be similarly proved using elliptic Harnack's inequality Corollary 4.22.

Chapter 5

Wave Equation

In this chapter we investigate the wave equation

$$u_{tt} - \Delta u = 0 \tag{5.1}$$

and the nonhomogeneous wave equation

$$u_{tt} - \Delta u = f(x,t) \tag{5.2}$$

subject to appropriate initial and boundary conditions. Here $x \in \Omega \subset \mathbb{R}^n$, $t > 0$; the unknown function $u = u(x,t) : \Omega \times [0, \infty) \to \mathbb{R}$.

We shall discover that solutions to the wave equation behave quite differently from solutions of Laplaces equation or the heat equation. For example, these solutions are generally not C^∞ and exhibit **finite speed of propagation**.

5.1. Derivation of the wave equation

The wave equation is a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In this physical interpretation $u(x,t)$ represents the displacement in some direction of the point at time $t \geq 0$.

Let V represent any smooth subregion of Ω. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V u dx = \int_V u_{tt} dx,$$

and the net force is

$$\int_{\partial V} \mathbf{F} \cdot \nu dS,$$

where $\mathbf{F}$ denoting the force acting on V through ∂V, ν is the unit outnormal on ∂V. Newton's law says (assume the mass is 1) that

$$\int_V u_{tt} = \int_{\partial V} \mathbf{F} \cdot \nu dS.$$

This identity is true for any region, hence the divergence theorem tells that

$$u_{tt} = \operatorname{div} \mathbf{F}.$$

For elastic bodies, F is a function of Du, i.e., $\mathbf{F} = F(Du)$. For small u and small Du, we use the linearization aDu to approximate $F(Du)$, and so

$$u_{tt} - a\Delta u = 0,$$

when $a = 1$, the resulting equation is the **wave equation**. The physical interpretation strongly suggests it will be mathematically appropriate to specify two initial conditions, $u(x, 0)$ and $u_t(x, 0)$.

5.2. One-dimensional wave equations and d'Alembert's formula

This section will devoted to solve the following Cauchy problem for one-dimensional wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{5.3}$$

where g, h are given functions.

Note that

$$u_{tt} - u_{xx} = (u_t - u_x)_t + (u_t - u_x)_x = v_t + v_x$$

where $v = u_t - u_x$. So $v(x, 0) = u_t(x, 0) - u_x(x, 0) = h(x) - g'(x) := a(x)$. From $v_t + v_x = 0$, we have $v(x, t) = a(x - t)$. Then u solves the nonhomogeneous transport equation

$$u_t - u_x = v(x, t), \quad u(x, 0) = g(x).$$

Solving this problem for u to obtain

$$\begin{aligned} u(x, t) &= g(x + t) + \int_0^t v(x + t - s, s)\, ds \\ &= g(x + t) + \int_0^t a(x + t - 2s)\, ds = g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} a(y)\, dy \\ &= g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} (h(y) - g'(y))\, dy, \end{aligned}$$

from which we deduce **d'Alembert's formula**:

$$u(x, t) = \frac{g(x + t) + g(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy. \tag{5.4}$$

Remark 5.1. (i) Any solution to the wave equation $u_{tt} = u_{xx}$ has the form

$$u(x, t) = F(x + t) + G(x - t)$$

for appropriate functions F and G. Note that $F(x + t)$ is called **traveling wave to the right** with speed 1. Similarly G is called **traveling wave to the left** with speed 1.

(ii) Even when the initial data g and h are not smooth, we still consider formula (5.4) defines a (weak) solution to the Cauchy problem. Note that if $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$ but is not in general smoother. Thus the wave equation does not have the smoothing effect like the heat equation has. However, solutions to the wave equation in one-dimension *do not* lose the regularity from the initial data; this property does not hold for higher-dimensional wave equations.

(iii) The value $u(x_0, t_0)$ depends only on the initial data from $x_0 - t_0$ to $x_0 + t_0$. In this sense, the interval $[x_0 - t_0, x_0 + t_0]$ is called the **domain of dependence** for the point (x_0, t_0). The Cauchy data at $(x_0, 0)$ influence the value of u in the region

$$I(x_0, t_0) = \{(x, t) \mid x_0 - t < x < x_0 + t, t > 0\},$$

which is called the **domain of influence** for $(x_0, 0)$. The **domain of determinacy** of (x_1, x_2) on $t = 0$ is given by

$$D(x_1, x_2) = \left\{(x, t) \;\middle|\; x_1 + t \le x \le x_2 - t,\ t \in \left[0, \frac{x_2 - x_1}{2}\right]\right\}.$$

Lemma 5.1 (Parallelogram property). *Any solution u of the one-dimensional wave equation satisfies*

$$u(A) + u(C) = u(B) + u(D), \tag{5.5}$$

where $ABCD$ is any parallelogram in $\mathbb{R} \times \bar{\mathbb{R}}^+$ with the slope 1 or -1, with A and C being two opposite points as shown in Figure 5.1.

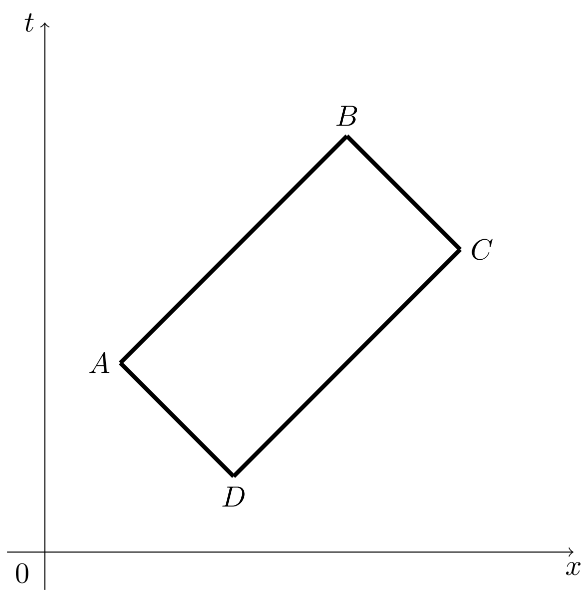

Figure 5.1. Any parallelogram $ABCD$ in Lemma 5.1.

We may use this property to solve certain initial and boundary problems.

EXAMPLE 5.2. Solve the initial boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} = 0, & x > 0,\ t > 0, \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x), & x > 0, \\ u(0, t) = k(t), & t > 0, \end{cases}$$

where g, h, k are given functions satisfying certain smoothness and compatibility conditions.

Solution. (See Figure 5.2.) If point $E = (x, t)$ is in the region (I); that is, $x \ge t > 0$ then $u(x, t)$ is given by (5.4):

$$u(x, t) = \frac{g(x + t) + g(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy.$$

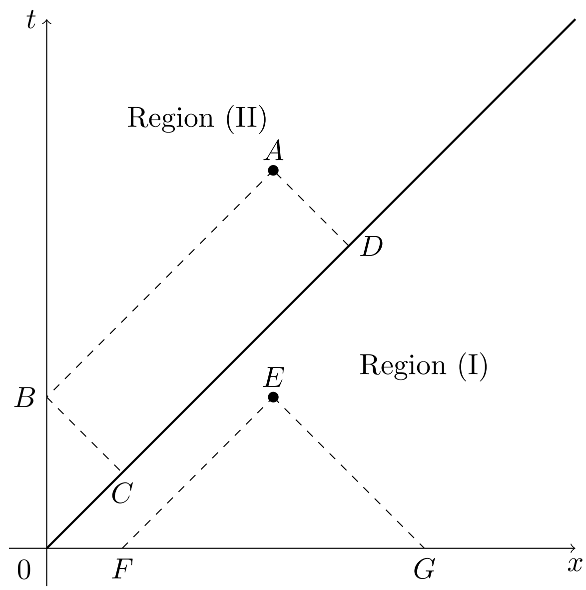

Figure 5.2. Mixed-value problem in $x > 0,\ t > 0$.

In particular,

$$u(x,x) = \frac{g(2x)+g(0)}{2} + \frac{1}{2}\int_0^{2x} h(y)dy.$$

If point $A = (x,t)$ is in the region (II); that is, $0 \le x < t$, then we use the parallelogram $ABCD$ as shown in the figure to obtain $u(x,t) = u(B) + u(D) - u(C)$, where

$$u(B) = u(0, t-x) = k(t-x),$$

$$u(D) = u(\frac{x+t}{2}, \frac{x+t}{2}) = \frac{g(x+t)+g(0)}{2} + \frac{1}{2}\int_0^{x+t} h(y)\,dy,$$

$$u(C) = u(\frac{t-x}{2}, \frac{t-x}{2}) = \frac{g(t-x)+g(0)}{2} + \frac{1}{2}\int_0^{t-x} h(y)\,dy.$$

Hence, for $0 \le x < t$,

$$u(x,t) = k(t-x) + \frac{g(x+t)-g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy.$$

Therefore the solution to this problem is given by

$$u(x,t) = \begin{cases} \dfrac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & (0 \le t \le x), \\ k(t-x) + \dfrac{g(x+t)-g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy & (0 \le x < t). \end{cases} \tag{5.6}$$

Of course we need some smoothness and compatibility conditions on g, h, k in order for $u(x,t)$ to be a true solution of the problem. Derive such conditions as an exercise. $\square$

Example 5.3. Solve the initial-boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,\pi),\ t > 0, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = 0, & t > 0. \end{cases}$$

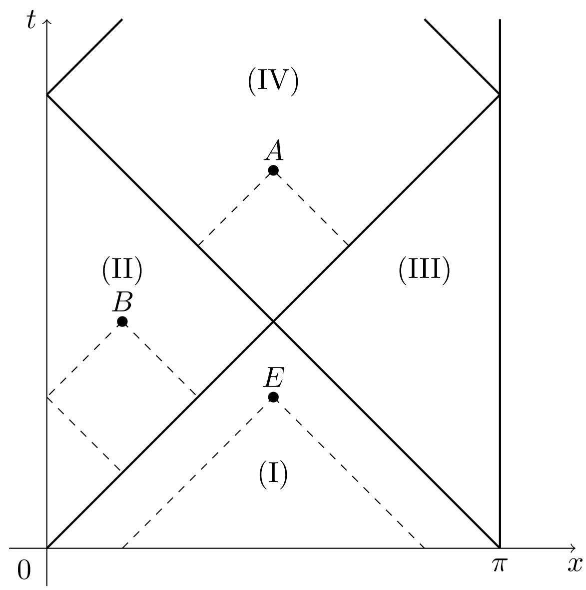

Figure 5.3. Mixed-value problem in $0 < x < \pi,\ t > 0$.

Solution. (See Figure 5.3.) We divide the strip $(0,\pi)\times(0,\infty)$ by line segments of slope ± 1 starting first at $(0,0)$ and $(\pi,0)$ and then at all the intersection points with the boundaries.

We can solve u in the region (I) by formula (5.4). In all other regions we use the parallelogram formula (5.5).

Another way to solve this problem is the method of **separation of variables**. First try $u(x,t) = X(x)T(t)$, then we should have

$$X''(x)T(t) = T''(t)X(x), \quad X(0) = X(\pi) = 0,$$

which implies that

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

where λ is a constant. From

$$X''(x) = \lambda X(x), \quad X(0) = X(\pi) = 0,$$

we find that $\lambda = -j^2$ with all $j = 1, 2, \ldots$, and

$$X_j(x) = \sin(jx), \quad T_j(t) = a_j \cos(jt) + b_j \sin(jt).$$

To make sure that u satisfies the initial condition, we consider

$$u(x,t) = \sum_{j=1}^{\infty} [a_j \cos(jt) + b_j \sin(jt)] \sin(jx). \tag{5.7}$$

To determine coefficients a_j and b_j, we use

$$u(x,0) = g(x) = \sum_{j=1}^{\infty} a_j \sin(jx), \quad u_t(x,0) = h(x) = \sum_{j=1}^{\infty} j b_j \sin(jx).$$

i.e., the a_j and jb_j are Fourier coefficients of functions $g(x)$ and $h(x)$. That is,

$$a_j = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(jx)dx, \quad b_j = \frac{2}{j\pi}\int_0^{\pi} h(x)\sin(jx)dx.$$

Substitute these coefficients into (5.7) and we obtain a formal solution u in terms of trigonometric series; the issue of convergence will not be discussed here. □

5.3. The method of spherical means

Suppose u is a nice solution to the Cauchy problem of n-dimensional wave equation

$$
\begin{cases} u_{tt} - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}^n. \end{cases} \tag{5.8}
$$

The idea is to reduce the problem to a problem of one-dimensional wave equation. This reduction requires the **method of spherical means**.

For $x \in \mathbb{R}$, $r > 0$, $t > 0$, define

$$
U(x;r,t) = ⨍_{\partial B(x,r)} u(y,t)\, dS_y = M_{u(\cdot,t)}(x,r),
$$

$$
G(x;r) = ⨍_{\partial B(x,r)} g(y)\, dS_y = M_g(x,r),
$$

$$
H(x;r) = ⨍_{\partial B(x,r)} h(y)\, dS_y = M_h(x,r).
$$

Note that

$$
U(x;r,t) = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} u(y,t) dS_y = \frac{1}{n\alpha_n} \int_{\partial B(0,1)} u(x + r\xi, t) dS_\xi.
$$

For fixed x, U becomes a function of $t \in \mathbb{R}^+$ and $r \in \mathbb{R}$.

5.3.1. The Euler-Poisson-Darboux equation.

Theorem 5.4 (Euler-Poisson-Darboux equation)**.** *Let $u \in C^m(\mathbb{R}^n \times [0,\infty))$ with $m \geq 2$ solve* (5.8)*. Then, for each $x \in \mathbb{R}^n$, the function $U(x;r,t) \in C^m([0,\infty) \times [0,\infty))$ and solves the Cauchy problem of the* **Euler-Poisson-Darboux equation**

$$
\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \textit{in } (0,\infty) \times (0,\infty), \\ U = G,\ U_t = H & \textit{on } (0,\infty) \times \{t = 0\}. \end{cases} \tag{5.9}
$$

Proof. 1. Note that the regularity of U on t follows easily from that of u on t. To show the regularity of U on r, we compute for $r > 0$

$$
U_r(x;r,t) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\, dy.
$$

From this we deduce $\lim_{r\to 0^+} U_r(x;r,t) = 0$, and computations show that

$$
U_{rr}(x;r,t) = ⨍_{\partial B(x,r)} \Delta u(y,t)\, dS_y + \left(\frac{1}{n} - 1\right) ⨍_{B(x,r)} \Delta u(y,t)\, dy.
$$

Thus $U_{rr}(x;0^+,t) = \frac{1}{n}\Delta u(x,t)$. We can further compute U_{rrr} and other higher-order derivatives of U up to the m-th order at $r = 0$, and this proves that $U \in C^m([0,\infty) \times [0,\infty))$.

2. Using the formula above, we have

$$
U_r = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\, dy = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)\, dy.
$$

Hence

$$r^{n-1}U_r = \frac{1}{n\alpha_n}\int_0^r \left(\int_{\partial B(x,\rho)} u_{tt}(y,t)dS_y\right)d\rho$$

and so, differentiating with respect to r yields

$$(r^{n-1}U_r)_r = \frac{1}{n\alpha_n}\int_{\partial B(x,r)} u_{tt}(y,t)dS_y = r^{n-1}U_{tt},$$

which expands into the Euler-Poisson-Darboux equation. The initial condition is satisfied easily; this completes the proof of the theorem. $\square$

5.3.2. Solution in $\mathbb{R}^3$ and Kirchhoff's formula. For the most important case of three-dimensional wave equations, we can easily see that the Euler-Poisson-Darboux equation implies

$$(rU)_{rr} = rU_{rr} + 2U_r = \frac{1}{r}(r^2U_r)_r = rU_{tt} = (rU)_{tt}.$$

That is, for fixed $x \in \mathbb{R}^3$, the function $\tilde{U}(r,t) = rU(x;r,t)$ solves the 1-D wave equation $\tilde{U}_{tt} = \tilde{U}_{rr}$ in $r > 0$, $t > 0$, with

$$\tilde{U}(0,t) = 0, \quad \tilde{U}(r,0) = rG := \tilde{G}, \quad \tilde{U}(r,0) = rH := \tilde{H}.$$

This is the mixed-value problem studied in Example 5.2; hence, by (5.6) we have

$$\tilde{U}(r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy \quad (0 \le r \le t).$$

We recover $u(x,t)$ by

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\tilde{U}(r,t)}{r} \\ &= \lim_{r\to 0^+}\left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy\right] = \tilde{G}'(t) + \tilde{H}(t). \end{aligned}$$

Therefore we obtain **Kirchhoff's formula for 3-D wave equation**:

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g(y)\,dS_y\right) + t\fint_{\partial B(x,t)} h(y)\,dS_y \\ &= \fint_{\partial B(x,t)} (th(y) + g(y) + Dg(y)\cdot(y-x))\,dS_y \\ &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)} (th(y) + g(y) + Dg(y)\cdot(y-x))\,dS_y \quad (x \in \mathbb{R}^3,\ t > 0). \end{aligned} \tag{5.10}$$

5.3.3. Solution in $\mathbb{R}^2$ by Hadamard's method of descent. Assume $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves problem (5.8) with $n = 2$. We would like to derive a formula of u in terms of g and h. The trick is to consider u as a solution to a 3-dimensional wave problem with one added dimension x_3 and use Kirchhoff's formula to find u. This is the well-known **Hadamard's method of descent**.

Define $\tilde{u}(\tilde{x},t) = u(x,t)$ for $\tilde{x} = (x,x_3) \in \mathbb{R}^3$, $t > 0$, where $x = (x_1,x_2) \in \mathbb{R}^2$. Then $\tilde{u}$ solves

$$\begin{cases} \tilde{u}_{tt} - \tilde{\Delta}\tilde{u} = 0, & (\tilde{x},t) \in \mathbb{R}^3 \times (0,\infty), \\ \tilde{u}(\tilde{x},0) = \tilde{g}(x),\ \tilde{u}_t(\tilde{x},0) = \tilde{h}(x), & \tilde{x} \in \mathbb{R}^3, \end{cases}$$

where $\tilde{\Delta}$ is the Laplacian in $\mathbb{R}^3$, $\tilde{g}(\tilde{x}) = g(x)$ and $\tilde{h}(\tilde{x}) = h(x)$. Let $\bar{x} = (x, 0) \in \mathbb{R}^3$. Then, by Kirchhoff's formula,

$$u(x,t) = \tilde{u}(\bar{x},t) = \frac{\partial}{\partial t}\left(t \fint_{\partial \tilde{B}(\bar{x},t)} \tilde{g}(\tilde{y})\, dS_{\tilde{y}} \right) + t \fint_{\partial \tilde{B}(x,t)} \tilde{h}(\tilde{y})\, dS_{\tilde{y}},$$

where $\tilde{B}(\bar{x},t)$ is the ball in $\mathbb{R}^3$ centered at $\bar{x}$ of radius t. To evaluate this formula, we parametrize $\partial\tilde{B}(\bar{x},t)$ by parameter $z = (z_1, z_2) \in B(x,t) \subset \mathbb{R}^2$ as

$$\tilde{y} = (z, \pm\gamma(z)), \quad \gamma(z) = \sqrt{t^2 - |z-x|^2} \quad (z \in B(x,t)).$$

Note that $D\gamma(z) = (x-z)/\sqrt{t^2 - |z-x|^2}$ and thus

$$dS_{\tilde{y}} = \sqrt{1 + |D\gamma(z)|^2}\, dz = \frac{t\, dz}{\sqrt{t^2 - |z-x|^2}};$$

hence, noting that $\partial B(\tilde{x},t)$ has the top and bottom parts with $y_3 = \pm\gamma(z)$,

$$\begin{aligned}
\fint_{\partial \tilde{B}(\bar{x},t)} \tilde{g}(\tilde{y})\, dS_{\tilde{y}} &= \frac{1}{4\pi t^2} \int_{\partial \tilde{B}(\bar{x},t)} \tilde{g}(\tilde{y})\, dS_{\tilde{y}} \\
&= \frac{2}{4\pi t^2} \int_{B(x,t)} \frac{g(z) t\, dz}{\sqrt{t^2 - |z-x|^2}} \\
&= \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(z)\, dz}{\sqrt{t^2 - |z-x|^2}} \\
&= \frac{t}{2} \fint_{B(x,t)} \frac{g(z)\, dz}{\sqrt{t^2 - |z-x|^2}}.
\end{aligned}$$

Similarly we obtain the formula for $\fint_{\partial \tilde{B}(\bar{x},t)} \tilde{h}(\tilde{y})\, dS_{\tilde{y}}$ and consequently, we obtain **Poisson's formula for 2-D wave equation**:

$$\begin{aligned}
u(x,t) &= \frac{\partial}{\partial t}\left(\frac{t^2}{2} \fint_{B(x,t)} \frac{g(z)\, dz}{\sqrt{t^2 - |z-x|^2}} \right) + \frac{t^2}{2} \fint_{B(x,t)} \frac{h(z)\, dz}{\sqrt{t^2 - |z-x|^2}} \\
&= \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t g(z) + t^2 h(z) + t Dg(z)\cdot(z-x)}{\sqrt{t^2 - |z-x|^2}}\, dz \quad (x \in \mathbb{R}^2,\ t > 0).
\end{aligned} \tag{5.11}$$

Remark 5.2. (i) There are fundamental differences of wave equations for one-dimension and dimensions $n = 2, 3$. In both Kirchhoff's formula ($n = 3$) and Poisson's formula ($n = 2$), the solution u depends on the derivative Dg of the initial data $u(x,0) = g(x)$. For example, if $g \in C^m$, $h \in C^{m-1}$ for some $m \geq 1$ then u is only in C^{m-1} and hence u_t is only in C^{m-2} (but $u_t(x,0) = h(x) \in C^{m-1}$); therefore, there is **loss of regularity** for wave equations when $n = 2, 3$ (in fact for all $n \geq 2$). This does not happen when $n = 1$, where u, u_t are at least as smooth as g, h.

(ii) There are also fundamental differences between 3-D wave equation and 2-D wave equation. In $\mathbb{R}^2$, we need the information of initial data g, h in the whole disk $B(x,t)$ to compute the value $u(x,t)$, while in $\mathbb{R}^3$ we only need the information of g, h on the sphere $\partial B(x,t)$ to compute the value $u(x,t)$. In $\mathbb{R}^3$, a "disturbance" initiated at x_0 propagates along the sharp wavefront $\partial B(x_0,t)$ and does not affect the value of u elsewhere; this is known as the **strong Huygens's principle**. In $\mathbb{R}^2$, a "disturbance" initiated at x_0 will affect the values $u(x,t)$ in the whole region $|x - x_0| \leq t$. In both cases $n = 2, 3$ (in fact all cases $n \geq 1$), the domain of influence of initial data grows (with time t) at speed 1; therefore, wave equation has **finite speed of propagation**.

(iii) To understand the differences between $n = 2$ and $n = 3$ of the wave equation, imagine you are at position x in $\mathbb{R}^n$ and there is a sharp initial disturbance at position x_0 away from you at time $t = 0$. If $n = 3$ then you will only feel the disturbance (e.g., hear a screaming) *once*, exactly at time $t = |x - x_0|$; however, if $n = 2$, you will feel the disturbance (e.g., you are on a boat in a large lake and feel the wave) at *all* times $t \geq |x - x_0|$, although the effect on you will die out at $t \to \infty$.

5.4. Solution of wave equation for general dimensions

As above, we can find the solution $u(x, t)$ of problem (5.8) in $\mathbb{R}^n$ by solving its spherical mean $U(x; r, t)$ from the Euler-Poisson-Darboux equation. We need the following useful result.

Lemma 5.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^{m+1}, where $m \geq 1$. Then for $k = 1, 2, \ldots, m$,*

$$(i) \quad (\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}f(r)) = (\frac{1}{r}\frac{d}{dr})^k(r^{2k}f'(r));$$

$$(ii) \quad (\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}f(r)) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j f}{dr^j}(r),$$

where $\beta_0^k = (2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$ and β_j^k are independent of f.

Proof. Homework. □

5.4.1. Solution for odd-dimensional wave equation. Assume that $n = 2k+1$ ($k \geq 1$) and $u \in C^{k+1}(\mathbb{R}^n \times [0, \infty))$ is a solution to the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0, \infty)$. As above, let $U(x; r, t)$ be the spherical mean of $u(x, t)$. Then for each $x \in \mathbb{R}^n$ the function $U(x; r, t)$ is C^{k+1} in $(r, t) \in [0, \infty) \times [0, \infty)$ and solves the Euler-Poisson-Darboux equation. Set

$$V(r, t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x; r, t)) \quad (r > 0,\ t \geq 0).$$

Lemma 5.6. *We have that $V_{tt} = V_{rr}$ and $V(0^+, t) = 0$ for $r > 0,\ t > 0$.*

Proof. By part (i) of Lemma 5.5, we have

$$\begin{aligned} V_{rr} &= (\frac{\partial^2}{\partial r^2})(\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U) = (\frac{1}{r}\frac{\partial}{\partial r})^k(r^{2k}U_r) \\ &= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[\frac{1}{r}(r^{2k}U_r)_r] = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[r^{2k-1}U_{tt}] = V_{tt}, \end{aligned}$$

where we used the equivalent form of Euler-Poisson-Darboux equation $(r^{2k}U_r)_r = r^{2k}U_{tt}$. That $V(0^+, t) = 0$ follows from part (ii) of Lemma 5.5. □

Now by (5.6), we have

$$V(r, t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{r+t}\tilde{H}(y)dy \quad (0 \leq r \leq t), \tag{5.12}$$

where

$$\tilde{G}(r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x; r, 0)), \quad \tilde{H}(r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U_t(x; r, 0)). \tag{5.13}$$

By (ii) of Lemma 5.5,

$$V(r,t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,t)) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1}\frac{\partial^j}{\partial r^j}U(x;r,t)$$

and hence

$$\begin{aligned} u(x,t) = U(x;0^+,t) &= \lim_{r\to 0^+} \frac{V(r,t)}{\beta_0^k\, r} \\ &= \frac{1}{\beta_0^k}\lim_{r\to 0^+}\left[\frac{\tilde G(r+t)-\tilde G(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{r+t}\tilde H(y)dy\right] \\ &= \frac{1}{\beta_0^k}[\tilde G'(t)+\tilde H(t)]. \end{aligned}$$

Therefore, a solution of the Cauchy problem of **odd-dimensional wave equation**

$$\begin{cases} u_{tt} = \Delta u, \quad (x,t)\in\mathbb{R}^{2k+1}\times(0,\infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x), \quad x\in\mathbb{R}^{2k+1} \end{cases}$$

is given by the formula

$$(5.14) \qquad u(x,t) = \frac{1}{\beta_0^k}\left\{\frac{\partial}{\partial t}\left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1}\Big(t^{2k-1}M_g(x,t)\Big)\right] + (\frac{1}{t}\frac{\partial}{\partial t})^{k-1}\Big(t^{2k-1}M_h(x,t)\Big)\right\}.$$

When $n=3$ (so $k=1$) this formula agrees with Kirchhoff's formula derived earlier.

Theorem 5.7 (Solution of wave equation in odd-dimensions)**.** *If $n = 2k+1 \geq 3$, $g \in C^{k+2}(\mathbb{R}^n)$ and $h \in C^{k+1}(\mathbb{R}^n)$, then the function $u(x,t)$ defined by (5.14) belongs to $C^2(\mathbb{R}^n\times(0,\infty))$, solves the wave equation $u_{tt}=\Delta u$ in $\mathbb{R}^n\times(0,\infty)$, and satisfies the Cauchy condition in the sense that, for each $x_0\in\mathbb{R}^n$,*

$$\lim_{x\to x_0,\, t\to 0^+} u(x,t) = g(x_0), \qquad \lim_{x\to x_0,\, t\to 0^+} u_t(x,t) = h(x_0).$$

Proof. We may separate the proof in two cases: (a) $g\equiv 0$, and (b) $h\equiv 0$. The proof in case (a) is given in the text. So we give a similar proof for case (b) by assuming $h\equiv 0$.

1. The function $u(x,t)$ defined by (5.14) becomes

$$u(x,t) = \frac{1}{\beta_0^k}\frac{\partial}{\partial t}\left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1}\Big(t^{2k-1}G(x;t)\Big)\right], \quad G(x;t) = M_g(x,t).$$

By Lemma 5.5(ii),

$$u(x,t) = \frac{1}{\beta_0^k}\sum_{j=0}^{k-1}\beta_j^k\left[(j+1)t^j\frac{\partial^j G}{\partial t^j} + t^{j+1}\frac{\partial^{j+1}G}{\partial t^{j+1}}\right] \to G(x_0,0^+) = g(x_0)$$

as $(x,t)\to(x_0,0^+)$. Also from this formula,

$$\lim_{x\to x_0, t\to 0^+} u_t(x,t) = \frac{2}{\beta_0^k}\lim_{x\to x_0,t\to 0^+} G_t(x,t).$$

Note that

$$G_t(x,t) = \frac{t}{n}⨍_{B(x,t)}\Delta g(y)\,dy = \frac{1}{n\alpha_n t^{2k}}\int_{B(x,t)}\Delta g(y)\,dy.$$

Hence $u_t(x,t)\to 0$ as $(x,t)\to(x_0,0^+)$.

2. By Lemma 5.5(i),

$$u_t(x,t) = \frac{1}{\beta_0^k}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k (t^{2k}G_t), \quad u_{tt}(x,t) = \frac{1}{\beta_0^k}\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k (t^{2k}G_t). \tag{5.15}$$

Since

$$G_t(x,t) = \frac{1}{n\alpha_n t^{2k}}\int_{B(x,t)} \Delta g(y)\,dy = \frac{1}{n\alpha_n t^{2k}}\int_0^t \left(\int_{\partial B(x,\rho)} \Delta g(y)\,dS_y\right) d\rho,$$

we have

$$(t^{2k}G_t)_t = t^{2k} ⨍_{\partial B(x,t)} \Delta g(y)\,dS_y.$$

Hence

$$u_{tt}(x,t) = \frac{1}{\beta_0^k}\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(\frac{1}{t}(t^{2k}G_t)_t\Big) = \frac{1}{\beta_0^k}\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(t^{2k-1}⨍_{\partial B(x,t)} \Delta g\,dS\Big).$$

On the other hand,

$$\Delta u(x,t) = \frac{1}{\beta_0^k}\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{k-1}\Big(t^{2k-1}\Delta G(x;t)\Big)$$

and

$$\Delta G(x;t) = ⨍_{\partial B(x,t)} \Delta g(y)\,dS_y.$$

This proves $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$. □

Remark 5.3. (i) Let $n = 2k+1 \geq 3$, to compute $u(x,t)$ we need the information of g, $Dg, \cdots, D^k g$ and that of $h, Dh, \cdots, D^{k-1}h$ only on $\partial B(x,t)$ not in the whole ball $B(x,t)$.

(ii) If $n = 1$, in order for u to be C^2, we need $g \in C^2$ and $h \in C^1$. However, if $n = 2k+1 \geq 3$, in order for u to be C^2, we need $g \in C^{k+2}$ and $h \in C^{k+1}$. In general there is a loss of smoothness of k-orders from the initial data.

5.4.2. Solution for even-dimensional wave equation. Assume that $n = 2k$ is even. Suppose u is a C^{k+1} solution to the Cauchy problem (5.8). Again we use **Hadamard's method of descent** similarly as in the case $n = 2$.

Let $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$. Set

$$\tilde{u}(\tilde{x},t) = u(x,t), \quad \tilde{g}(\tilde{x}) = g(x), \quad \tilde{h}(\tilde{x}) = h(x).$$

Then $\tilde{u}$ is a C^{k+1} solution to the wave equation in $\mathbb{R}^{n+1}\times(0,\infty)$ with initial data $\tilde{u}(\tilde{x},0) = \tilde{g}(\tilde{x})$, $\tilde{u}_t(\tilde{x},0) = \tilde{h}(\tilde{x})$. Since $n+1 = 2k+1$ is odd, we use (5.14) to obtain $\tilde{u}(\tilde{x},t)$ and then $u(x,t) = \tilde{u}(\bar{x},t)$, where $\bar{x} = (x,0) \in \mathbb{R}^{n+1}$. In this way, we obtain

$$u(x,t) = \frac{1}{(2k-1)!!}\left\{\frac{\partial}{\partial t}\left[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1}\Big(t^{2k-1}M_{\tilde{g}}(\bar{x};t)\Big)\right] + (\frac{1}{t}\frac{\partial}{\partial t})^{k-1}\Big(t^{2k-1}M_{\tilde{h}}(\bar{x};t)\Big)\right\}.$$

Note that

$$\begin{aligned} M_{\tilde{g}}(\bar{x},t) &= \frac{1}{t^n(n+1)\alpha_{n+1}}\int_{y_{n+1}^2+|x-y|^2=t^2} \tilde{g}(y,y_{n+1})dS \\ &= \frac{2}{t^{n-1}(n+1)\alpha_{n+1}}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy, \end{aligned}$$

since $dS = \frac{t}{\sqrt{t^2-|y-x|^2}}dy$ on the surface $y_{n+1}^2 + |y-x|^2 = t^2$. Similarly we have

$$M_{\tilde{h}}(\bar{x},t) = \frac{2}{t^{n-1}(n+1)\alpha_{n+1}} \int_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy.$$

Therefore, we have the following representation formula for even n:

$$\begin{aligned} u(x,t) = \frac{2}{(n+1)!!\alpha_{n+1}} \Big\{ \frac{\partial}{\partial t}\Big[(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}} \int_{B(x,t)} \frac{g(y)\,dy}{\sqrt{t^2-|y-x|^2}} \Big] \\ + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}} \int_{B(x,t)} \frac{h(y)\,dy}{\sqrt{t^2-|y-x|^2}} \Big\}. \end{aligned} \tag{5.16}$$

When $n = 2$, this reduces to **Poisson's formula for 2-D wave equation** obtained above.

Theorem 5.8 (Solution of wave equation in even-dimensions)**.** *If $n = 2k \ge 2$, $g \in C^{k+2}(\mathbb{R}^n)$ and $h \in C^{k+1}(\mathbb{R}^n)$, then the function $u(x,t)$ defined by* (5.16) *belongs to $C^2(\mathbb{R}^n \times (0,\infty))$, solves the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$, and satisfies the Cauchy condition in the sense that, for each $x_0 \in \mathbb{R}^n$,*

$$\lim_{x\to x_0,\, t\to 0^+} u(x,t) = g(x_0), \qquad \lim_{x\to x_0,\, t\to 0^+} u_t(x,t) = h(x_0).$$

Proof. This follows from the theorem in odd dimensions. $\square$

Remark 5.4. (i) Let $n = 2k \ge 2$, to compute $u(x,t)$ we need the information of g, $Dg, \cdots, D^k g$ and that of $h, Dh, \cdots, D^{k-1}h$ in the solid ball $B(x,t)$.

(ii) If $n = 2k \ge 2$, in order for u to be C^2, we need $g \in C^{k+2}$ and $h \in C^{k+1}$. In general there is again a loss of smoothness of k-orders from the initial data.

5.4.3. Solution of wave equation from the heat equation*. We study another method of solving odd-dimensional wave equations by the heat equation.

Suppose u is a bounded, smooth solution to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), \quad u_t(x,0) = 0 & \text{on } x \in \mathbb{R}^n, \end{cases} \tag{5.17}$$

where n is odd and g is smooth with nice decay at ∞. We extend u to negative times by *even extension* of t and then define

$$v(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} u(x,s)e^{-\frac{s^2}{4t}}\,ds \quad (x \in \mathbb{R}^n,\ t > 0).$$

Then v is bounded,

$$\begin{aligned} \Delta v(x,t) &= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \Delta u(x,s)e^{-\frac{s^2}{4t}}\,ds \\ &= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} u_{ss}(x,s)e^{-\frac{s^2}{4t}}\,ds \\ &= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} u_s(x,s)\frac{s}{2t}e^{-\frac{s^2}{4t}}\,ds \\ &= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} u(x,s)\Big(\frac{s^2}{4t^2} - \frac{1}{2t}\Big)e^{-\frac{s^2}{4t}}\,ds \end{aligned}$$

and

$$v_t(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} u(x,s)\Big(\frac{s^2}{4t^2} - \frac{1}{2t}\Big)e^{-\frac{s^2}{4t}}\,ds.$$

Moreover

$$\lim_{t\to 0^+} v(x,t) = g(x) \quad (x \in \mathbb{R}^n).$$

Therefore, v solves the Cauchy problem for the heat equation:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = g(x) & \text{on } x \in \mathbb{R}^n. \end{cases}$$

As v is bounded, by uniqueness, we have

$$v(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|y-x|^2}{4t}}\, dy \quad (x \in \mathbb{R}^n,\ t>0).$$

We have two formulas for $v(x,t)$ and take $4t = 1/\lambda$ in the two formulas to obtain

$$\begin{aligned} \int_0^\infty u(x,s) e^{-\lambda s^2}\, ds &= \frac{1}{2}\Big(\frac{\lambda}{\pi}\Big)^{\frac{n-1}{2}} \int_{\mathbb{R}^n} e^{-\lambda |y-x|^2} g(y)\, dy \\ &= \frac{n\alpha_n}{2}\Big(\frac{\lambda}{\pi}\Big)^{\frac{n-1}{2}} \int_0^\infty e^{-\lambda r^2} r^{n-1} G(x;r)\, dr \end{aligned} \tag{5.18}$$

for all $\lambda > 0$, where

$$G(x;r) = M_g(x,r) = ⨍_{\partial B(x,r)} g(y)\, dS_y.$$

So far, we have not used the odd dimension assumption. We will solve for u from (5.18) when $n = 2k+1 \geq 3$ is odd. Noticing that $-\frac{1}{2r}\frac{d}{dr}(e^{-\lambda r^2}) = \lambda e^{-\lambda r^2}$, we have

$$\begin{aligned} \lambda^{\frac{n-1}{2}} \int_0^\infty e^{-\lambda r^2} r^{n-1} G(x;r)\, dr &= \int_0^\infty \lambda^k e^{-\lambda r^2} r^{2k} G(x;r)\, dr \\ &= \frac{(-1)^k}{2^k} \int_0^\infty \Big[\Big(\frac{1}{r}\frac{d}{dr}\Big)^k (e^{-\lambda r^2})\Big] r^{2k} G(x;r)\, dr \\ &= \frac{1}{2^k} \int_0^\infty r\Big[\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^k (r^{2k-1} G(x;r))\Big] e^{-\lambda r^2}\, dr, \end{aligned}$$

where we integrated by parts k times (be careful with the operator $(\frac{1}{r}\frac{d}{dr})^k$). We can then write (5.18) as

$$\int_0^\infty u(x,r) e^{-\lambda r^2}\, dr = \frac{n\alpha_n}{\pi^k 2^{k+1}} \int_0^\infty r\Big[\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^k (r^{2k-1} G(x;r))\Big] e^{-\lambda r^2}\, dr$$

for all $\lambda > 0$. If we think of r^2 as τ, then this equation says that the Laplace transforms of two functions of τ are the same; therefore, the two functions of τ must be the same, which also implies the two functions of r are also the sam. So we obtain

$$u(x,t) = \frac{n\alpha_n}{\pi^k 2^{k+1}} t\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^k (t^{2k-1} G(x;t)) = \frac{n\alpha_n}{\pi^k 2^{k+1}} \frac{\partial}{\partial t}\Big[(\frac{1}{t}\frac{\partial}{\partial t})^{k-1}\Big(t^{2k-1} G(x;t)\Big)\Big], \tag{5.19}$$

which, except for the constant, agrees with the formula (5.14) with $h \equiv 0$. In fact the constant here, using $\alpha_n = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}$, $\Gamma(\frac{1}{2}) = \pi^{1/2}$ and $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$,

$$\frac{n\alpha_n}{\pi^k 2^{k+1}} = \frac{(2k+1)\pi^{1/2}}{2^{k+1}\Gamma(k+1+\frac{1}{2})} = \frac{1}{(2k-1)!!} = \frac{1}{\beta_0^k}$$

is also in agreement with the constant in (5.14).

5.5. Nonhomogeneous wave equations and Duhamel's principle

We now turn to the initial value problem for nonhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0)=0, \;\; u_t(x,0)=0 & \text{on } x \in \mathbb{R}^n, \end{cases} \tag{5.20}$$

where $f(x,t)$ is a given function.

Motivated by **Duhamel's principle** used to solve nonhomogeneous heat equations, for each $s \ge 0$, let $U(x,t;s)$ be the solution to the homogeneous Cauchy problem

$$\begin{cases} U_{tt}(x,t;s) - \Delta U(x,t;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty), \\ U(x,s;s)=0, \;\; U_t(x,s;s)=f(x,s) & \text{on } x \in \mathbb{R}^n. \end{cases} \tag{5.21}$$

Define

$$u(x,t) = \int_0^t U(x,t;s)ds \tag{5.22}$$

Note that if $v = v(x,t;s)$ is the solution to the Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0)=0, \;\; v_t(x,0)=f(x,s) & \text{on } x \in \mathbb{R}^n, \end{cases} \tag{5.23}$$

then $U(x,t;s) = v(x,t-s;s)$ for all $x \in \mathbb{R}^n, \; t \ge s$. Therefore,

$$u(x,t) = \int_0^t U(x,t;s)\,ds = \int_0^t v(x,t-s;s)\,ds \quad (x \in \mathbb{R}^n, \; t>0).$$

Theorem 5.9. *Assume $n \ge 2$ and $f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times [0,\infty))$. Let $U(x,t;s)$ be the solution of (5.21). Then the function u defined by (5.22) is in $C^2(\mathbb{R}^n \times [0,\infty))$ and a solution to (5.20).*

Proof. 1. The regularity of f guarantees a solution $U(x,t;s)$ is given by (5.14) if n is odd or (5.16) if n is even. In either case, $u \in C^2(\mathbb{R}^n \times [0,\infty))$.

2. A direct computation shows that

$$u_t(x,t) = U(x,t;t) + \int_0^t U_t(x,t;s)ds = \int_0^t U_t(x,t;s)ds,$$

$$u_{tt}(x,t) = U_t(x,t;t) + \int_0^t U_{tt}(x,t;s)ds = f(x,t) + \int_0^t U_{tt}(x,t;s)ds,$$

$$\Delta u(x,t) = \int_0^t \Delta U(x,t;s)ds.$$

Hence $u_{tt} - \Delta u = f(x,t)$. Clearly $u(x,0) = u_t(x,0) = 0$. □

EXAMPLE 5.10. Find a solution of the following problem

$$\begin{cases} u_{tt} - u_{xx} = te^x, \;\; (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0)=0, \; u_t(x,0)=0. \end{cases}$$

Solution. First, for each $s \ge 0$ solve

$$\begin{cases} v_{tt} = v_{xx} \;\; \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0)=0, \;\; v_t(x,0)=se^x. \end{cases}$$

From d'Alembert's formula, we have

$$v = v(x,t;s) = \frac{1}{2}\int_{x-t}^{x+t} se^y dy = \frac{1}{2}s(e^{x+t} - e^{x-t}).$$

Hence $U(x,t;s) = v(x,t-s;s) = \frac{1}{2}s(e^{x+t-s} - e^{x+s-t})$ and so

$$\begin{aligned} u(x,t) &= \int_0^t U(x,t;s)ds = \frac{1}{2}\int_0^t s(e^{x+t-s} - e^{x+s-t})\, ds \\ &= \frac{1}{2}[e^{x+t}(-te^{-t} - e^{-t} + 1) - e^{x-t}(te^t - e^t + 1)] \\ &= \frac{1}{2}(-2te^x + e^{x+t} - e^{x-t}). \end{aligned}$$

□

Example 5.11. Find a solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f(x,t), & x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Solution. By Kirchhoff's formula, the solution v of the Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = 0, & x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = f(x,s), & x \in \mathbb{R}^3 \end{cases}$$

is given by

$$v = v(x,t;s) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y,s)\, dS_y \quad (x \in \mathbb{R}^3,\ t > 0).$$

Hence

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t-s;s)\, ds = \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dS_y ds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS_y dr \\ &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\, dy. \end{aligned}$$

Note that the domain of dependence (on f) is solid ball $B(x,t)$. □

5.6. Energy methods and the uniqueness

There are some subtle issues about the uniqueness of the Cauchy problem of wave equations. The formulas (5.14) for odd n or (5.16) for even n hold under more and more smoothness conditions of the initial data g, h as the dimension n gets larger and larger. For initial data not too smooth we cannot use such formulas to claim the uniqueness. Instead we use certain quantities that behave nicely for the wave equation. One such quantity is the **energy**.

5.6.1. Domain of dependence. Let $u \in C^2$ be a solution to the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$. Fix $x_0 \in \mathbb{R}^n,\ t_0 > 0$, and consider the **backward wave cone** with apex (x_0, t_0):

$$K(x_0,t_0) = \{(x,t) \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}.$$

Theorem 5.12 (Domain of dependence)**.** *Let $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C^1(\mathbb{R}^n \times [0,\infty))$ solve the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$. If $u = u_t = 0$ on $B(x_0,t_0) \times \{t=0\}$, then $u \equiv 0$ within $K(x_0,t_0)$.*

Proof. Define the local energy

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} \Big(u_t^2(x,t) + |Du(x,t)|^2\Big)dx \quad (0 \le t \le t_0).$$

Then

$$e(t) = \frac{1}{2}\int_0^{t_0-t}\int_{\partial B(x_0,\rho)} \Big(u_t^2(x,t) + |Du(x,t)|^2\Big)dS_x\, d\rho$$

and so, by the divergence theorem,

$$\begin{aligned}
e'(t) &= \int_0^{t_0-t}\int_{\partial B(x_0,\rho)} \Big(u_t u_{tt}(x,t) + Du\cdot Du_t(x,t)\Big)dS_x d\rho \\
&\quad - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \Big(u_t^2(x,t) + |Du(x,t)|^2\Big)dS_x \\
&= \int_{B(x_0,t_0-t)} (u_t u_{tt} + Du\cdot Du_t)dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \Big(u_t^2(x,t) + |Du(x,t)|^2\Big)dS \\
&= \int_{B(x_0,t_0-t)} u_t(u_{tt} - \Delta u)dx + \int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial\nu}dS \\
&\quad - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \Big(u_t^2(x,t) + |Du(x,t)|^2\Big)dS \\
&= \int_{\partial B(x_0,t_0-t)} \Big(u_t\frac{\partial u}{\partial\nu} - \frac{1}{2}u_t^2(x,t) - \frac{1}{2}|Du(x,t)|^2\Big)dS \le 0
\end{aligned} \tag{5.24}$$

because the last integrand is less than zero; in fact,

$$u_t\frac{\partial u}{\partial\nu} - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2 \le |u_t||\frac{\partial u}{\partial\nu}| - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2 \le -\frac{1}{2}(|u_t| - |Du|)^2 \le 0.$$

Now that $e'(t) \le 0$ implies that $e(t) \le e(0) = 0$ for all $0 \le t \le t_0$. Thus $u_t = Du = 0$, and consequently $u \equiv 0$ in $K(x_0,t_0)$. □

Theorem 5.13 (Uniqueness of Cauchy problem for wave equation)**.** *Given any f, g, h the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u = f(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^n \end{cases}$$

can have at most one solution u in $C^2(\mathbb{R}^n \times (0,\infty)) \cap C^1(\mathbb{R}^n \times [0,\infty))$.

Example 5.14. (a) Show that there exists a constant K such that

$$|u(x,t)| \le \frac{K}{t}U(0) \quad \forall\, 0 < t < T$$

whenever $T > 0$ and u is a smooth solution to the Cauchy problem of 3-D wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^3,\ 0 < t < T, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^3, \end{cases}$$

where $U(0) = \int_{\mathbb{R}^3}(|g| + |h| + |Dg| + |Dh| + |D^2 g|)\, dy$.

(b) Let u be a smooth solution to the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3 \times (0,\infty)$ satisfying

$$\lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^3} (|u(x,t)| + |Du(x,t)| + |u_t(x,t)| + |Du_t(x,t)| + |D^2u(x,t)|)\, dx = 0.$$

Show that $u \equiv 0$ on $\mathbb{R}^3 \times (0,\infty)$.

Proof. The details are left as Homework. Part (b) follows from (a) by considering $\tilde{u}(x,t) = u(x, T-t)$ on $\mathbb{R}^n \times (0,T)$. □

5.6.2. Energy method for mixed-value problem of wave equation. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and let $\Omega_T = \Omega \times (0,T]$. Let

$$\Gamma_T = \partial' \Omega_T = \overline{\Omega_T} \setminus \Omega_T.$$

We are interested in the uniqueness of initial-boundary value problem (mixed-value problem)

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } \Gamma_T, \\ u_t = h & \text{on } \Omega \times \{t=0\}. \end{cases} \tag{5.25}$$

Theorem 5.15 (Uniqueness of mixed-value problem for wave equation)**.** *There exists at most one solution $u \in C^2(\Omega_T) \cap C^1(\overline{\Omega_T})$ of* **mixed-value problem** (5.25).

Proof. Let u_1, u_2 be any two solutions; then $v = u_1 - u_2$ is in $C^2(\Omega_T) \cap C^1(\overline{\Omega_T})$ and solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \Omega_T, \\ v = 0 & \text{on } \Gamma_T, \\ v_t = 0 & \text{on } \Omega \times \{t=0\}. \end{cases}$$

Define the energy

$$e(t) = \frac{1}{2} \int_\Omega \left(v_t^2(x,t) + |Dv(x,t)|^2 \right) dx \quad (0 \le t \le T).$$

Then, by the divergence theorem

$$e'(t) = \int_\Omega (v_t v_{tt} + Dv \cdot Dv_t) dx = \int_\Omega v_t (v_{tt} - \Delta v) dx + \int_{\partial\Omega} v_t \frac{\partial v}{\partial \nu} dS = 0$$

since $v = 0$ on $\partial\Omega$ for all $0 < t < T$ implies that $v_t = 0$ on $\partial\Omega$ for all $0 < t < T$. Therefore $v \equiv 0$ from $v = 0$ on Γ_T. □

5.6.3. Other initial and boundary value problems. Uniqueness of wave equation can be used to find the solutions to some mixed-value problems. Since solution is unique, any solution found in special forms will be the unique solution.

Example 5.16. Solve the Cauchy problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = 0, \;\; u_t(x,0) = h(|x|), \end{cases}$$

where $h(r)$ is a given function.

Solution. In theory, we could use Kirchhoff's formula to find the solution; however, the computation would be too complicated. Instead, we can try to find a solution in the form of $u(x,t) = v(|x|,t)$ by solving an equation for v, which becomes exactly the Euler-Poisson-Darboux equation that can be solved easily when $n = 3$; some condition on h is needed in order to have a classical solution. Details are left as an exercise. □

Example 5.17. Let $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Solve

$$\begin{cases} u_{tt} - \Delta u + \lambda u = 0 \ \text{ in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x). \end{cases}$$

Solution. We use the idea of Hadamard's descent method. We first make u a solution to the wave equation in $\mathbb{R}^{n+1} \times (0,\infty)$ and recover u by this solution.

If $\lambda = \mu^2 > 0$ (the equation is called the **Klein-Gordon equation**), let $v(\tilde{x},t) = u(x,t)\cos(\mu x_{n+1})$, where $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$ and $x \in \mathbb{R}^n$.

If $\lambda = -\mu^2 < 0$, let $v(\tilde{x},t) = u(x,t)e^{\mu x_{n+1}}$. Then, in both cases, $v(\tilde{x},t)$ solves the wave equation and can be solved by using the formula (5.14) or (5.16). Then we recover $u(x,t)$ in both cases by $u(x,t) = v(\bar{x},t)$, where $\bar{x} = (x,0) \in \mathbb{R}^{n+1}$. $\square$

Example 5.18. Solve

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^+,\ t > 0, \\ u(x,0) = g(x), \ \ u_t(x,0) = h(x), & x_n > 0, \\ u(x',0,t) = 0, & x' \in \mathbb{R}^{n-1}. \end{cases}$$

Solution. We extend the functions g, h to odd functions $\tilde{g}$ and $\tilde{h}$ in x_n; e.g., $\tilde{g}(x', -x_n) = -\tilde{g}(x', x_n)$ for all $x_n \in \mathbb{R}$ and $\tilde{g}(x', x_n) = g(x', x_n)$ when $x_n > 0$. We then solve

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \tilde{u}(x,0) = \tilde{g}(x), \ \ \tilde{u}_t(x,0) = \tilde{h}(x), & x \in \mathbb{R}^n. \end{cases}$$

Since $V(x,t) = \tilde{u}(x', x_n, t) + \tilde{u}(x', -x_n, t)$ solves

$$V_{tt} - \Delta V = 0, \ \ V(x,0) = 0, \ \ V_t(x,0) = 0,$$

the uniqueness result implies $V \equiv 0$, i.e., $\tilde{u}$ is an odd function in x_n. Hence $u = \tilde{u}|_{x_n > 0}$ is the solution to the original problem. $\square$

Example 5.19. Let Ω be a bounded domain. Solve

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{in } \Omega \times (0,\infty), \\ u(x,0) = g(x), \ \ u_t(x,0) = h(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0. \end{cases}$$

Solution. Use the method of separation variables and try to find the solution of the form

$$u = \sum_{j=1}^{\infty} u_j(x) T_j(t),$$

where

$$-\Delta u_j = \lambda_j u_j, \ \ u_j|_{\partial\Omega} = 0,$$
$$T_j''(t) - \lambda_j T(t) = w_j(t), \ \ T_j(0) = a_j, \ \ T_j'(0) = b_j,$$

By the elliptic theory, eigenfunctions $\{u_j(x)\}_{j=1}^{\infty}$ form an orthonormal basis of $L^2(\Omega)$, and $w_j(t)$, a_j, b_j are the Fourier coefficients of $w(x,t)$, $g(x)$ and $h(x)$ with respect to $u_j(x)$, respectively.

The question is whether the series gives indeed a true solution; we do not study such questions in this course.

$\square$

5.7. Finite speed of propagation for second-order linear hyperbolic equations

We study a class of special second-order linear partial differential equations of the form

$$u_{tt} + Lu = 0 \quad (x \in \mathbb{R}^n,\ t > 0),$$

where L has a special form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u,$$

with smooth symmetric coefficients $(a^{ij}(x))$ satisfying uniform ellipticity condition on $\mathbb{R}^n$. In this case, we say the operator $\partial_{tt} + L$ is **uniformly hyperbolic**.

Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Assume $q(x)$ is a continuous function on $\mathbb{R}^n$, positive and smooth in $\mathbb{R}^n \setminus \{x_0\}$ and $q(x_0) = 0$. Consider a *curved backward cone*

$$C = \{(x,t) \in \mathbb{R}^n \times (0, t_0) \mid q(x) < t_0 - t\}$$

and for each $0 \le t \le t_0$ let

$$C_t = \{x \in \mathbb{R}^n \mid q(x) < t_0 - t\}.$$

Assume $\partial C_t = \{x \in \mathbb{R}^n \mid q(x) = t_0 - t\}$ is a smooth surface for each $t \in [0, t_0)$. In addition, we assume

$$\sum_{i,j=1}^{n} a^{ij}(x) D_i q(x) D_j q(x) \le 1 \quad (x \in \mathbb{R}^n \setminus \{x_0\}). \tag{5.26}$$

Lemma 5.20. *Let $\beta(x,t)$ be a smooth function and*

$$\alpha(t) = \int_{C_t} \beta(x,t)\, dx \quad (0 < t < t_0).$$

Then

$$\alpha'(t) = \int_{C_t} \beta_t(x,t)\, dx - \int_{\partial C_t} \frac{\beta(x,t)}{|Dq(x)|}\, dS_x.$$

Proof. This follows from the **co-area formula**. □

Theorem 5.21 (Domain of dependence)**.** *Let u be a smooth solution to $u_{tt} + Lu = 0$ in $\mathbb{R}^n \times (0, \infty)$. If $u = u_t = 0$ on C_0, then $u \equiv 0$ with the cone C.*

Proof. Define the local energy

$$e(t) = \frac{1}{2} \int_{C_t} \Big(u_t^2 + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u \Big) dx \quad (0 \le t \le t_0).$$

Then the lemma above implies

$$\begin{aligned} e'(t) &= \int_{C_t} \Big(u_t u_{tt} + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u_t \Big) dx - \frac{1}{2} \int_{\partial C_t} \Big(u_t^2 + \sum_{i,j=1}^{n} a^{ij} D_i u D_j u \Big) \frac{1}{|Dq|} dS \\ &:= A - B. \end{aligned}$$

Note that $a^{ij}D_iuD_ju_t = D_j(a^{ij}u_tD_iu) - u_tD_j(a^{ij}D_iu)$ (no sum); hence, integrating by parts and by the equation $u_{tt} + Lu = 0$, we have

$$\begin{aligned} A &= \int_{C_t} u_t\Big(u_{tt} - \sum_{i,j=1}^n D_j(a^{ij}D_iu)\Big)dx + \int_{\partial C_t}\sum_{i,j=1}^n u_ta^{ij}(D_iu)\nu_j dS \\ &= -\int_{C_t} u_t \sum_{i,j=1}^n D_iuD_ja^{ij}\,dx + \int_{\partial C_t}\sum_{i,j=1}^n u_ta^{ij}(D_iu)\nu_j dS, \end{aligned}$$

where $\nu = (\mu_1, \nu_2, \cdots, \nu_n)$ is the outer unit normal on ∂C_t. In fact,

$$\nu_j = \frac{D_jq}{|Dq|} \quad (j = 1, 2, \cdots, n) \quad \text{on } \partial C_t.$$

Since $\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{i,j=1}^n a^{ij}v_iw_j$ defines an inner product on $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, by **Cauchy-Schwartz's inequality**,

$$\Big|\sum_{i,j=1}^n a^{ij}(D_iu)\nu_j\Big| \le \Big(\sum_{i,j=1}^n a^{ij}D_iuD_ju\Big)^{1/2}\Big(\sum_{i,j=1}^n a^{ij}\nu_i\nu_j\Big)^{1/2}.$$

Therefore

$$\begin{aligned} |A| &\le Ce(t) + \int_{\partial C_t} |u_t|\Big(\sum_{i,j=1}^n a^{ij}D_iuD_ju\Big)^{1/2}\Big(\sum_{i,j=1}^n a^{ij}\nu_i\nu_j\Big)^{1/2}dS \\ &\le Ce(t) + \frac{1}{2}\int_{\partial C_t}\Big(u_t^2 + \sum_{i,j=1}^n a^{ij}D_iuD_ju\Big)\Big(\sum_{i,j=1}^n a^{ij}\nu_i\nu_j\Big)^{1/2}dS. \end{aligned}$$

However, since $\nu_j = (D_jq)/|Dq|$, by (5.26), we have

$$\Big(\sum_{i,j=1}^n a^{ij}\nu_i\nu_j\Big)^{1/2} \le \frac{1}{|Dq|} \quad \text{on } \partial C_t.$$

Consequently, we derive that $|A| \le Ce(t) + B$ and thus

$$e'(t) \le Ce(t) \quad (0 < t < t_0).$$

Sine $e(0) = 0$, this gives $e(t) \equiv 0$. Hence $u \equiv 0$ within C. □

Index