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Partial Differential Equations II
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by

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Chapter 1

Part I – Sobolev Spaces

Lecture 1 – 1/7/19

1.1. Overview

The course will cover Chapters 5, 6, and part of 8 of Evans's book, but will have plenty of additional materials and change of order of the material.

These chapters will cover Sobolev spaces, second-order elliptic equations, and some basic material on the calculus of variations.

1.1.1. Motivation: Poisson's equation.

Consider

$$\begin{cases} \Delta u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

For nice domain Ω and nice f, g, this can be solved by Green's formula or Perron's method. What about f not even continuous?

Example 1.1. If $\Omega = (-1, 1)$, $f = 2\mathrm{sgn}(x)$ and $g(-1) = -1, g(1) = 1$, then any solution u must have $u(x) = x^2$ for $x > 0$ and $u(x) = -x^2$ for $x < 0$, but then $u''(0)$ does not exist; hence the problem does not have a classical solution – namely, solutions having all orders of derivatives appearing in the equation at every point of the domain.

Much of the modern theory of PDE is built upon a treatment of the PDE in some reasonable (or physical) ways to lower the order of derivatives for the functions appearing in the equation and define a suitable sense that these functions solve the equation *weakly*; the nutshell is that such a definition of weak solutions recovers the classical solutions when the weak solutions are smooth.

Since for all $u \in C^2(\bar{\Omega})$ and $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \phi \Delta u \, dx = -\int_\Omega Du \cdot D\phi \, dx,$$

so, a $C^2(\bar{\Omega})$ solution to the Poisson equation above must satisfy

$$\int_\Omega (Du \cdot D\phi + f\phi) \, dx = 0 \quad \forall \phi \in C_0^\infty(\Omega). \tag{1.1}$$

This identity only needs, for example, Du exists and is integrable and also f is integrable. This is a way to lower the order of required derivatives in the equation. Even then we still encounter a problem whether given such a f (and nice g) there exists a function u with Du integrable that satisfies the above identity. The space of $u \in C^1(\bar{\Omega})$ is in general not enough for the existence of such u; this requires the study of Sobolev space.

Related to the Poisson equation, let us also consider the following quantity

$$I(u) = \int_\Omega \Big(\frac{1}{2}|Du|^2 + fu\Big)\,dx.$$

This quantity is well-defined if $Du, u, f \in L^2(\Omega)$. If we have such a u that minimizes $I(v)$ among all such v's satisfying the given boundary condition, then we would have that the (quadratic) function

$$h(\epsilon) = I(u + \epsilon\phi) \quad \forall\, \epsilon \in \mathbb{R},\ \phi \in C_0^\infty(\Omega),$$

takes minimum at $\epsilon = 0$. This gives $h'(0) = 0$, which again becomes the identity (1.1) above.

1.1.2. Examples of function spaces. Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 1$. The set $C(\Omega)$ of (real-valued) continuous functions defined on Ω is an infinite dimensional vector space with the usual definitions of addition and scalar multiplication:

$$(f+g)(x) = f(x) + g(x) \quad \text{for} \quad f, g \in C(\Omega),\ x \in \Omega$$

$$(\alpha f)(x) = \alpha f(x) \quad \text{for} \quad \alpha \in \mathbb{R},\ f \in C(\Omega),\ x \in \Omega.$$

$C(\bar{\Omega})$ consists of those functions which are uniformly continuous on Ω. Each such function has a continuous extension to $\bar{\Omega}$. $C_0(\Omega)$ consists of those functions which are continuous in Ω and have compact support in Ω. (The **support** of a function f defined on Ω is the closure of the set $\{x \in \Omega : f(x) \neq 0\}$ and is denoted by $\operatorname{supp}(f)$.) The latter two spaces are clearly subspaces of $C(\Omega)$.

For each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we denote by D^α the partial derivative

$$D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_i = \partial/\partial x_i$$

of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $|\alpha| = 0$, then $D^0 = I$(identity).

For integers $m \geq 0$, let $C^m(\Omega)$ be the collection of all $f \in C(\Omega)$ such that $D^\alpha f \in C(\Omega)$ for all α with $|\alpha| \leq m$. We write $f \in C^\infty(\Omega)$ iff $f \in C^m(\Omega)$ for all $m \geq 0$. For $m \geq 0$, define $C_0^m(\Omega) = C_0(\Omega) \cap C^m(\Omega)$ and let $C_0^\infty(\Omega) = C_0(\Omega) \cap C^\infty(\Omega)$. The spaces $C^m(\Omega), C^\infty(\Omega), C_0^m(\Omega), C_0^\infty(\Omega)$ are all subspaces of the vector space $C(\Omega)$. Similar definitions can be given for $C^m(\bar{\Omega})$ etc.

For $m \geq 0$, define X to be the set of all $f \in C^m(\Omega)$ for which

$$\|f\|_{m,\infty} \equiv \sum_{|\alpha| \leq m} \sup_\Omega |D^\alpha f(x)| < \infty.$$

Then X is a Banach space with norm $\|\cdot\|_{m,\infty}$. To prove, for example, the completeness when $m = 0$, we let $\{f_n\}$ be a Cauchy sequence in X, i.e., assume for any $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that for all $x \in \Omega$

$$\sup_{x\in\Omega} |f_n(x) - f_m(x)| < \varepsilon \quad \text{if} \quad m, n > N(\varepsilon).$$

But this means that $\{f_n(x)\}$ is a uniformly Cauchy sequence of bounded continuous functions, and thus converges uniformly to a bounded continuous function $f(x)$. Letting $m \to \infty$ in the above inequality shows that $\|f_n - f\|_{m,\infty} \to 0$.

Note that the same proof is valid for the set of bounded continuous scalar-valued functions defined on a nonempty subset of a normed space X.

EXAMPLE 1.2. Let Ω be a nonempty Lebesgue measurable set in $\mathbb{R}^n$. For $p \in [1, \infty)$, we denote by $L^p(\Omega)$ the set of equivalence classes of Lebesgue measurable functions on Ω for which

$$\|f\|_p \equiv \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

(Two functions belong to the same equivalence class, i.e., are equivalent, if they differ only on a set of measure 0.) Let $L^\infty(\Omega)$ denote the set of equivalence classes of Lebesgue measurable functions on Ω for which

$$\|f\|_\infty \equiv \text{ess-sup}_{x \in \Omega} |f(x)| < \infty.$$

Then $L^p(\Omega), 1 \leq p \leq \infty$, are Banach spaces with norms $\| \cdot \|_p$. For $p \in [1, \infty]$ we write $f \in L^p_{loc}(\Omega)$ iff $f \in L^p(K)$ for each compact set $K \subset \Omega$.

For the sake of convenience, we will also consider $L^p(\Omega)$ as a set of functions. With this convention in mind, we can assert that $C_0(\Omega) \subset L^p(\Omega)$. In fact, if $p \in [1, \infty)$, then as we shall show later, $C_0(\Omega)$ is dense in $L^p(\Omega)$. The space $L^p(\Omega)$ is also separable if $p \in [1, \infty)$. This follows easily, when Ω is compact, from the last remark and the Weierstrass approximation theorem.

Recall that if $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, then **Hölder's inequality** is that if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

This extends to the **general Hölder's inequality**: If $p_i \in [1, \infty]$ and $\sum_{i=1}^k \frac{1}{p_i} = 1$ then for $f_i \in L^{p_i}(\Omega)$

$$\|f_1 f_2 \cdots f_k\|_{L^1(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} \cdots \|f_k\|_{L^{p_k}(\Omega)}.$$

1.1.3. Banach spaces. A (real) **vector space** is a set X, whose elements are called **vectors**, and in which two operations, **addition** and **scalar multiplication**, are defined as follows:

(a) To every pair of vectors x and y corresponds a vector $x + y$ in such a way that

$$x + y = y + x \qquad \text{and} \qquad x + (y + z) = (x + y) + z.$$

X contains a unique vector 0 (the **zero vector** or **origin** of X) such that $x + 0 = x$ for every $x \in X$, and to each $x \in X$ corresponds a unique vector $-x$ such that $x + (-x) = 0$.

(b) To every pair (α, x), with $\alpha \in \mathbb{R}$ and $x \in X$, corresponds a vector αx in such a way that

$$1x = x, \qquad \alpha(\beta x) = (\alpha\beta)x$$

and such that the two distributive laws

$$\alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x$$

hold.

A nonempty subset M of a vector space X is called a **subspace** of X if $\alpha x + \beta y \in M$ for all $x, y \in M$ and all $\alpha, \beta \in \mathbb{R}$.

A subset M of a vector space X is said to be **convex** if $tx + (1-t)y \in M$ whenever $t \in (0,1)$, $x, y \in M$. (Clearly, every subspace of X is convex.)

Let $x_1, \ldots, x_n$ be elements of a vector space X. The set of all $\alpha_1 x_1 + \cdots + \alpha_n x_n$, with $\alpha_i \in \mathbb{R}$, is called the **span** of $x_1, \ldots, x_n$ and is denoted by $\operatorname{span}\{x_1, \ldots, x_n\}$. The elements $x_1, \ldots, x_n$ are said to be **linearly independent** if $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ implies that $\alpha_i = 0$ for each i; otherwise, the elements $x_1, \ldots, x_n$ are said to be **linearly dependent**. An arbitrary collection of vectors is said to be linearly independent if every finite subset of distinct elements is linearly independent.

The **dimension** of a vector space X, denoted by $\dim X$, is either 0, a positive integer or ∞. If $X = \{0\}$ then $\dim X = 0$; if there exist linearly independent $\{u_1, \ldots, u_n\}$ such that $\operatorname{span}\{x_1, \ldots, x_n\} = X$, then $\dim X = n$ and $\{u_1, \ldots, u_n\}$ is called a **basis** for X; in all other cases $\dim X = \infty$.

1.1.4. Normed Spaces. A (real) vector space X is said to be a **normed space** if to every $x \in X$ there is associated a nonnegative real number $\|x\|$, called the **norm** of x, in such a way that

(a) $\|x + y\| \leq \|x\| + \|y\|$ for all x and y in X (**Triangle inequality**)
(b) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and all $\alpha \in \mathbb{R}$
(c) $\|x\| > 0$ if $x \neq 0$.

Note that (b) and (c) imply that $\|x\| = 0$ iff $x = 0$. Moreover, it easily follows from (a) that

$$|\|x\| - \|y\|| \leq \|x - y\| \quad \forall\, x, y \in X.$$

1.1.5. Completeness and Banach Spaces. A sequence $\{x_n\}$ in a normed space X is called a **Cauchy sequence** if, for each $\epsilon > 0$, there exists an integer N such that $\|x_m - x_n\| < \epsilon$ for all $m, n \geq N$.

We say a sequence $\{x_n\}$ converges to x in X and write $x_n \to x$ if $\lim_{n\to\infty} \|x_n - x\| = 0$ and, in this case, x is called the **limit** of $\{x_n\}$.

A normed space X is called **complete** if every Cauchy sequence in X converges to a limit in X.

A complete (real) normed space is called a (real) **Banach space**. A Banach space is **separable** if it contains a countable dense set. It can be shown that a subspace of a separable Banach space is itself separable.

1.1.6. Hilbert Spaces. Let H be a real vector space. H is said to be an **inner product space** if to every pair of vectors x and y in H there corresponds a real-valued function (x, y), called the **inner product** of x and y, such that

(a) $(x, y) = (y, x)$ for all $x, y \in H$
(b) $(x + y, z) = (x, z) + (y, z)$ for all $x, y, z \in H$
(c) $(\lambda x, y) = \lambda(x, y)$ for all $x, y \in H$, $\lambda \in \mathbb{R}$
(d) $(x, x) \geq 0$ for all $x \in H$, and $(x, x) = 0$ if and only if $x = 0$.

For $x \in H$ we set

$$\|x\| = (x, x)^{1/2}. \tag{1.2}$$

Theorem 1.3. *If H is an inner product space, then for all x and y in H, it follows that*

(a) $|(x, y)| \leq \|x\| \, \|y\|$ (**Cauchy-Schwarz inequality**);

(b) $\|x+y\| \le \|x\| + \|y\|$ (**Triangle inequality**);
(c) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (**Parallelogram law**).

Proof. (a) is obvious if $x = 0$, and otherwise it follows by taking $\delta = -(x,y)/\|x\|^2$ in

$$0 \le \|\delta x + y\|^2 = |\delta|^2\|x\|^2 + 2\delta(x,y) + \|y\|^2.$$

This identity, with $\delta = 1$, and (a) imply (b). (c) follows easily by using (1.2). $\square$

If H is complete under this norm, then H is said to be a **Hilbert space**.

EXAMPLE 1.4. The space $L^2(\Omega)$ is a Hilbert space with inner product

$$(f,g) = \int_\Omega f(x)g(x)\,dx \quad \forall\, f,g \in L^2(\Omega).$$

LECTURE 2 – 1/9/19

1.2. Sobolev Spaces

1.2.1. Hölder Spaces. Let $f\colon \Omega \to \mathbb{R}$ and $0 < \gamma \le 1$. The the **α-th Hölder seminorm** of f is defined by

$$[f]_{C^{0,\gamma}(\Omega)} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

Definition 1.1. For $k \in \mathbb{N}^+, 0 < \gamma \le 1$, the **Hölder space** $C^{k,\gamma}(\bar\Omega)$ is the set of all functions $f \in C^k(\bar\Omega)$ for which the **norm**

$$\|f\|_{C^{k,\gamma}(\bar\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha f\|_{C^k(\bar\Omega)} + \sum_{|\alpha|=k} [D^\alpha f]_{C^{0,\gamma}(\Omega)}$$

is finite.

Theorem 1.5. *The Hölder space $C^{k,\gamma}(\bar\Omega)$ is a Banach space under the given norm.*

Proof. Exercise. $\square$

1.2.2. Weak Derivatives. Let Ω be a nonempty open set in $\mathbb{R}^n$. Suppose $u \in C^m(\Omega)$ and $\varphi \in C_0^m(\Omega)$. Then by integration by parts

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx, \quad |\alpha| \le m \tag{1.3}$$

where $v = D^\alpha u$. Motivated by (1.3), we now enlarge the class of functions for which the notion of derivative can introduced.

Definition 1.2. Let $u \in L^1_{loc}(\Omega)$. A function $v \in L^1_{loc}(\Omega)$ is called the **α^{th}-weak partial derivative** of u, written $v = D^\alpha u$, provided

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx \text{ for all } \varphi \in C_0^\infty(\Omega). \tag{1.4}$$

Lemma 1.6. *An α^{th}-weak partial derivative $D^\alpha u$, if exists, then must be unique in $L^1_{loc}(\Omega)$.*

Proof. Suppose $v, \tilde{v}$ are both the α^{th}-weak partial derivative of u. Then

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx = (-1)^{|\alpha|} \int_\Omega \tilde{v}\varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Then

$$\int_\Omega (v - \tilde{v})\varphi dx = 0 \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

This implies $v = \tilde{v}$ in $L^1_{loc}(\Omega)$. □

If a function u has an ordinary α^{th}-partial derivative lying in $L^1_{loc}(\Omega)$, then it is clearly the α^{th}-weak partial derivative.

In contrast to the corresponding classical derivative, the weak partial derivative $D^\alpha u$ is defined at once for the order α without assuming the existence of the corresponding partial derivatives of lower orders. In fact, the weak partial derivatives of lower orders may not exist.

Example 1.7. Let $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ and let

$$u(x) = \begin{cases} 1 & x \in (0,1) \times (0,1), \\ -1 & x \in (-1,0) \times (-1,0), \\ 0 & \text{elsewhere.} \end{cases}$$

Show the weak partial derivatives $D^{(1,0)}u = u_{x_1}$ and $D^{(0,1)}u = u_{x_2}$ do not exist, but $D^{(1,1)}u = u_{x_1 x_2} = 0$ exists.

Proof. Exercise. □

Example 1.8. (a) The function $u(x) = |x_1|$ has in the ball $\Omega = B(0,1)$ the weak partial derivatives $u_{x_1} = \text{sgn } x_1, u_{x_i} = 0, i = 2, \ldots, n$. In fact, for any $\varphi \in C_0^1(\Omega)$

$$\int_\Omega |x_1| \varphi_{x_1} dx = \int_{\Omega^+} x_1 \varphi_{x_1} dx - \int_{\Omega^-} x_1 \varphi_{x_1} dx$$

where $\Omega^+ = \Omega \cap (x_1 > 0)$, $\Omega^- = \Omega \cap (x_1 < 0)$. Since $x_1\varphi = 0$ on $\partial\Omega$ and also for $x_1 = 0$, an application of the divergence theorem yields

$$\int_\Omega |x_1| \varphi_{x_1} dx = -\int_{\Omega^+} \varphi dx + \int_{\Omega^-} \varphi dx = -\int_\Omega (\text{sgn } x_1)\varphi dx.$$

Hence $|x_1|_{x_1} = \text{sgn } x_1$. Similarly, since for $i \geq 2$

$$\int_\Omega |x_1| \varphi_{x_i} dx = \int_\Omega (|x_1|\varphi)_{x_i} dx = -\int_\Omega 0\varphi dx$$

$|x_1|_{x_i} = 0$ for $i = 2, \ldots, n$. Note that the function $|x_1|$ has no classical derivative with respect to x_1 in Ω.

(b) By the above computation, the function $u(x) = |x|$ has the weak derivative $u'(x) = \text{sgn } x$ on the interval $\Omega = (-1,1)$. On the other hand, $\text{sgn } x$ does not have a weak derivative on Ω due to the discontinuity at $x = 0$.

LECTURE 3 – 1/11/19

1.2.3. Sobolev Spaces. Fix $1 \le p \le \infty$ and let k be a nonnegative integer. We define

$$W^{k,p}(\Omega) = \{u : u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega),\ 0 < |\alpha| \le k\},$$

where $D^\alpha u$ denotes the α^{th} weak derivative. When $k = 0$, $W^{k,p}(\Omega)$ will mean $L^p(\Omega)$.

A norm on $W^{k,p}(\Omega)$ is defined by

$$\|u\|_{k,p} = \|u\|_{W^{k,p}(\Omega)} = \begin{cases} (\int_\Omega \sum_{|\alpha|\le k} |D^\alpha u|^p dx)^{1/p} & \text{if } 1 \le p < \infty, \\ \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{if } p = \infty. \end{cases} \tag{1.5}$$

The space $W^{k,p}(\Omega)$ with this norm is called the **Sobolev space of order** k **and power** p.

We define the space $W_0^{k,p}(\Omega)$ to be the closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{k,p}$.

Remark 1.3. The spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are special since they become a Hilbert space under the inner product

$$(u,v)_{k,2} = (u,v)_{W^{k,2}(\Omega)} = \int_\Omega \sum_{|\alpha|\le k} D^\alpha u D^\alpha v dx.$$

Since we shall be dealing mostly with these spaces in the sequel, we introduce the special notation:

$$H^k(\Omega) = W^{k,2}(\Omega),\quad H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

EXAMPLE 1.9. Let $\Omega = B(0,1/2) \subset \mathbb{R}^2$ and define $u(x) = \ln(\ln(2/r)),\ x \in \Omega$, where $r = |x| = (x_1^2 + x_2^2)^{1/2}$. Then $u \in H^1(\Omega)$ but $u \notin L^\infty(\Omega)$.

Proof. First of all u is unbounded near $x = 0$; next $u \in L^2(\Omega)$, for

$$\int_\Omega |u|^2 dx = \int_0^{2\pi} \int_0^{1/2} r[\ln(\ln(2/r))]^2 dr d\theta$$

and a simple application of L'Hopital's rule shows that the integrand is bounded and thus the integral is finite. Similarly, it is easy to check that the classical partial derivative

$$u_{x_1} = \frac{-\cos\theta}{r\ln(2/r)},\ \text{ where } x_1 = r\cos\theta$$

also belongs to $L^2(\Omega)$. Now we show that the defining equation for the weak derivative is met. So, u_{x_1} is also the weak x_1-derivative of u and $u_{x_1} \in L^2(\Omega)$. Similarly, weak $x-2$-derivative $u_{x_2} \in L^2(\Omega)$; hence by definition $u \in W^{1,2}(\Omega) = H^1(\Omega)$.

To show u_{x_1} is the weak x_1-derivative of u, let $\Omega_\varepsilon = \{x : \varepsilon < r < 1/2\}$ and choose $\varphi \in C_0^\infty(\Omega)$. Then by the divergence theorem and the absolute continuity of integrals

$$\int_\Omega u\varphi_{x_1} dx = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} u\varphi_{x_1} dx = \lim_{\varepsilon\to 0} \left[-\int_{\Omega_\varepsilon} u_{x_1}\varphi dx + \int_{r=\varepsilon} u\varphi n_1 ds\right]$$

where $n = (n_1, n_2)$ is the unit outward normal to Ω_ε on $r = \varepsilon$. But $(ds = \varepsilon d\theta)$

$$|\int_{r=\varepsilon} u\varphi n_1 ds| \le \int_0^{2\pi} |u(\varepsilon)|\,|\varphi|\varepsilon d\theta \le 2\pi\varepsilon c\ln(\ln(2/\varepsilon)) \to 0$$

as $\varepsilon \to 0$. Thus

$$\int_\Omega u\varphi_{x_1} dx = -\int_\Omega u_{x_1}\varphi dx.$$

□

Lecture 4 – 1/14/19

From the definition of weak partial derivatives, we can verify certain properties of Sobolev functions that are usually true for smooth functions.

Theorem 1.10. **(Properties of Sobolev functions)** *Assume $u, v \in W^{k,p}(\Omega)$. Then*

(1) *$D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi-indices α, β with $|\alpha| + |\beta| \leq k$.*

(2) *For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ for all $|\alpha| \leq k$.*

(3) *If U is an open subset of Ω then $u \in W^{k,p}(U)$, with the same $D^\alpha u$ as the restrictions on U.*

(4) *If $\varphi \in C_0^k(\Omega)$ then $\varphi u \in W^{k,p}(\Omega)$ and the* **Leibniz formula** *holds: for all $|\alpha| \leq k$,*

$$D^\alpha(\varphi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta \varphi)(D^{\alpha-\beta} u), \tag{1.6}$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Proof. Direct deduction from the definition of weak partial derivatives. The property (2) asserts that $W^{k,p}(\Omega)$ is a vector space. □

Theorem 1.11. *$W^{k,p}(\Omega)$ is a Banach space under the norm* (1.5).

Proof. We prove that $W^{k,p}(\Omega)$ is complete with respect to the norm (1.5). We prove this for $1 \leq p < \infty$; the case $p = \infty$ is similar. Let $\{u_n\}$ be a Cauchy sequence of elements in $W^{k,p}(\Omega)$, i.e.,

$$\|u_n - u_m\|_{k,p}^p = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u_n - D^\alpha u_m|^p dx \to 0 \ \text{ as } \ m, n \to \infty.$$

Then for any $\alpha,\ |\alpha| \leq k$, when $m, n \to \infty$

$$\int_\Omega |D^\alpha u_n - D^\alpha u_m|^p dx \to 0$$

and, in particular, when $|\alpha| = 0$

$$\int_\Omega |u_n - u_m|^p dx \to 0.$$

Since $L^p(\Omega)$ is complete, it follows that there are functions $u^\alpha \in L^p(\Omega),\ \ |\alpha| \leq k$ such that $D^\alpha u_n \to u^\alpha$ in $L^p(\Omega)$. Note that, for each $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega u D^\alpha \varphi dx \leftarrow \int_\Omega u_n D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega \varphi D^\alpha u_n dx \to (-1)^{|\alpha|} \int_\Omega u^\alpha \varphi dx.$$

Hence $u^\alpha = D^\alpha u^0 \in L^p(\Omega)$, and thus $u^0 \in W^{k,p}(\Omega)$. As $D^\alpha u_n \to u^\alpha = D^\alpha u^0$ in $L^p(\Omega)$ for all $|\alpha| \leq k$, it follows that $\|u_n - u^0\|_{k,p} \to 0$ as $n \to \infty$. This proves the completeness of $W^{k,p}(\Omega)$; hence it is a Banach space. □

1.3. Approximations

1.3.1. Mollifiers. Let $x \in \mathbb{R}^n$ and let $B(x,h)$ denote the open ball with center at x and radius h. For each $h > 0$, let $\omega_h(x) \in C^\infty(\mathbb{R}^n)$ satisfy

$$\omega_h(x) \geq 0;\ \ \operatorname{supp}\omega_h \subseteq \bar{B}(0,h),\ \ \int_{\mathbb{R}^n} \omega_h(x)dx = \int_{B(0,h)} \omega_h(x)dx = 1.$$

Such functions are called **mollifiers**. For example, let

$$\omega(x) = \begin{cases} k \exp\,[(|x|^2-1)^{-1}], & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $k > 0$ is chosen so that $\int_{\mathbb{R}^n} \omega(x)\,dx = 1$. Then, a family of mollifiers can be taken as $\omega_h(x) = h^{-n}\omega(x/h)$ for $h > 0$.

Let Ω be a nonempty open set in $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. We set $u = 0$ outside Ω. Define for each $h > 0$ the **mollified function**

$$u_h(x) = \int_\Omega \omega_h(x-y)u(y)dy$$

where ω_h is a mollifier.

Remark 1.4. There are two other forms in which u_h can be represented, namely

$$u_h(x) = \int_{\mathbb{R}^n} \omega_h(x-y)u(y)dy = \int_{B(x,h)} \omega_h(x-y)u(y)dy \tag{1.7}$$

the latter equality being valid since ω_h vanishes outside the (open) ball $B(x,h)$. Thus the values of $u_h(x)$ depend only on the values of u on the ball $B(x,h)$. In particular, if $\operatorname{dist}(x, \operatorname{supp}(u)) \geq h$, then $u_h(x) = 0$.

Theorem 1.12. *Let Ω be a nonempty open set in $\mathbb{R}^n$. Then*

(a) $u_h \in C^\infty(\mathbb{R}^n)$.

(b) *If* $\operatorname{supp}(u)$ *is a compact subset of Ω, then $u_h \in C_0^\infty(\Omega)$ for all h sufficiently small.*

Proof. Since u is integrable and $\omega_h \in C^\infty$, the Lebesgue theorem on differentiating integrals implies that for $|\alpha| < \infty$

$$D^\alpha u_h(x) = \int_\Omega u(y)D^\alpha\omega_h(x-y)dy$$

i.e., $u_h \in C^\infty(\mathbb{R}^n)$. Statement (b) follows from the remark preceding the theorem. □

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With respect to a bounded set Ω we construct another set $\Omega^{(h)}$ as follows: with each point $x \in \Omega$ as center, draw a ball of radius h; the union of these balls is then $\Omega^{(h)}$. Clearly $\Omega^{(h)} \supset \Omega$. Moreover, u_h can be nonzero only in $\Omega^{(h)}$.

Corollary 1.13. *Let Ω be a nonempty bounded open set in $\mathbb{R}^n$ and let $h > 0$ be any number. Then there exists a function $\eta \in C^\infty(\mathbb{R}^n)$ such that*

$$0 \leq \eta(x) \leq 1;\ \ \eta(x) = 1,\ x \in \Omega^{(h)};\ \ \eta(x) = 0,\ x \in (\Omega^{(3h)})^c.$$

Such a function is called a **cut-off function** *for Ω.*

Proof. Let $\chi(x)$ be the characteristic function of the set $\Omega^{(2h)}$: $\chi(x) = 1$ for $x \in \Omega^{(2h)}$, $\chi(x) = 0$ for $x \notin \Omega^{(2h)}$ and set

$$\eta(x) \equiv \chi_h(x) = \int_{\mathbb{R}^n} \omega_h(x-y)\chi(y)dy.$$

Then

$$\eta(x) = \int_{\Omega^{(2h)}} \omega_h(x-y)dy \in C^\infty(\mathbb{R}^n),$$

$$0 \le \eta(x) \le \int_{\mathbb{R}^n} \omega_h(x-y)dy = 1,$$

and

$$\eta(x) = \int_{B(x,h)} \omega_h(x-y)\chi(y)dy = \begin{cases} \int_{B(x,h)} \omega_h(x-y)dy = 1, & x \in \Omega^{(h)}, \\ 0, & x \in (\Omega^{(3h)})^c. \end{cases}$$

□

In particular, if $\Omega' \subset\subset \Omega$, then there is a function $\eta \in C_0^\infty(\Omega)$ such that

$$\eta(x) = 1 \text{ for } x \in \overline{\Omega'}, \text{ and } 0 \le \eta(x) \le 1 \text{ in } \Omega.$$

Henceforth, the notation $\Omega' \subset\subset \Omega$ means that Ω', Ω are open sets, Ω' is bounded, and that $\overline{\Omega'} \subset \Omega$.

1.3.2. Approximation Theorems.

Lemma 1.14. *Let Ω be a nonempty bounded open set in $\mathbb{R}^n$. Then every $u \in L^p(\Omega)$ is p-**mean continuous**, i.e.,*

$$\int_\Omega |u(x+z) - u(x)|^p dx \to 0 \text{ as } z \to 0.$$

Proof. Choose $a > 0$ large enough so that Ω is strictly contained in the ball $B(0,a)$. Then the function

$$U(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in B(0,2a) \setminus \Omega \end{cases}$$

belongs to $L^p(B(0,2a))$. For $\varepsilon > 0$, there is a function $\bar{U} \in C(\bar{B}(0,2a))$ which satisfies the inequality $\|U - \bar{U}\|_{L^p(B(0,2a))} < \varepsilon/3$. By multiplying $\bar{U}$ by an appropriate cut-off function, it can be assumed that $\bar{U}(x) = 0$ for $x \in B(0,2a) \setminus B(0,a)$. Therefore for $|z| \le a$,

$$\|U(x+z) - \bar{U}(x+z)\|_{L^p(B(0,2a))} = \|U(x) - \bar{U}(x)\|_{L^p(B(0,a))} \le \varepsilon/3.$$

Since function $\bar{U}$ is uniformly continuous in $B(0,2a)$, there is a $0 < \delta < a$ such that $\|\bar{U}(x+z) - \bar{U}(x)\|_{L^p(B(0,2a))} \le \varepsilon/3$ whenever $|z| < \delta$. Hence for $|z| < \delta$ we easily see that $\|u(x+z) - u(x)\|_{L^p} = \|U(x+z) - U(x)\|_{L^p(B(0,2a))} \le \varepsilon$. □

Theorem 1.15. *Let Ω be a nonempty open set in $\mathbb{R}^n$. If $u \in L^p(\Omega)$ $(1 \le p < \infty)$, then*

(a) $\|u_h\|_p \le \|u\|_p$

(b) $\|u_h - u\|_p \to 0$ *as* $h \to 0$.

If $u \in C^k(\bar{\Omega})$ and $\bar{\Omega}$ is compact, then, for all $\Omega' \subset\subset \Omega$,

(c) $\|u_h - u\|_{C^k(\bar{\Omega}')} \to 0$ *as* $h \to 0$.

Proof. 1. If $1 < p < \infty$, let $q = p/(p-1)$. Then $\omega_h = \omega_h^{1/p}\omega_h^{1/q}$ and Hölder's inequality implies

$$\begin{aligned} |u_h(x)|^p &\leq \int_\Omega \omega_h(x-y)|u(y)|^p dy \left(\int_\Omega \omega_h(x-y)dy\right)^{p/q} \\ &\leq \int_\Omega \omega_h(x-y)|u(y)|^p dy \end{aligned}$$

which obviously holds also for $p = 1$. An application of Fubini's Theorem gives

$$\int_\Omega |u_h(x)|^p dx \leq \int_\Omega \left(\int_\Omega \omega_h(x-y)dx\right)|u(y)|^p dy \leq \int_\Omega |u(y)|^p dy$$

which implies (a).

2. To prove (b), let $\omega(x) = h^n \omega_h(hx)$. Then $\omega(x) \in C^\infty(\mathbb{R}^n)$ and satisfies

$$\omega(x) \geq 0; \;\; \omega(x) = 0 \;\text{ for }\; |x| \geq 1$$

$$\int_{\mathbb{R}^n} \omega(x)dx = \int_{B(0,1)} \omega(x)dx = 1.$$

Using the change of variable $z = (x-y)/h$ we have

$$\begin{aligned} u_h(x) - u(x) &= \int_{B(x,h)} [u(y) - u(x)]\omega_h(x-y)dy \\ &= \int_{B(0,1)} [u(x-hz) - u(x)]\omega(z)dz. \end{aligned}$$

Hence by Hölder's inequality

$$|u_h(x) - u(x)|^p \leq d\int_{B(0,1)} |u(x-hz) - u(x)|^p dz$$

and so by Fubini's Theorem

$$\int_\Omega |u_h(x) - u(x)|^p dx \leq d\int_{B(0,1)} (\int_\Omega |u(x-hz) - u(x)|^p dx)dz.$$

The right-hand side goes to zero as $h \to 0$ since every $u \in L^p(\Omega)$ is p-mean continuous.

3. We now prove (c) for $k = 0$. Let Ω', Ω'' be such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Let h_0 be the shortest distance between $\partial\Omega'$ and $\partial\Omega''$. Take $h < h_0$. Then

$$u_h(x) - u(x) = \int_{B(x,h)} [u(y) - u(x)]\omega_h(x-y)dy.$$

If $x \in \bar{\Omega}'$, then in the above integral $y \in \bar{\Omega}''$. Now u is uniformly continuous in $\bar{\Omega}''$ and $\omega_h \geq 0$, and therefore for an arbitrary $\varepsilon > 0$ we have

$$|u_h(x) - u(x)| \leq \varepsilon \int_{B(x,h)} \omega_h(x-y)dy = \varepsilon$$

provided h is sufficiently small. The case $k \geq 1$ is handled similarly and is left as an exercise. □

Remark 1.5. In (c) of the theorem above, we cannot replace Ω' by Ω. Let $u \equiv 1$ for $x \in [0,1]$ and consider $u_h(x) = \int_0^1 \omega_h(x-y)dy$, where $\omega_h(y) = \omega_h(-y)$. Now $\int_{-h}^h \omega_h(y)dy = 1$ and so $u_h(0) = 1/2$ for all $h < 1$. Thus $u_h(0) \to 1/2 \neq 1 = u(0)$. Moreover, for $x \in (0,1)$ and h sufficiently small, $(x-h, x+h) \subset (0,1)$ and so $u_h(x) = \int_{x-h}^{x+h} \omega_h(x-y)dy = 1$ which implies $u_h(x) \to 1$ for all $x \in (0,1)$.

Corollary 1.16. *Let Ω be a nonempty open set in $\mathbb{R}^n$. Then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \le p < \infty$.*

Proof. Suppose first that Ω is bounded and let $\Omega' \subset\subset \Omega$. For a given $u \in L^p(\Omega)$ set

$$v(x) = \begin{cases} u(x), & x \in \Omega' \\ 0, & x \in \Omega \backslash \Omega'. \end{cases}$$

Then

$$\int_\Omega |u-v|^p dx = \int_{\Omega\backslash\Omega'} |u|^p dx.$$

By the absolute continuity of integrals, we can choose Ω' so that the integral on the right is arbitrarily small, i.e., $\|u-v\|_p < \varepsilon/2$. Since $\mathrm{supp}(v)$ is a compact subset of Ω, Theorems 1.12(b) and 1.15(b) imply that for h sufficiently small, $v_h(x) \in C_0^\infty(\Omega)$ with $\|v - v_h\|_p < \varepsilon/2$, and therefore $\|u - v_h\|_p < \varepsilon$. If Ω is unbounded, choose a ball B large enough so that

$$\int_{\Omega\backslash\Omega'} |u|^p dx < \varepsilon/2$$

where $\Omega' = \Omega \cap B$, and repeat the proof just given. $\square$

1.3.3. Local Interior Approximation. We now consider the following local approximation theorem.

Theorem 1.17. *Let Ω be a nonempty open set in $\mathbb{R}^n$ and suppose $u, v \in L^1_{loc}(\Omega)$. Then $v = D^\alpha u$ if and only if for each compact set $S \subset \Omega$ there exists a sequence of functions $\{u_h\}$ in $C^\infty(\Omega)$ such that $\|u_h - u\|_{L^1(S)} \to 0$, $\|D^\alpha u_h - v\|_{L^1(S)} \to 0$ as $h \to 0$.*

Proof. 1. (Necessity) Suppose $v = D^\alpha u$. Let $S \subset \Omega$ be compact, and choose $d > 0$ small enough so that the sets $\Omega' \equiv S^{(d/2)}, \Omega'' \equiv S^{(d)}$ satisfy $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. For $x \in \mathbb{R}^n$ define

$$u_h(x) = \int_{\Omega''} \omega_h(x-y)u(y)dy, \quad v_h(x) = \int_{\Omega''} \omega_h(x-y)v(y)dy.$$

Clearly, $u_h, v_h \in C^\infty(\mathbb{R}^n)$ for $h > 0$. Moreover, from Theorem 1.15 we have $\|u_h - u\|_{L^1(S)} \le \|u_h - u\|_{L^1(\Omega'')} \to 0$. Now we note that if $x \in \Omega'$ and $0 < h < d/2$, then $\omega_h(x-y) \in C_0^\infty(\Omega'')$. Thus by Theorem 1.12 and the definition of weak derivative,

$$\begin{aligned} D^\alpha u_h(x) &= \int_{\Omega''} u(y) D_x^\alpha \omega_h(x-y)dy = (-1)^{|\alpha|} \int_{\Omega''} u(y) D_y^\alpha \omega_h(x-y)dy \\ &= \int_{\Omega''} \omega_h(x-y) \cdot v(y)dy = v_h(x). \end{aligned}$$

Thus, $\|D^\alpha u_h - v\|_{L^1(S)} \to 0$.

2. (Sufficiency) Let $\varphi \in C_0^\infty(\Omega)$; we claim

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx.$$

To see this, choose a compact subset S of Ω such that $S \supset \mathrm{supp}(\varphi)$. Let $\{u_h\}$ be the sequence as given. Then as $h \to \infty$

$$\int_\Omega u D^\alpha \varphi\, dx \leftarrow \int_\Omega u_h D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega \varphi D^\alpha u_h dx \to (-1)^{|\alpha|} \int_\Omega v\varphi dx,$$

which is the claim. $\square$

Theorem 1.18. *Let Ω be a domain in $\mathbb{R}^n$. If $u \in L^1_{loc}(\Omega)$ has a weak derivative $D^\alpha u = 0$ in Ω whenever $|\alpha| = 1$, then $u =$const. a.e. in Ω.*

Proof. Given a subdomain $\Omega' \subset\subset \Omega$, choose $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and define

$$u_h(x) = \int_{\Omega''} \omega_h(x-y)u(y)dy \qquad (x \in \mathbb{R}^n).$$

Let $x \in \Omega'$ and $0 < h < \mathrm{dist}(\Omega', \partial\Omega'')$. Then the function $\varphi(y) = \omega_h(x-y)$ is in $C_0^\infty(\Omega'')$. Since $Du = 0$ in Ω weakly, it follows that

$$Du_h(x) = \int_{\Omega''} D\omega_h(x-y)u(y)\,dy = -\int_\Omega u(y)D\varphi(y)\,dy = \int_\Omega Du(y)\varphi(y)\,dy = 0$$

for all $x \in \Omega'$ and $0 < h < \mathrm{dist}(\Omega', \partial\Omega'')$. Thus $u_h(x) = c(h)$, a constant, in Ω' for each $0 < h < \mathrm{dist}(\Omega', \partial\Omega'')$. Since $\|u_h - u\|_{L^1(\Omega')} = \|c(h) - u\|_{L^1(\Omega')} \to 0$ as $h \to 0$, it follows that

$$\|c(h_1) - c(h_2)\|_{L^1(\Omega')} = |c(h_1) - c(h_2)|\mathrm{mes}(\Omega') \to 0 \ \text{ as } \ h_1, h_2 \to 0.$$

Consequently, $c(h) \to c$ in $\mathbb{R}$ as $h \to 0$. Hence $u(x) = c$ a.e. in Ω'; therefore, we have proved that u is constant on any subdomain $\Omega' \subset\subset \Omega$. However, since Ω is connected, u must be constant on Ω. □

We now note some properties of $W^{k,p}(\Omega)$ which follow easily from the results of this and the previous section.

(a) If $\Omega' \subset \Omega$ and if $u \in W^{k,p}(\Omega)$, then $u \in W^{k,p}(\Omega')$.

(b) If $u \in W^{k,p}(\Omega)$ and $|a(x)|_{k,\infty} < \infty$, then $au \in W^{k,p}(\Omega)$. In this case any weak derivative $D^\alpha(au)$ is computed according to the usual Leibniz' rule of differentiating the product of functions.

(c) If $u \in W^{k,p}(\Omega)$ and u_h is its mollified function, then for any compact set $S \subset \Omega$, $\|u_h - u\|_{W^{k,p}(S)} \to 0$ as $h \to 0$. If in addition, u has compact support in Ω, then $\|u_h - u\|_{k,p} \to 0$ as $h \to 0$.

1.3.4. Chain Rules.

Theorem 1.19. *Let Ω be an open set in $\mathbb{R}^n$. Let $f \in C^1(\mathbb{R})$, $|f'(s)| \le M$ for all $s \in \mathbb{R}$ and suppose u has a weak partial derivative $D^\alpha u$ for some $|\alpha| = 1$. Then the composite function $f \circ u$ has the α-weak partial derivative $D^\alpha(f \circ u) = f'(u)D^\alpha u$. Moreover, if $f(0) = 0$ and if $u \in W^{1,p}(\Omega)$, then $f \circ u \in W^{1,p}(\Omega)$.*

Proof. 1. Let $S \subset \Omega$ be any compact set. According to Theorem 1.17, there exists a sequence $\{u_h\} \subset C^\infty(\Omega)$ such that $\|u_h - u\|_{L^1(S)} \to 0$, $\|D^\alpha u_h - D^\alpha u\|_{L^1(S)} \to 0$ as $h \to 0$. Thus

$$\int_S |f(u_h) - f(u)|dx \le \sup|f'| \int_{\Omega'} |u_h - u|dx \to 0 \text{ as } h \to 0,$$

and

$$\int_S |f'(u_h)D^\alpha u_h - f'(u)D^\alpha u|dx \le \sup|f'| \int_S |D^\alpha u_h - D^\alpha u|dx$$
$$+ \int_S |f'(u_h) - f'(u)||D^\alpha u|dx.$$

Since $\|u_h - u\|_{L^1(S)} \to 0$, there exists a subsequence of $\{u_h\}$, which we call $\{u_h\}$ again, which converges a.e. in S to u. Moreover, since f' is continuous, one has $f'(u_h(x)) \to f'(u(x))$ a.e. in S. Hence the last integral tends to zero by the dominated convergence theorem. Consequently, the sequences $\{f(u_h)\}$, $\{f'(u_h)D^\alpha u_h\}$ converge to $f(u)$, $f'(u)D^\alpha u$ in $L^1(S)$ respectively, and the first conclusion follows by an application of Theorem 1.17 again.

2. If $f(0) = 0$, the mean value theorem implies $|f(s)| \le M|s|$ for all $s \in \mathbb{R}$. Thus, $|f(u(x))| \le M|u(x)|$ for all $x \in \Omega$ and so $f \circ u \in L^p(\Omega)$ if $u \in L^p(\Omega)$. Similarly, $f'(u(x))D^\alpha u \in L^p(\Omega)$ if $u \in W^{1,p}(\Omega)$, which shows that $f \circ u \in W^{1,p}(\Omega)$. □

Corollary 1.20. *Let Ω be a bounded open set in $\mathbb{R}^n$. If u has an α^{th} weak derivative $D^\alpha u$ for some $|\alpha| = 1$, then so does $|u|$ and*

$$D^\alpha |u| = \begin{cases} D^\alpha u & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -D^\alpha u & \text{if } u < 0 \end{cases}$$

i.e., $D^\alpha|u| = (sgnu)D^\alpha u$ with the properly defined sgn function. In particular, if $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$.

Proof. The positive and negative parts of u are defined by

$$u^+ = \max\{u, 0\}, \quad u^- = \min\{u, 0\}.$$

We show that if $D^\alpha u$ exists then $D^\alpha u^+$ exists and that

$$D^\alpha u^+ = \begin{cases} D^\alpha u & \text{if } u > 0, \\ 0 & \text{if } u \le 0. \end{cases} \tag{1.8}$$

Then the result for $|u|$ follows easily from the relations $|u| = u^+ - u^-$ and $u^- = -(-u)^+$. To prove (1.8), for $h > 0$ define

$$f_h(s) = \begin{cases} (s^2 + h^2)^{\frac{1}{2}} - h & \text{if } s > 0 \\ 0 & \text{if } s \le 0. \end{cases}$$

Clearly $f_h \in C^1(\mathbb{R})$ and f_h' is bounded on $\mathbb{R}$. By Theorem 1.19, $f_h(u)$ has a weak derivative, and for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f_h(u) D^\alpha \varphi dx = -\int_\Omega D^\alpha(f_h(u))\varphi dx = -\int_{u>0} \varphi \frac{uD^\alpha u}{(u^2 + h^2)^{\frac{1}{2}}} dx.$$

Upon letting $h \to 0$, it follows that $f_h(u) \to u^+$, and so by the dominating convergence theorem

$$\int_\Omega u^+ D^\alpha \varphi dx = -\int_{u>0} \varphi D^\alpha u dx = -\int_\Omega v\varphi dx,$$

where

$$v = \begin{cases} D^\alpha u & \text{if } u > 0, \\ 0 & \text{if } u \le 0, \end{cases}$$

which establishes the desired result for u^+. □

Theorem 1.21. *If $u_m \to u$ in $W^{1,p}(\Omega)$ then $u_m^+ \to u^+$ in $W^{1,p}(\Omega)$. Moreover, if $u \in W_0^{1,p}(\Omega)$, then $u^+, u^-, |u| \in W_0^{1,p}(\Omega)$.*

Proof. Without loss of generality, we assume $u_m(x) \to u(x)$ as $m \to \infty$ for all $x \in \Omega'$, where $\Omega' \subset \Omega$ is measurable and $|\Omega \setminus \Omega'| = 0$. Clearly, from $|a^+ - b^+| \le |a - b|$ for all $a, b \in \mathbb{R}$, we have

$$\|u_m^+ - u^+\|_{L^p(\Omega)} \le \|u_m - u\|_{L^p(\Omega)} \to 0.$$

We only prove the case when $1 \le p < \infty$; the case $p = \infty$ can be proved by taking $p \to \infty$. By (1.8), we have

$$\|Du_m^+ - Du^+\|_{L^p(\Omega)}^p = \|Du_m - Du\|_{L^p(A_m)}^p + \|Du\|_{L^p(B_m)}^p + \|Du_m\|_{L^p(C_m)}^p$$

$$\le \|Du_m - Du\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(B_m)} + (\|Du_m - Du\|_{L^p(C_m)} + \|Du\|_{L^p(C_m)})^p$$

$$\le (1+2^p)\|Du_m - Du\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(B_m)} + 2^p\|Du\|^p_{L^p(C_m)},$$

where $A_m = \{x \in \Omega' \mid u_m(x) > 0,\ u(x) \ge 0\}$ and

$$B_m = \{x \in \Omega' \mid u_m(x) \le 0,\ u(x) > 0\}, \quad C_m = \{x \in \Omega' \mid u_m(x) > 0,\ u(x) < 0\}.$$

As $u_m(x) \to u(x)$ on Ω', it follows that (**Real Analysis Exercise!**)

$$\lim_{m\to\infty} |B_m| = \lim_{m\to\infty} |C_m| = 0.$$

Hence $\|Du\|_{L^p(B_m)} + \|Du\|_{L^p(C_m)} \to 0$ and thus

$$\lim_{m\to\infty} \|Du_m^+ - Du^+\|_{L^p(\Omega)} = 0.$$

This proves $u_m^+ \to u^+$ in $W^{1,p}(\Omega)$.

Finally, let $u \in W_0^{1,p}(\Omega)$ and $u_m \in C_0^\infty(\Omega)$ be such that $u_m \to u$ in $W^{1,p}(\Omega)$. Clearly $u_m^+ \in W_0^{1,p}(\Omega)$ and $u_m^+ \to u^+$ in $W^{1,p}(\Omega)$; this proves $u^+ \in W_0^{1,p}(\Omega)$. The conclusion for u^- and $|u|$ follows easily from the identities $u^- = -(-u)^+$ and $|u| = u^+ - u^-$. □

1.3.5. Partition of Unity. To establish a global interior approximation for a Sobolev function in $W^{k,p}(\Omega)$, we need the following well-known result of **partition of unity**.

Theorem 1.22. (Partition of Unity) *Let $\Omega = \cup_{\lambda\in\Lambda}\Omega_\lambda$, where $\{\Omega_\lambda\}$ is a collection of open sets. Then there exist C^∞ functions $\psi_i(x)$ $(i = 1, 2, \dots)$ such that for each i*

(a) $0 \le \psi_i(x) \le 1$,

(b) $\operatorname{supp}\psi_i \subset\subset \Omega_{\lambda_i}$ *for some* $\lambda_i \in \Lambda$,

(c) $\sum_{i=1}^\infty \psi_i(x) = 1$ *for every* $x \in \Omega$,

(d) $\forall\, x \in \Omega\ \exists\, B(x,r)$ *such that* $\exists\, k,\ \psi_i|_{B(x,r)} \equiv 0\ \ \forall\, i \ge k$.

Such a sequence $\{\psi_i\}$ is called a C^∞ **partition of unity for Ω subordinate to** *$\{\Omega_\lambda\}$.*

Proof. Let $U_m = B(0,m) \cap \Omega$ for $m = 1, 2, \dots$ and $U_0 = U_{-1} = \emptyset$. Then, for each $x \in \Omega$, there exist $m = m(x) \ge 0$ and $\lambda = \lambda(x) \in \Lambda$ such that

$$x \in \bar U_{m+1} \setminus U_m \subset U_{m+2} \setminus \bar U_{m-1}, \quad x \in Q_{\lambda(x)}.$$

So we choose $r_x > 0$ such that $\bar B(x, r_x) \subset \Omega_{\lambda(x)} \cap (U_{m+2} \setminus \bar U_{m-1})$. Let $\phi_x \in C^\infty(B(x,r_x))$ be such that $0 \le \phi_x \le 1$ and $\phi_x|_{B(x,r_x/2)} = 1$.

For each $m \ge 0$, since $\bar U_{m+1} \setminus U_m$ is compact, we choose a finite covering of this set by balls $\{B(x, r_x/2)\}$. Let $\{B(x_i, r_{x_i}/2)\}$ be the collection of all such finite coverings over all $m = 0, 1, 2, \dots$, and let ϕ_i be the corresponding cut-off functions on $B(x_i, r_{x_i})$. Clearly, a small ball $B(x,r)$ will only intersect a finite number of the balls $\{B(x_i, r_{x_i})\}$, so for some k, $\phi_i|_{B(x,r)} \equiv 0$ for all $i \ge k$. This implies that the sum $\psi(x) = \sum_{i=1}^\infty \phi_i(x)$ is locally a finite sum and thus is smooth in Ω, and also $\psi(x) > 0$ for each $x \in \Omega$. Thus the functions $\psi_i = \frac{\phi_i}{\psi}$ provide the partition of unity. □

1.3.6. Global Interior Approximation. The following global interior approximation theorem is known as the **Meyers-Serrin Theorem**: $H = W$.

Theorem 1.23. (Meyers-Serrin Theorem) *Let Ω be open in $\mathbb{R}^n$, $u \in W^{k,p}(\Omega)$ and $1 \le p < \infty$. Then there exist functions $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that*

$$u_m \to u \ \ in \ \ W^{k,p}(\Omega).$$

In other words, $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Proof. 1. We have $\Omega = \cup_{i=1}^\infty \Omega_i$, where $\Omega_i = \{x \in \Omega : |x| < i, \ \mathrm{dist}(x, \partial\Omega) > 1/i\}$. Set $V_i = \Omega_{i+3} \setminus \bar{\Omega}_{i+1}$. Choose an open set $V_0 \subset\subset \Omega$ so that $\Omega = \cup_{i=0}^\infty V_i$.

2. Let $\{\zeta_i\}_{i=0}^\infty$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^\infty$; that is,

$$\zeta_i \in C_0^\infty(V_i), \ \ 0 \le \zeta_i \le 1, \ \ \sum_{i=0}^\infty \zeta_i = 1 \ \text{ on } \Omega.$$

Let $u \in W^{k,p}(\Omega)$. Then $\zeta_i u \in W^{k,p}(\Omega)$ and $\mathrm{supp}(\zeta_i u) \subset V_i$.

3. Fix $\delta > 0$. Choose $\epsilon_i > 0$ so small that $u^i = \omega_{\epsilon_i} * (\zeta_i u)$ satisfies

$$\|u^i - \zeta_i u\|_{W^{k,p}(\Omega)} < \frac{\delta}{2^{i+1}}, \quad \mathrm{supp}\, u^i \subset W_i := \Omega_{i+4} \setminus \bar{\Omega}_i.$$

Note that $V_i \subset W_i$. Let $v = \sum_{i=0}^\infty u^i$. For each $U \subset\subset \Omega$, $W_i \cap U = \emptyset$ for all sufficiently large i; so $v|_U$ is a finite sum and thus $v \in C^\infty(U)$, which implies $v \in C^\infty(\Omega)$. Also, as $u = \sum_{i=0}^\infty \zeta_i u$ in Ω, we have

$$\|v - u\|_{W^{k,p}(U)} \le \sum_{i=0}^\infty \|u^i - \zeta_i u\|_{W^{k,p}(\Omega)} \le \delta \sum_{i=0}^\infty \frac{1}{2^{i+1}} = \delta.$$

This is independent of $U \subset\subset \Omega$; hence $\|v - u\|_{W^{k,p}(\Omega)} \le \delta$. □

Remark 1.6. The result holds when $\Omega = \mathbb{R}^n$. From this one can show that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$ for $1 \le p < \infty$. However for bounded domains Ω it follows that $W^{1,p}(\Omega) \ne W_0^{1,p}(\Omega)$ (see the trace operator later).

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Example 1.24. Is the approximation true when $p = \infty$?

No. Let $\Omega = (-1, 1)$ and $u(x) = |x|$. Show there is no function $v \in C^\infty(\Omega)$ such that $\|u' - v'\|_{L^\infty} < 1/4$.

1.3.7. Approximation up to the Boundary.

Theorem 1.25. *Let Ω be open, bounded and $\partial\Omega \in C^1$. Let $u \in W^{k,p}(\Omega)$, $1 \le p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{\Omega})$ such that*

$$u_m \to u \ \ in \ \ W^{k,p}(\Omega).$$

In other words, $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Proof. See Evans's textbook. □

Exercise 1.7. Prove the product rule for weak derivatives:

$$D_i(uv) = (D_i u)v + u(D_i v)$$

where $u, D_i u$ are locally $L^p(\Omega)$, $v, D_i v$ are locally $L^q(\Omega)$ ($p > 1, 1/p + 1/q = 1$).

Exercise 1.8. (a) If $u \in W_0^{k,p}(\Omega)$ and $v \in C^k(\bar{\Omega})$, prove that $uv \in W_0^{k,p}(\Omega)$.

(b) If $u \in W^{k,p}(\Omega)$ and $v \in C_0^k(\Omega)$, prove that $uv \in W_0^{k,p}(\Omega)$.

LECTURE 7 – 1/23/19

1.4. Extensions

If $\Omega \subset \Omega'$, then any function $u(x) \in C_0^k(\Omega)$ has an obvious extension $U(x) \in C_0^k(\Omega')$ by zero outside Ω. From the definition of $W_0^{k,p}(\Omega)$ it follows that the function $u(x) \in W_0^{k,p}(\Omega)$ and extended as being equal to zero in $\Omega' \backslash \Omega$ belongs to $W_0^{k,p}(\Omega')$. In general, a function $u \in W^{k,p}(\Omega)$ and extended by zero to Ω' will not belong to $W^{k,p}(\Omega')$. (Consider the function $u(x) \equiv 1$ in Ω.)

We now consider a more general extension result.

Theorem 1.26. *Let Ω, Ω' be bounded open sets in $\mathbb{R}^n$, $\Omega \subset\subset \Omega'$, $k \geq 1$, $1 \leq p \leq \infty$, and let $\partial\Omega$ be C^k.*

(a) *There exists a linear operator $E_1\colon C^k(\bar{\Omega}) \to C_0^k(\Omega')$ such that for each $u \in C^k(\bar{\Omega})$,*

(i) *$E_1 u = u$ in Ω,*

(ii) *$\|E_1 u\|_{C^k(\bar{\Omega}')} \leq C\|u\|_{C^k(\bar{\Omega})}$ and $\|Eu\|_{W^{k,p}(\Omega')} \leq C\|u\|_{W^{k,p}(\Omega)}$ for a constant C depending only on k, p, Ω and Ω'.*

(b) *There exists a linear operator $E\colon W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$ such that for each $u \in W^{k,p}(\Omega)$,*

(i) *$Eu = u$ almost everywhere (a.e.) in Ω,*

(ii) *$\|Eu\|_{W^{k,p}(\Omega')} \leq C\|u\|_{W^{k,p}(\Omega)}$ for a constant C depending only on k, p, Ω and Ω'.*

(c) *There exists a linear operator $E_2\colon C^k(\partial\Omega) \to C^k(\bar{\Omega})$ such that for each $u \in C^k(\partial\Omega)$,*

(i) *$E_2 u = u$ on $\partial\Omega$,*

(ii) *$\|E_2 u\|_{C^k(\bar{\Omega})} \leq C\|u\|_{C^k(\partial\Omega)}$ for a constant C depending only on k, Ω and Ω'.*

Proof. 1. Suppose first that $u \in C^k(\bar{\Omega})$. Let $y = \psi(x)$ define a C^k diffeomorphism that straightens the boundary near $x^0 = (x_1^0, \ldots, x_n^0) \in \partial\Omega$. In particular, we assume there is a ball $B = B(x^0, r)$ such that $\psi(B \cap \Omega) \subset \mathbb{R}_+^n$ (i.e., $y_n > 0$), $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}_+^n$. (e.g., we could choose $y_i = x_i - x_i^0$ for $i = 1, \ldots, n-1$ and $y_n = x_n - \varphi(x_1, \ldots, x_{n-1})$, where φ is of class C^k. Moreover, without loss of generality, we can assume $y_n > 0$ if $x \in B \cap \Omega$.)

2. Let G and $G^+ = G \cap \mathbb{R}_+^n$ be respectively, a ball and half-ball in the image of ψ such that $\psi(x^0) \in G$. Setting $\bar{u}(y) = u \circ \psi^{-1}(y)$ and $y = (y_1, \ldots, y_{n-1}, y_n) = (y', y_n)$, we define an extension $\bar{U}(y)$ of $\bar{u}(y)$ into $y_n < 0$ by

$$\bar{U}(y', y_n) = \sum_{i=1}^{k+1} c_i \bar{u}(y', -y_n/i), \quad y_n < 0$$

where the c_i are constants determined by the system of equations

$$\sum_{i=1}^{k+1} c_i(-1/i)^m = 1, \quad m = 0, 1, \ldots, k. \tag{1.9}$$

Note that the determinant of the system (1.9) is nonzero since it is a Vandemonde determinant. One verifies readily that the extended function $\bar{U}$ is continuous with all derivatives up to order k in G. For example,

$$\lim_{y\to(y',0)} \bar{U}(y) = \sum_{i=1}^{k+1} c_i \bar{u}(y',0) = \bar{u}(y',0)$$

by virtue of (1.9) with $m = 0$. A similar computation shows that

$$\lim_{y\to(y',0)} \bar{U}_{y_i}(y) = \bar{u}_{y_i}(y',0), \quad i = 1,\dots,n-1.$$

Finally

$$\lim_{y\to(y',0)} \bar{U}_{y_n}(y) = \sum_{i=1}^{k+1} c_i(-1/i)\bar{u}_{y_n}(y',0) = \bar{u}_{y_n}(y',0)$$

by virtue of (1.9) with $m = 1$. Similarly we can handle the higher derivatives. Thus $w = \bar{U}\circ\psi \in C^k(\overline{B'})$ for some ball $B' = B'(x^0)$ and $w = u$ in $B'\cap\Omega$, (If $x \in B'\cap\Omega$, then $\psi(x) \in G^+$ and $w(x) = \bar{U}(\psi(x)) = \bar{u}(\psi(x)) = u(\psi^{-1}\psi(x)) = u(x)$) so that w provides a C^k extension of u into $\Omega \cup B'$. Moreover,

$$\sup_{G^+} |\bar{u}(y)| = \sup_{G^+} |u(\psi^{-1}(y))| \le \sup_{\Omega} |u(x)|$$

and since $x \in B'$ implies $\psi(x) \in G$

$$\sup_{B'} |\bar{U}(\psi(x))| \le c \sup_{G^+} |\bar{u}(y)| \le c \sup_{\Omega} |u(x)|.$$

Since a similar computation for the derivatives holds, it follows that there is a constant $c > 0$, independent of u, such that

$$\|w\|_{C^k(\bar{\Omega}\cup B')} \le c\|u\|_{C^k(\bar{\Omega})}, \quad \|w\|_{W^{k,p}(\bar{\Omega}\cup B')} \le C\|u\|_{W^{k,p}(\Omega)}.$$

3. Now consider a finite covering of $\partial\Omega$ by balls B_i, $i = 1,\dots,N$, such as B in the preceding, and let $\{w_i\}$ be the corresponding C^k extensions. We may assume the balls B_i are so small that their union with Ω is contained in Ω'. Let $\Omega_0 \subset\subset \Omega$ be such that Ω_0 and the balls B_i provide a finite open covering of Ω. Let $\{\eta_i\}$, $i = 1,\dots,N$, be a partition of unity subordinate to this covering and set

$$w = u\eta_0 + \sum w_i\eta_i$$

with the understanding that $w_i\eta_i = 0$ if $\eta_i = 0$. Then $w = E_1u$ is an extension of u into Ω' and has the required properties. Thus (a) is established.

4. We now prove (b). We only give the proof for the case $1 \le p < \infty$; the case $p = \infty$ is left as an exercise. (*Hint:* track all the constants involved for each $p < \infty$ and show they do not blow up as $p \to \infty$.). So let $1 \le p < \infty$ and $u \in W^{k,p}(\Omega)$. Then by Theorem 1.25, there exist functions $u_m \in C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $W^{k,p}(\Omega)$. Let $\Omega \subset \Omega'' \subset \Omega'$, and let U_m be the extension of u_m to Ω'' as given in (a). Then

$$\|U_m - U_l\|_{W^{k,p}(\Omega'')} \le c\|u_m - u_l\|_{W^{k,p}(\Omega)}$$

which implies that $\{U_m\}$ is a Cauchy sequence and so converges to a $U \in W_0^{k,p}(\Omega'')$, since $U_m \in C_0^k(\Omega'')$. Now extend U_m, U by 0 to Ω'. It is easy to see that $U = Eu$ is the desired extension.

5. We now prove (c). At any point $x^0 \in \partial\Omega$ let the mapping ψ and the ball G be defined as in (a). By definition, $u \in C^k(\partial\Omega)$ implies that $\bar{u} = u \circ \psi^{-1} \in C^k(G \cap \partial\mathbb{R}^n_+)$. We define

$\bar{\Phi}(y', y_n) = \bar{u}(y')$ in G and set $\Phi(x) = \bar{\Phi} \circ \psi(x)$ for $x \in \psi^{-1}(G)$. Clearly, $\Phi \in C^k(\bar{B})$ for some ball $B = B(x^0)$ and $\Phi = u$ on $B \cap \partial\Omega$. Now let $\{B_i\}$ be a finite covering of $\partial\Omega$ by balls such as B and let Φ_i be the corresponding C^k functions defined on B_i. For each i, we define the function $U_i(x)$ as follows: in the ball B_i take it equal to Φ_i, outside B_i take it equal to zero if $x \notin \partial\Omega$ and equal to $u(x)$ if $x \in \partial\Omega$. The proof can now be completed as in (a) by use of an appropriate partition of unity. $\square$

1.5. Trace Theorem

Unless otherwise stated, Ω will denote a bounded open connected set in $\mathbb{R}^n$, i.e., a bounded domain. Let Γ be a surface which lies in $\bar{\Omega}$ and has the representation

$$x_n = \varphi(x'), \quad x' = (x_1, \ldots, x_{n-1})$$

where $\varphi(x')$ is Lipschitz continuous in $\bar{U}$. Here U is the projection of Γ onto the coordinate plane $x_n = 0$. Let $p \geq 1$. A function u defined on Γ is said to belong to $L^p(\Gamma)$ if

$$\|u\|_{L^p(\Gamma)} \equiv \left(\int_\Gamma |u(x)|^p dS \right)^{\frac{1}{p}} < \infty,$$

where

$$\int_\Gamma |u(x)|^p dS = \int_U |u(x', \varphi(x'))|^p \Big(1 + \sum_{i=1}^{n-1} (\frac{\partial \varphi}{\partial x_i}(x'))^2 \Big)^{\frac{1}{2}} dx'.$$

Thus $L^p(\Gamma)$ reduces to a space of the type $L^p(U)$ where U is a domain in $\mathbb{R}^{n-1}$.

1.5.1. The Trace Operator. For every function $u \in C(\bar{\Omega})$, its values $\gamma_0 u \equiv u|_\Gamma$ on Γ are uniquely given. The function $\gamma_0 u$ will be called the **trace** of the function u on Γ. Note that $u \in L^p(\Gamma)$ since $\gamma_0 u \in C(\Gamma)$.

On the other hand, if we consider a function u defined a.e. in Ω (i.e., functions are considered equal if they coincide a.e.), then the values of u on Γ are not uniquely determined since meas$(\Gamma) = 0$. In particular, since $\partial\Omega$ has measure 0, there exist infinitely many extensions of u to $\bar{\Omega}$ that are equal a.e. We shall therefore introduce the concept of trace for functions in $W^{1,p}(\Omega)$ so that if in addition, $u \in C(\bar{\Omega})$, the new definition of trace reduces to the definition given above.

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In the rest of this section, we assume $1 \leq p < \infty$.

Lemma 1.27. *Let Ω be bounded with $\partial\Omega \in C^1$. Then for $u \in C^1(\bar{\Omega})$,*

$$\|\gamma_0 u\|_{L^p(\partial\Omega)} \leq c\|u\|_{1,p} \tag{1.10}$$

where the constant $c > 0$ does not depend on u.

Proof. 1. Let $x^0 \in \partial\Omega$. Assume $\partial\Omega$ is flat near x^0, lying in the plane $\{x_n = 0\}$. Choose a ball B centered x^0 such that $B^+ = B \cap \{x_n > 0\} \subset \Omega$ and let $\hat{B}$ be the concentric ball of half radius with B. Select a cut-off function $\zeta \in C_0^\infty(B)$ with $0 \leq \zeta \leq 1$ and $\zeta|_{\hat{B}} = 1$. Let $\Gamma = \partial\Omega \cap \hat{B}$ and write $x' = (x_1, \ldots, x_{n-1})$. Then

$$\int_\Gamma |u|^p \, dx' \leq \int_{\{x_n=0\}} \zeta |u|^p \, dx' = -\int_{B^+} (\zeta |u|^p)_{x_n} \, dx$$

$$= -\int_{B^+} (|u|^p \zeta_{x_n} + p|u|^{p-1}(\mathrm{sgn} u) u_{x_n} \zeta)\, dx$$
$$\leq C \int_{B^+} (|u|^p + |Du|^p)\, dx \leq C \int_\Omega (|u|^p + |Du|^p)\, dx,$$

where we used **Young's inequality**.

2. Let $x^0 \in \partial\Omega$. If $\partial\Omega$ is not flat near x^0, then we flatten the boundary near x^0 by a function $x_n = \phi(x')$ as usual and use $dS = (1 + |D\phi|^2)^{1/2} dx'$ to still obtain

$$\int_\Gamma |u|^p\, dS \leq C \int_\Omega (|u|^p + |Du|^p)\, dx,$$

where Γ is some open set of $\partial\Omega$ containing x^0.

3. Since $\partial\Omega$ is compact, there exist finitely many (N) points $x_i^0 \in \partial\Omega$ and open sets Γ_i of $\partial\Omega$ such that $\partial\Omega = \cup_{i=1}^N \Gamma_i$ and

$$\int_{\Gamma_i} |u|^p\, dS \leq C \int_\Omega (|u|^p + |Du|^p)\, dx \quad (i = 1, \ldots, N).$$

Consequently, if we write $\gamma_0(u) = u|_{\partial\Omega}$, then

$$\|\gamma_0(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

□

Remark 1.9. If $1 < p < \infty$, by examining the Steps 1 and 2 of the proof above and using **Young's inequality with** ϵ, we also have

$$\|\gamma_0(u)\|^p_{L^p(\partial\Omega)} \leq \frac{c_1}{\beta^{\frac{1}{p-1}}} \|u\|^p_{L^p(\Omega)} + c_2\beta\|Du\|^p_{L^p(\Omega)} \quad \forall\, 0 < \beta < 1, \tag{1.11}$$

where constants c_1, c_2 depend only on p and Ω, but not on β and u.

Since $1 \leq p < \infty$ and thus $\overline{C^\infty(\bar\Omega)} = W^{1,p}(\Omega)$, the bounded linear operator $\gamma_0 : C^\infty(\bar\Omega) \subset W^{1,p}(\Omega) \to L^p(\partial\Omega)$ can be uniquely extended to a bounded linear operator $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that (1.10) and (1.11) remain true for all $u \in W^{1,p}(\Omega)$.

More precisely, we obtain $\gamma_0 u$ in the following way: Let $u \in W^{1,p}(\Omega)$. We choose a sequence $\{u_n\} \subset C^\infty(\bar\Omega)$ with $\|u_n - u\|_{1,p} \to 0$. Then

$$\|\gamma_0(u_m) - \gamma_0(u_n)\|_{L^p(\partial\Omega)} \leq c\|u_m - u_n\|_{1,p,\Omega} \to 0.$$

Hence $\{\gamma_0(u_n)\}$ is a Cauchy sequence in $L^p(\partial\Omega)$; so there exists a function in $L^p(\partial\Omega)$, defined to be $\gamma_0(u)$, such that $\|\gamma_0 u_n - \gamma_0 u\|_{L^p(\partial\Omega)} \to 0$. Furthermore, this function $\gamma_0(u)$ is independent of the sequence $\{u_n\}$ in the sense that if $v_j \in C^\infty(\bar\Omega)$ and $\|v_j - u\|_{1,p} \to 0$, then $\lim_{j\to\infty} \gamma_0(v_j) = \gamma_0(u)$; this is thanks to

$$\|\gamma_0(v_j) - \gamma_0(u_j)\|_{L^p(\partial\Omega)} \leq c\|v_j - u_j\|_{1,p,\Omega} \to 0.$$

The function $\gamma_0 u$ (as an element of $L^p(\partial\Omega)$) will be called the **trace** of the function $u \in W^{1,p}(\Omega)$ on the boundary $\partial\Omega$. ($\|\gamma_0 u\|_{L^p(\partial\Omega)}$ will be denoted by $\|u\|_{L^p(\partial\Omega)}$.) Thus the trace of a function is defined for any element $u \in W^{1,p}(\Omega)$.

The above discussion partly proves the following:

Theorem 1.28. (Trace operator) *Suppose $\partial\Omega \in C^1$. Then there is a unique bounded linear operator $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $\gamma_0 u = u|_{\partial\Omega}$ for $u \in C(\bar\Omega) \cap W^{1,p}(\Omega)$, and $\gamma_0(au) = \gamma_0 a \cdot \gamma_0 u$ for $a(x) \in C^1(\bar\Omega)$, $u \in W^{1,p}(\Omega)$. Moreover, $\mathcal{N}(\gamma_0) = W_0^{1,p}(\Omega)$ and $\overline{R(\gamma_0)} = L^p(\partial\Omega)$.*

Proof. 1. Suppose $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$. Then the function $u_m \in C^\infty(\bar{\Omega})$ constructed in the proof of the global approximation Theorem 1.25 also converges uniformly to u on $\bar{\Omega}$. Hence $u_m|_{\partial\Omega} \to u|_{\partial\Omega}$ uniformly on $\partial\Omega$; on the other hand, $u_m|_{\partial\Omega} = \gamma_0(u_m) \to \gamma_0(u)$ in $L^p(\partial\Omega)$, and hence $u|_{\partial\Omega} = \gamma_0 u$ in $L^p(\partial\Omega)$.

2. Now $au \in W^{1,p}(\Omega)$ if $a \in C^1(\bar{\Omega})$ and $u \in W^{1,p}(\Omega)$, and consequently, $\gamma_0(au)$ is defined. Let $\{u_n\} \subset C^1(\bar{\Omega})$ with $\|u_n - u\|_{1,p} \to 0$. Then

$$\gamma_0(au_n) = \gamma_0 a \cdot \gamma_0 u_n$$

and the desired product formula follows by virtue of the continuity of γ_0.

3. If $u \in W_0^{1,p}(\Omega)$, then there is a sequence $\{u_n\} \subset C_0^1(\Omega)$ with $\|u_n - u\|_{1,p} \to 0$. But $u_n|_{\partial\Omega} = 0$ and as $n \to \infty$, $u_n|_{\partial\Omega} \to \gamma_0 u$ in $L^p(\partial\Omega)$ which implies $\gamma_0 u = 0$. Hence $W_0^{1,p}(\Omega) \subset \mathcal{N}(\gamma_0)$. The opposite inclusion $\mathcal{N}(\gamma_0) \subset W_0^{1,p}(\Omega)$ is more difficult to prove.

4. Suppose $u \in \mathcal{N}(\gamma_0)$. If $u \in W^{1,p}(\Omega)$ has compact support in Ω, then by an earlier remark, $u \in W_0^{1,p}(\Omega)$. If u does not have compact support in Ω, then it can be shown that there exists a sequence of cut-off functions η_k such that $\eta_k u \in W^{1,p}(\Omega)$ has compact support in Ω, and moreover, $\|\eta_k u - u\|_{1,p} \to 0$. By using the corresponding mollified functions, it follows that $u \in W_0^{1,p}(\Omega)$ and $\mathcal{N}(\gamma_0) \subset W_0^{1,p}(\Omega)$. Details can be found in Evans's book.

5. Finally, to prove $\overline{R(\gamma_0)} = L^p(\partial\Omega)$, let $f \in L^p(\partial\Omega)$ and let $\varepsilon > 0$ be given. Then there is a $u \in C^1(\partial\Omega)$ such that $\|u - f\|_{L^p(\partial\Omega)} < \varepsilon$. If we let $U \in C^1(\bar{\Omega})$ be the extension of u into $\bar{\Omega}$, then clearly $\|\gamma_0 U - f\|_{L^p(\partial\Omega)} < \varepsilon$, which is the desired result since $U \in W^{1,p}(\Omega)$. □

Remark 1.10. We note that the function $u \equiv 1$ belongs to $W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and its trace on $\partial\Omega$ is 1. Hence this function does not belong to $W_0^{1,p}(\Omega)$, which establishes the earlier assertion that $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$. In fact, for all bounded open sets Ω, $1 \notin W_0^{1,p}(\Omega)$, which follows from the embedding theorems proved later.

1.5.2. Higher-Order Trace Operators. Let $u \in W^{k,p}(\Omega)$, $k > 1$. Since any weak derivative $D^\alpha u$ of order $|\alpha| < k$ belongs to $W^{1,p}(\Omega)$, this derivative has a trace $\gamma_0 D^\alpha u$ belonging to $L^p(\partial\Omega)$. Moreover

$$\|D^\alpha u\|_{L^p(\partial\Omega)} \le c\|D^\alpha u\|_{1,p} \le c\|u\|_{k,p}$$

for constant $c > 0$ independent of u.

Assuming the boundary $\partial\Omega \in C^1$, the unit outward normal vector n to $\partial\Omega$ exists and is bounded. Thus, the concept of traces makes it possible to introduce $\partial u/\partial n$ for $u \in W^{2,p}(\Omega)$. More precisely, if $u \in W^{2,p}(\Omega)$, then there exist traces of the functions u, $D_i u$ so that, if n_i are the direction cosines of the normal, we may define

$$\gamma_1 u = \sum_{i=1}^{n} (\gamma_0(D_i u)) n_i.$$

The trace operator $\gamma_1 : W^{2,p}(\Omega) \to L^p(\partial\Omega)$ is continuous and $\gamma_1 u = (\partial u/\partial n)|_{\partial\Omega}$ for $u \in C^1(\bar{\Omega}) \cap W^{2,p}(\Omega)$.

For a function $u \in C^k(\bar{\Omega})$ we define the various traces of normal derivatives given by

$$\gamma_j u = \frac{\partial^j u}{\partial n^j}|_{\partial\Omega}, \quad 0 \le j \le k-1.$$

Each γ_j can be extended by continuity to all of $W^{k,p}(\Omega)$ and we obtain the following:

Theorem 1.29. (Higher-order traces) *Suppose $\partial\Omega \in C^k$. Then there is a unique continuous linear operator $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{k-1}) : W^{k,p}(\Omega) \to \prod_{j=0}^{k-1} W^{k-1-j,p}(\partial\Omega)$ such that for $u \in C^k(\bar{\Omega})$*

$$\gamma_0 u = u|_{\partial\Omega},\ \gamma_j u = \frac{\partial^j u}{\partial n^j}|_{\partial\Omega},\ j = 1, \ldots, k-1.$$

Moreover, $\mathcal{N}(\gamma) = W_0^{k,p}(\Omega)$ and $\overline{\mathcal{R}(\gamma)} = \prod_{j=0}^{k-1} W^{k-1-j,p}(\partial\Omega)$.

The Sobolev space $W^{k-1-j,p}(\partial\Omega)$ can be defined locally. However, this space is not the range of γ_j; the range of γ_j is actually a fractional Sobolev space on the boundary $\partial\Omega$. For example, $\gamma_0(H^1(\Omega)) = H^{1/2}(\partial\Omega)$. We will not study such Sobolev spaces.

Lecture 9 – 1/28/19

1.5.3. Green's Identities. In this section we assume that $p = 2$ and we continue to assume Ω is a bounded domain.

Theorem 1.30. (Integration by Parts) *Let $u, v \in H^1(\Omega)$ and let $\partial\Omega \in C^1$. Then for any $i = 1, \ldots, n$*

$$\int_\Omega vD_i u dx = \int_{\partial\Omega} (\gamma_0 u \cdot \gamma_0 v) n_i dS - \int_\Omega uD_i v dx. \tag{1.12}$$

($D_i u, D_i v$ are weak derivatives.)

Proof. Let $\{u_n\}$ and $\{v_n\}$ be sequences of functions in $C^1(\bar{\Omega})$ with $\|u_n - u\|_{H^1(\Omega)} \to 0, \|v_n - v\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. Formula (1.12) holds for u_n, v_n

$$\int_\Omega v_n D_i u_n dx = \int_{\partial\Omega} u_n v_n n_i dS - \int_\Omega u_n D_i v_n dx$$

and upon letting $n \to \infty$ relation (1.12) follows. □

Corollary 1.31. *Let $\partial\Omega \in C^1$.*

(a) *If $v \in H^1(\Omega)$ and $u \in H^2(\Omega)$ then*

$$\int_\Omega v\Delta u dx = \int_{\partial\Omega} \gamma_0 v \cdot \gamma_1 u dS - \int_\Omega (\nabla u \cdot \nabla v) dx \quad (\textbf{Green's 1st identity}).$$

(b) *If $u, v \in H^2(\Omega)$ then*

$$\int_\Omega (v\Delta u - u\Delta v) dx = \int_{\partial\Omega} (\gamma_0 v \cdot \gamma_1 u - \gamma_0 u \cdot \gamma_1 v) dS \quad (\textbf{Green's 2nd identity}).$$

Proof. If in (1.12) we replace u by $D_i u$ and sum from 1 to n, then Green's 1st identity is obtained. Interchanging the roles of u, v in Green's 1st identity and subtracting the two identities yields Green's 2nd identity. □

Exercise 1.11. Establish the following one-dimensional version of the trace theorem: If $u \in W^{1,p}(\Omega)$, where $\Omega = (a, b)$, then

$$\|u\|_{L^p(\partial\Omega)} \equiv (|u(a)|^p + |u(b)|^p)^{1/p} \leq \text{ const } \|u\|_{W^{1,p}(\Omega)}$$

where the constant is independent of u.

1.6. Embedding Theorems

In what follows, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, with $X \subset Y$ as sets. We say that X is **continuously embedded** into Y and also write $X \subset Y$ if there exists a constant $C > 0$ such that

$$\|u\|_Y \leq C\|u\|_X \quad \forall\, u \in X.$$

We consider the following question: *Is $W^{k,p}(\Omega)$ continuously embedded into certain other spaces?* Certainly, by definition, $W^{k,p}(\Omega) \subset W^{j,p}(\Omega)$ for all $0 \leq j < k$, but such embeddings are not interesting. Are there other spaces not directly from defintion? The answer will be yes, but which other spaces depend upon whether $1 \leq kp < n$, $kp = n$, $n < kp < \infty$.

A series of special results will be needed. We start with $k = 1$.

1.6.1. Gagliardo-Nirenberg-Sobolev Inequality. Suppose $1 \leq p < n$. Do there exist constants $C > 0$ and $1 \leq q < \infty$ such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{1.13}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$? The point is that the constants C and q should not depend on u.

We shall show that if such an inequality holds, then q must have a specific form. For this, choose any $u \in C_0^\infty(\mathbb{R}^n)$, $u \not\equiv 0$, and define for $\lambda > 0$

$$u_\lambda(x) \equiv u(\lambda x) \quad (x \in \mathbb{R}^n).$$

Now

$$\int_{\mathbb{R}^n} |u_\lambda|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \frac{1}{\lambda^n}\int_{\mathbb{R}^n} |u(y)|^q dy$$

and

$$\int_{\mathbb{R}^n} |Du_\lambda|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n}\int_{\mathbb{R}^n} |Du(y)|^p dy.$$

Inserting these inequalities into (1.13) we find

$$\frac{1}{\lambda^{n/q}}\|u\|_{L^q(\mathbb{R}^n)} \leq C\frac{\lambda}{\lambda^{n/p}}\|Du\|_{L^p(\mathbb{R}^n)}$$

and so

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-n/p+n/q}\|Du\|_{L^p(\mathbb{R}^n)}. \tag{1.14}$$

If $1 - n/p + n/q > 0$ (or < 0), then we can upon sending λ to 0 (or ∞) in (1.14) obtain a contradiction ($u = 0$). Thus we must have $1 - n/p + n/q = 0$; that is, $q = p^*$, where

$$p^* = \frac{np}{n-p} \tag{1.15}$$

is called the **Sobolev conjugate** of p. Note that then

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p. \tag{1.16}$$

Next we prove that the inequality (1.13) indeed holds for $q = p^*$.

Lemma 1.32. (Gagliardo-Nirenberg-Sobolev Inequality) *Assume $1 \leq p < n$. Then there is a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{1.17}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

Proof. First assume $p=1$. Since u has compact support, for each $i=1,\ldots,n$ we have

$$u(x)=\int_{-\infty}^{x_i} u_{x_i}(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)dy_i$$

and so

$$|u(x)|\le\int_{-\infty}^{\infty}|Du(x_1,\ldots,y_i,\ldots,x_n)|dy_i \quad (i=1,\ldots,n).$$

Consequently

$$|u(x)|^{\frac{n}{n-1}}\le\prod_{i=1}^n\left(\int_{-\infty}^{\infty}|Du(x_1,\ldots,y_i,\ldots,x_n)|dy_i\right)^{\frac{1}{n-1}}. \tag{1.18}$$

Integrate this inequality with respect to x_1:

$$\begin{aligned}
\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1 &\le \int_{-\infty}^{\infty}\prod_{i=1}^n\left(\int_{-\infty}^{\infty}|Du|dy_i\right)^{\frac{1}{n-1}}dx_1\\
&= \left(\int_{-\infty}^{\infty}|Du|dy_1\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i=2}^n\left(\int_{-\infty}^{\infty}|Du|dy_i\right)^{\frac{1}{n-1}}dx_1\\
&\le \left(\int_{-\infty}^{\infty}|Du|dy_1\right)^{\frac{1}{n-1}}\left(\prod_{i=2}^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_i\right)^{\frac{1}{n-1}}
\end{aligned}$$

the last inequality resulting from the extended Hölder inequality.

We continue by integrating with respect to $x_2,\ldots,x_n$ and applying the extended Hölder inequality to eventually find (pull out an integral at each step)

$$\begin{aligned}
\int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-1}}dx &\le \prod_{i=1}^n\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|dx_1\ldots dy_i\ldots dx_n\right)^{\frac{1}{n-1}}\\
&= \left(\int_{\mathbb{R}^n}|Du|dx\right)^{\frac{n}{n-1}}
\end{aligned}$$

which is estimate (1.17) for $p=1$. Mr. Minh Le showed me an elegant proof of this case by using induction to show

$$\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}\,dx\le\left(\prod_{i=1}^n\|u_{x_i}\|_{L^1(\mathbb{R}^n)}\right)^{\frac{1}{n-1}} \qquad \forall\, u\in C_0^1(\mathbb{R}^n).$$

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Now consider the case that $1<p<n$. Let $v=|u|^\gamma$, where $\gamma>1$ is to be selected. Note that $D_iv=\gamma|u|^{\gamma-1}D_iu$; thus $v\in C_0^1(\mathbb{R}^n)$. So, using the above case of $p=1$ for $u=v$, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}|u(x)|^{\frac{\gamma n}{n-1}}dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n}|D|u|^\gamma|dx\\
&= \gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|dx\\
&\le \gamma\left(\int_{\mathbb{R}^n}|u|^{\frac{p(\gamma-1)}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|Du|^pdx\right)^{\frac{1}{p}}.
\end{aligned}$$

We choose γ so that the powers of $|u|$ on both sides are equal; namely, $\frac{\gamma n}{n-1} = \frac{p(\gamma-1)}{p-1}$. This gives $\gamma = \frac{p(n-1)}{n-p} > 1$, in which case

$$\frac{\gamma n}{n-1} = \frac{p(\gamma-1)}{p-1} = \frac{np}{n-p} = p^*.$$

Thus, the above estimate becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}.$$

□

Theorem 1.33. (Gagliardo-Nirenberg-Sobolev-Poincaré's Inequality) *Let $\Omega \subset \mathbb{R}^n$ be any open set and $1 \leq p < n$. If $u \in W_0^{1,p}(\Omega)$, then $u \in L^{p^*}(\Omega)$ and*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|Du\|_{L^p(\Omega)}, \tag{1.19}$$

where the constant C depends only on p and n.

Proof. Let $u \in W_0^{1,p}(\Omega)$ and let $u_m \in C_0^\infty(\Omega)$ be such that $u_m \to u$ in $W^{1,p}(\Omega)$. Extend u_m by zero outside Ω so that $u_m \in C_0^\infty(\mathbb{R}^n)$. By Lemma 1.32, we have

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)} \quad \forall\, l, m;$$

hence $\{u_m\}$ is a Cauchy sequence in $L^{p*}(\mathbb{R}^n)$ and thus $u_m \to \tilde{u}$ in $L^{p^*}(\mathbb{R}^n)$. One must have $\tilde{u} = u$ a.e. in Ω and thus $u = \tilde{u} \in L^{p*}(\Omega)$. Again Lemma 1.32 gives

$$\|u_m\|_{L^{p^*}(\Omega)} = \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)} = C\|Du_m\|_{L^p(\Omega)},$$

and taking $m \to \infty$ proves (1.19). □

Theorem 1.34. (Poincaré's Inequality) *Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Assume $1 \leq p < n$ and $q \in [1, p^*]$. If $u \in W_0^{1,p}(\Omega)$, then $u \in L^q(\Omega)$ and*

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)},$$

where the constant C depends only on p, q, n and Ω.

Proof. The result follows from Theorem 1.33 since, from $|\Omega| < \infty$ and **Hölder's inequality**,

$$\|u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p*}}\|u\|_{L^{p^*}(\Omega)}$$

for every $q \in [1, p^*]$. □

Remark 1.12. A Poincaré type of inequality is an estimate of the L^q norm of certain quantities involving u by the L^p norm of Du. We will have the other type of Poincaré's inequalities later.

Theorem 1.35. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega \in C^1$. Assume $1 \leq p < n$, and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$ and*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \tag{1.20}$$

where the constant C depends only on p,n and Ω.

Proof. Since $\partial\Omega \in C^1$, there exists an extension $U \in W^{1,p}(\mathbb{R}^n)$ such that $U = u$ in Ω, U has compact support and

$$\|U\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}. \tag{1.21}$$

Moreover, since U has compact support, there exist mollified functions $u_m \in C_0^\infty(\mathbb{R}^n)$ such that $u_m \to U$ in $W^{1,p}(\mathbb{R}^n)$. Now according to Lemma 1.32,

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$$

for all $l, m \ge 1$; whence $u_m \to U$ in $L^{p^*}(\mathbb{R}^n)$ as well. Since Lemma 1.32 also implies

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du_m\|_{L^p(\mathbb{R}^n)},$$

we get in the limit that

$$\|U\|_{L^{p^*}(\mathbb{R}^n)} \le C\|DU\|_{L^p(\mathbb{R}^n)}.$$

This inequality and (1.21) complete the proof. □

The case $p = n$. Since $W^{1,n}(\Omega) \subset W^{1,p}(\Omega)$ for all $p < n$, we have the continuous embedding $W^{1,n}(\Omega) \subset L^r(\Omega)$ for all $1 \le r < \infty$. However, we do not have the embedding $W^{1,n}(\Omega) \subset L^\infty(\Omega)$; for example, function $u = \ln\ln(1 + \frac{1}{|x|}) \in W^{1,n}(B(0,1))$ but not to $L^\infty(B(0,1))$ if $n \ge 2$.

In fact, the space $W^{1,n}(\Omega)$ is embedded into the space of functions of **bounded mean oscillation** in Ω, namely, $BMO(\Omega)$; however, we shall not study this embedding in this course.

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1.6.2. Morrey's Inequality. We now turn to the case $n < p \le \infty$.

The next result shows that if $u \in W^{1,p}(\Omega)$, then u is in fact Hölder continuous, after possibly being redefined on a set of measure zero.

Theorem 1.36. (Morrey's Inequality) *Assume $n < p \le \infty$. Then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall\, u \in C^1(\mathbb{R}^n). \tag{1.22}$$

Proof. We first prove the following inequality: for all $x \in \mathbb{R}^n$, $r > 0$ and all $u \in C^1(\mathbb{R}^n)$,

$$\int_{B(x,r)} |u(y) - u(x)|\, dy \le \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy. \tag{1.23}$$

To prove this, note that, for any w with $|w| = 1$ and $0 < s < r$,

$$\begin{aligned} |u(x+sw) - u(x)| &= \left| \int_0^s \frac{d}{dt} u(x+tw)dt \right| \\ &= \left| \int_0^s Du(x+tw)\cdot w dt \right| \\ &\le \int_0^s |Du(x+sw)|\, dt. \end{aligned}$$

Now we integrate w over $\partial B(0,1)$ to obtain

$$\begin{aligned}\int_{\partial B(0,1)} |u(x+sw)-u(x)|\,dS &\le \int_0^s \int_{\partial B(0,1)} |Du(x+sw)|\,dSdt \\ &= \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \\ &\le \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy.\end{aligned}$$

Multiply both sides by s^{n-1} and integrate over $s \in (0,r)$ and we obtain (1.23).
To establish the bound on $\|u\|_{C^0(\mathbb{R}^n)}$, we observe that, by (1.23), for $x \in \mathbb{R}^n$,

$$\begin{aligned}|u(x)| &\le \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)-u(x)|\,dy + \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)|\,dy \\ &\le C\left(\int_{\mathbb{R}^n} |Du(y)|^p\,dy\right)^{1/p}\left(\int_{B(x,1)} |y-x|^{\frac{(1-n)p}{p-1}}\,dy\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)} \\ &\le C\|u\|_{W^{1,p}(\mathbb{R}^n)}\end{aligned}$$

for $n < p \le \infty$, where C is a constant depending on p, n; here we have used the fact

$$\int_{B(x,1)} |y-x|^{\frac{(1-n)p}{p-1}}\,dy = \omega_n \frac{p-1}{p-n} < \infty \quad \forall\, n < p \le \infty.$$

To establish the bound on the semi-norm $[u]_\gamma$, $\gamma = 1 - \frac{n}{p}$, take any two points $x, y \in \mathbb{R}^n$. Let $r = |x-y|$ and $W = B(x,r) \cap B(y,r)$. Then

$$|u(x)-u(y)| \le \frac{1}{|W|}\int_W |u(x)-u(z)|\,dz + \frac{1}{|W|}\int_W |u(y)-u(z)|\,dz. \tag{1.24}$$

Note that $|W| = \beta r^n$, $r = |x-y|$ and $\int_W \le \min\{\int_{B(x,r)}, \int_{B(y,r)}\}$. Hence, using (1.23), by Hölder's inequality, we obtain

$$\begin{aligned}\int_W |u(x)-u(z)|\,dz &\le \int_{B(x,r)} |u(x)-u(z)|\,dz \le \frac{r^n}{n}\int_{B(x,r)} |Du(z)||x-z|^{1-n}\,dz \\ &\le \frac{r^n}{n}\left(\int_{B(x,r)} |Du(z)|^p\,dz\right)^{1/p}\left(\int_{B(x,r)} |z-x|^{\frac{(1-n)p}{p-1}}\,dz\right)^{\frac{p-1}{p}} \\ &\le C\,r^n\,\|Du\|_{L^p(B(x,r))}\left(\int_0^r s^{\frac{(1-n)p}{p-1}}\,s^{n-1}\,ds\right)^{\frac{p-1}{p}} \\ &\le C\,r^{n+\gamma}\|Du\|_{L^p(B(x,r))}\end{aligned}$$

for $n < p \le \infty$, where $\gamma = 1 - \frac{n}{p}$ and $C = C_n(\frac{p-1}{p-n})^{\frac{p-1}{p}}$; similarly,

$$\int_W |u(y)-u(z)|\,dz \le C\,r^{n+\gamma}\|Du\|_{L^p(B(y,r))}.$$

Hence, by (1.24) and noting that $B(x,r) \cup B(y,r) \subset B(x,2r)$, we have

$$|u(x)-u(y)| \le C\,|x-y|^\gamma \|Du\|_{L^p(B(x,2r))} \quad \forall\, y \in B(x,r). \tag{1.25}$$

This inequality, also of independent importance itself, and the bound on $\|u\|_{C^0}$ above complete the proof. $\square$

Theorem 1.37. (Estimates for $W^{1,p}(\Omega)$, $n < p \le \infty$) *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, with $\partial\Omega \in C^1$. Assume $n < p \le \infty$, and $u \in W^{1,p}(\Omega)$. Then, after possibly redefining u on a null set, $u \in C^{0,1-\frac{n}{p}}(\bar{\Omega})$ and*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\bar{\Omega})} \le C\|u\|_{W^{1,p}(\Omega)}$$

where the constant C depends only on p, n and Ω.

Proof. Since $\partial\Omega \in C^1$, there exists an extension $U \in W^{1,p}(\mathbb{R}^n)$ such that $U = u$ in Ω, U has compact support K and

$$\|U\|_{W^{1,p}(\mathbb{R}^n)} \le C_1(n, p, \Omega, K)\|u\|_{W^{1,p}(\Omega)}. \tag{1.26}$$

1. First assume $n < p < \infty$. Since U has compact support, the mollified functions $u_m \in C_0^\infty(\mathbb{R}^n)$ of U satisfy that $u_m \to U$ in $W^{1,p}(\mathbb{R}^n)$. According to Morrey's inequality,

$$\|u_m - u_l\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C_2(n, p)\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $l, m \ge 1$; whence there is a function $u^* \in C^{0,1-n/p}(\mathbb{R}^n)$ such that $u_m \to u^*$ in $C^{0,1-n/p}(\mathbb{R}^n)$. Thus $u^* = u$ a.e. in Ω. Since we also have

$$\|u_m\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C_2(n, p)\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$$

we get in the limit that

$$\|u^*\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C_2(n, p)\|U\|_{W^{1,p}(\mathbb{R}^n)}. \tag{1.27}$$

This inequality and (1.26) complete the proof if $n < p < \infty$.

2. Assume $p = \infty$. Note that the constants C_1 and C_2 in (1.26) and (1.27) remain bounded as $p \to \infty$. Thus u^* determined above is also in $C^{0,1}(\mathbb{R}^n)$ with the $C^{0,1}$-norm less than or equal to $C\|u\|_{W^{1,\infty}(\Omega)}$. □

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1.6.3. General Embedding Theorems. We can now combine the above estimates to obtain more general embedding theorems.

We summarize these results in the following theorem.

Theorem 1.38. (General Sobolev Inequalities) *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega \in C^1$. Assume $1 \le p \le \infty$ and k is a positive integer.*

(a) *Let $0 \le j < k$, $1 \le p, q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{k-j}{n}$. Then*

$$W^{k,p}(\Omega) \subset W^{j,q}(\Omega).$$

In particular, if $kp < n$ and $q = np/(n - kp)$, then

$$W^{k,p}(\Omega) \subset L^q(\Omega);$$

that is,

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)}, \tag{1.28}$$

where the constant C depends only on k, p, n and Ω.

(b) *If $kp = n$ and $1 \le r < \infty$, then*

$$W^{k,p}(\Omega) \subset L^r(\Omega);$$

that is,

$$\|u\|_{L^r(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)}, \tag{1.29}$$

where the constant C depends only on k, p, r, n and Ω.

(c) *If $kp > n$ and $m = k - [\frac{n}{p}] - 1$, then*

$$W^{k,p}(\Omega) \subset C^{m,\gamma}(\bar{\Omega});$$

that is,

$$\|u\|_{C^{m,\gamma}(\bar{\Omega})} \le C\|u\|_{W^{k,p}(\Omega)}, \tag{1.30}$$

where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number} < 1 & \text{if } \frac{n}{p} \text{ is an integer,} \end{cases}$$

and the constant C depends only on k, p, n, γ and Ω.

All above results are valid for $W_0^{k,p}(\Omega)$ spaces on arbitrary bounded domains Ω.

Proof. (a) Assume $0 \le j < k$, $1 \le p, q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{k-j}{n}$. Let $u \in W^{k,p}(\Omega)$. Since $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \le k$, the Gagliardo-Nirenberg-Sobolev inequality implies

$$\|D^\beta u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)}$$

if $|\beta| \le k - 1$, and so $u \in W^{k-1,p^*}(\Omega)$. Moreover, $\|u\|_{k-1,p^*} \le c\|u\|_{k,p}$. Similarly, we find $u \in W^{k-2,p^{**}}(\Omega)$, where

$$\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}.$$

Moreover, $\|u\|_{k-2,p^{**}} \le c\|u\|_{k-1,p^*}$. Continuing, this proves $W^{k,p}(\Omega) \subset W^{j,q}(\Omega)$.
In particular, with $j = 0$, we have that $W^{k,p}(\Omega) \subset W^{0,q}(\Omega) = L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.
The stated estimate (1.28) follows from combining the relevant estimates at each stage of the above argument.

(b) Assume $kp = n$ and $1 \le r < \infty$. Then there exists a $p' \in (1, p)$ such that $kp' < n$ and $q' = \frac{np'}{n-kp'} > r$. Thus

$$W^{k,p}(\Omega) \subset W^{k,p'}(\Omega) \subset L^{q'}(\Omega) \subset L^r(\Omega).$$

(c) Assume $kp > n$ and $m = k - [\frac{n}{p}] - 1$. As proved above, $W^{k,p}(\Omega) \subset W^{j,q}(\Omega)$ if

$$0 < \frac{1}{q} = \frac{1}{p} - \frac{k-j}{n} \le 1. \tag{1.31}$$

(i) Assume $\frac{n}{p}$ is not an integer. Let $j = k - [\frac{n}{p}] = m+1$. Thus $q > n$ and hence, by **Morrey's inequality** and induction, $W^{j,q}(\Omega) \subset C^{j-1,1-\frac{n}{q}}(\bar{\Omega})$; hence $W^{k,p}(\Omega) \subset C^{j-1,1-\frac{n}{q}}(\bar{\Omega})$. But $j - 1 = m$ and $1 - \frac{n}{q} = \gamma$ in this case.

(ii) Assume $\frac{n}{p}$ is an integer. In this case, let $j = k + 1 - \frac{n}{p} = m + 2$ in (1.31); then $q = n$. So $W^{k,p}(\Omega) \subset W^{m+2,n}(\Omega) \subset W^{m+1,r}(\Omega) \subset C^{m,1-\frac{n}{r}}(\bar{\Omega})$ for all $n < r < \infty$. The result follows in this case if $\gamma = 1 - \frac{n}{r}$.

In a similar manner the embeddings for $W_0^{k,p}(\Omega)$ can be established without the smoothness of $\partial\Omega$.

□

1.7. Compactness

We now consider the compactness of the embeddings. Note that if X and Y are Banach spaces with $X \subset Y$ then we say that X is **compactly embedded** in Y, written $X \subset\subset Y$, provided

(i) $\|u\|_Y \leq C\|u\|_X$ ($u \in X$) for some constant C; that is, the embedding is continuous;

(ii) each bounded sequence in X has a convergent subsequence in Y.

We summarize the compactness results in the following theorem. Parts (a) and (b) are also called the **Rellich-Kondrachov Compactness Theorem**.

Theorem 1.39. (Compactness Theorem) *Let $\Omega \subset \mathbb{R}^n$ be bounded and open.*
If $1 \leq p < n$, then

(a) *the embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ is compact for each $1 \leq q < np/(n-p)$;*

(b) *assuming $\partial\Omega \in C^1$, the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ is compact for each $1 \leq q < np/(n-p)$.*

If $p = n$, then

(c) *assuming $\partial\Omega \in C^1$, the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ is compact for each $1 \leq q < \infty$.*

If $p > n$, then

(d) *assuming $\partial\Omega \in C^1$, the embedding $W^{1,p}(\Omega) \subset C^{0,\alpha}(\bar{\Omega})$ is compact for each $0 \leq \alpha < 1 - (n/p)$.*

If $1 < p < \infty$, then

(e) *assuming $\partial\Omega \in C^1$, $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact.*

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Proof. (a)&(b) We prove (b) as the proof of (a) is similar. Let $1 \leq p < n$ and $1 \leq q < p^*$. Assume $\{u_m\}$ is a bounded sequence in $W^{1,p}(\Omega)$. By extension, we assume each u_m has compact support in a bounded open set V in $\mathbb{R}^n$ and $\{u_m\}$ is a bounded sequence in $W^{1,p}(V)$. Let

$$u_m^\varepsilon = \omega_\varepsilon * u_m$$

be the mollifying sequence of u_m. We also assume each u_m^ε has compact support in V as well.

(i) We first claim that

$$\lim_{\varepsilon\to 0} \|u_m^\varepsilon - u_m\|_{L^q(\Omega)} = 0 \quad \text{uniformly in } m. \tag{1.32}$$

To prove this, note that if u_m is smooth then

$$u_m^\varepsilon(x) - u_m(x) = \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \omega(\frac{x-z}{\varepsilon})(u_m(z) - u_m(x))\, dz$$

$$= -\varepsilon \int_{B(0,1)} \omega(y) \int_0^1 Du_m(x - \varepsilon ty) \cdot y\, dtdy.$$

Thus

$$\int_V |u_m^\varepsilon(x) - u_m(x)|\, dx \le \varepsilon \int_{B(0,1)} \omega(y) \int_0^1 \int_V |Du_m(x - \varepsilon ty)|\, dxdtdy \le \varepsilon \int_V |Du_m(z)|\, dz.$$

By approximation, this estimate holds if $u_m \in W^{1,p}(V)$. Hence

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)} \le \varepsilon C \|Du_m\|_{L^p(V)}.$$

(In fact, the similar estimate also shows that $\|u_m^\varepsilon - u_m\|_{L^p(V)} \le \varepsilon \|Du_m\|_{L^p(V)}$, which is enough for (1.32) if $1 \le q \le p$.) In general, to estimate the L^q-norm, we use the **interpolation inequality** for L^p norms to have

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\theta \in (0,1]$ is such that $\frac{1}{q} = \theta + (1-\theta)\frac{1}{p^*}$; namely $\theta = \frac{p^*-q}{q(p^*-1)}$. Therefore, since $\{u_m\}$ and $\{u_m^e\}$ are bounded in $W^{1,p}(V)$, we have

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le C\varepsilon^\theta$$

for a constant C independent of m, which proves (1.32).

(ii) Next we claim that for each $\varepsilon > 0$ the sequence $\{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous in $C(\bar V)$. This is easy from

$$\|u_m^\varepsilon\|_{L^\infty(V)} = \|\omega_\varepsilon * u_m\|_{L^\infty(V)} \le \|\omega_\varepsilon\|_{L^\infty} \|u_m\|_{L^1(V)} \le \frac{C}{\varepsilon^n}$$

and

$$\|Du_m^\varepsilon\|_{L^\infty(V)} = \|D\omega_\varepsilon * u_m\|_{L^\infty(V)} \le \|D\omega_\varepsilon\|_{L^\infty} \|u_m\|_{L^1(V)} \le \frac{C}{\varepsilon^{n+1}}.$$

(iii) Given each $\delta > 0$, we claim that there exists a subsequence $\{u_{m_j}\}$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta.$$

To see this, first select $\varepsilon > 0$ such that $\|u_m^\varepsilon - u_m\|_{L^q(V)} < \delta/2$ for all m. Then, since $\{u_m^\varepsilon\}$ is uniformly bounded and equicontinuous in $C(\bar V)$, by the **Arzela-Ascoli theorem**, we obtain a subsequence $\{u_{m_j}^\varepsilon\}$ of $\{u_m^\varepsilon\}$ which converges uniformly on $\bar V$. In partcular,

$$\limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0.$$

Hence, by the triangle inequality,

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta.$$

(iv) To obtain a subsequence of $\{u_m\}$ which converges in $L^q(V)$, we use $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and a standard diagonalization process. This completes the proof of (a)&(b).

(c) Let $\{u_m\}$ be a bounded sequence in $W^{1,n}(\Omega)$ and $1 \le q < \infty$. Take a number $p \in (1,n)$ such that $q < p^* = \frac{np}{n-p}$. Then $\{u_m\}$ is also bounded in $W^{1,p}(\Omega)$. By (b), $\{u_m\}$ has a subsequence which converges in $L^q(\Omega)$; this proves (c).

(d) By Morrey's inequality, the embedding is continuous if $\beta = 1 - (n/p)$. Now use the fact that $C^{0,\beta}$ is compact in $C^{0,\alpha}$ if $\alpha < \beta$. *Hint:* Use the **interpolation inequality** for Hölder nomrs

$$[u]_{C^{0,\alpha}} \le [u]_{C^{0,\beta}}^{\alpha/\beta} \|u\|_{L^\infty}^{1-\alpha/\beta}.$$

(Exercise!)

(e) Let $1 < p < \infty$ and $\{u_m\}$ be a bounded sequence in $W^{1,p}(\Omega)$. By the inequality (1.11),

$$\|\gamma_0 u_m\|^p_{L^p(\partial\Omega)} \le \frac{c_1}{\beta^{\frac{1}{p-1}}}\|u_m\|^p_{L^p(\Omega)} + C_2\beta \quad \forall\, 0 < \beta < 1, \tag{1.33}$$

where the constants c_1, C_2 do not depend on u_m or β. By (b)-(d), $\{u_m\}$ has a subsequence $\{u_{m_j}\}$ which is Cauchy in $L^p(\Omega)$: given $0 < \varepsilon < 1$, an N can be found such that

$$\|u_{m_j} - u_{m_k}\|_{L^p(\Omega)} < \varepsilon^{\frac{1}{p-1}} \quad \forall\, j, k \ge N.$$

Now choose $\beta = \varepsilon$ and apply the inequality (1.33) to $u_{m_j} - u_{m_k}$ to obtain

$$\|\gamma_0 u_{m_j} - \gamma_0 u_{m_k}\|^p_{L^p(\partial\Omega)} \le (c_1 + C_2)\varepsilon \quad \forall\, j, k \ge N;$$

this shows that the sequence $\{\gamma_0 u_{m_j}\}$ is Cauchy and thus converges in $L^p(\partial\Omega)$. □

Noncompact Embeddings. We point out the following noncompact emdedding results.

(i) *Unbounded domains.* The boundedness of Ω is essential in the above theorem. For example, let $I = (0,1)$ and $I_j = (j, j+1)$. Let $f \in C^1_0(I)$ and define f_j to be the same function defined on I_j by translation. We can normalize f so that $\|f\|_{W^{1,p}(I)} = 1$. The same is then true for each f_j and thus $\{f_j\}$ is a bounded sequence in $W^{1,p}(\mathbb{R})$. Clearly $f \in L^q(\mathbb{R})$ for every $1 \le q \le \infty$. Further, if

$$\|f\|_{L^q(\mathbb{R})} = \|f\|_{L^q(I)} = a > 0$$

then for any $j \ne k$ we have

$$\|f_j - f_k\|^q_{L^q(\mathbb{R})} = \int_j^{j+1} |f_j|^q + \int_k^{k+1} |f_k|^q = 2a^q$$

and so f_i cannot have a convergent subsequence in $L^q(\mathbb{R})$. Thus none of the embeddings $W^{1,p}(\mathbb{R}) \subset L^q(\mathbb{R})$ can be compact. This example generalizes to n dimensional space and to open sets like a half-space.

(ii) *Critical powers.* The embedding $W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega)$ is not compact if Ω is bounded open and $1 \le p < n$. For example, let $\{B(a_i, r_i)\}$ be a family of disjoint open balls compactly supported in Ω. Take a nontrivial function $\phi \in C^\infty_0(B(0,1))$. Let

$$\phi_i(x) = \begin{cases} r_i^{1-\frac{n}{p}}\phi(\frac{x-a_i}{r_i}) & \text{if } x \in B(a_i, r_i), \\ 0 & \text{if } x \in \Omega \setminus B(a_i, r_i). \end{cases}$$

Then $\phi_i \in W^{1,p}_0(\Omega)$ and $D\phi_i(x) = r_i^{-\frac{n}{p}} D\phi(\frac{x-a_i}{r_i})\chi_{B(a_i,r_i)}(x)$; thus

$$\|\phi_i\|_{L^p(\Omega)} = r_i\|\phi\|_{L^p(B(0,1))}, \quad \|D\phi_i\|_{L^p(\Omega)} = \|D\phi\|_{L^p(B(0,1))}.$$

Since $\{r_i\}$ is bounded, it follows that $\{\phi_i\}$ is a bounded sequence in $W^{1,p}_0(\Omega)$. But, for all $i \ne j$, since $B(a_i, r_i) \cap B(a_j, r_j) = \emptyset$,

$$\|\phi_i - \phi_j\|^{p^*}_{L^{p^*}(\Omega)} = \|\phi_i\|^{p^*}_{L^{p^*}(\Omega)} + \|\phi_j\|^{p^*}_{L^{p^*}(\Omega)} = 2\|\phi\|^{p^*}_{L^{p^*}(B(0,1))}.$$

Hence $\{\phi_i\}$ cannot have a subsequence which is Cauchy in $L^{p^*}(\Omega)$.

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1.8. Additional Topics

1.8.1. Equivalent Norms of $W^{1,p}(\Omega)$.

Definition 1.13. Two norms $\|\cdot\|$ and $|\cdot|$ on a vector space X are **equivalent** if there exist constants $c_1, c_2 \in (0,\infty)$ such that

$$c_1\|x\| \le |x| \le c_2\|x\| \quad \text{for all } x \in X.$$

Note that the property of a set to be open, closed, compact, or complete in a normed space is not affected if the norm is replaced by an equivalent norm.

Recall that a **seminorm** q on a vector space has all the properties of a norm except that $q(u) = 0$ need not imply $u = 0$.

Theorem 1.40. *Let Ω be bounded domain (open, connected) in $\mathbb{R}^n$ with $\partial\Omega \in C^1$ and $1 \le p < \infty$. Set*

$$\|u\| = \left(\int_\Omega \sum_{i=1}^n |D_i u|^p dx + (q(u))^p\right)^{1/p}$$

where $q : W^{1,p}(\Omega) \to \mathbb{R}$ is a seminorm with the following two properties:

(i) *There is a constant $d > 0$ such that*

$$q(u) \le d\|u\|_{1,p} \quad \forall\, u \in W^{1,p}(\Omega).$$

(ii) *If $u =$ constant, then $q(u) = 0$ implies $u = 0$.*

Then $\|\cdot\|$ is an equivalent norm on $W^{1,p}(\Omega)$.

Proof. It is easy to check that $\|\cdot\|$ satisfies the triangle inequality and $\|ku\| = |k|\|u\|$ for all $k \in \mathbb{R}, u \in W^{1,p}(\Omega)$. Furthermore, if $\|u\| = 0$ then $Du = 0$ and $q(u) = 0$. Since Ω is connected, this implies $u = C$ is constant (**Exercise!**) and $q(C) = 0$; thus by (ii), $C = 0$. This proves $\|u\|$ defines a norm on $W^{1,p}(\Omega)$.

To show $\|u\|$ is equivalent to $\|u\|_{1,p}$, by (i), it suffices to prove that there is a constant $c > 0$ such that

$$\|u\|_{1,p} \le c\|u\| \quad \forall\, u \in W^{1,p}(\Omega). \tag{1.34}$$

We use a *compactness proof*. Suppose (1.34) is false. Then there exists a sequence $v_n \in W^{1,p}(\Omega)$ such that $\|v_n\|_{1,p} > n\|v_n\|$. Set $u_n = v_n/\|v_n\|_{1,p}$. So

$$\|u_n\|_{1,p} = 1 \quad \text{and} \quad 1 > n\|u_n\|. \tag{1.35}$$

According to Theorem 1.39, there is a subsequence, call it again $\{u_n\}$, which converges to u in $L^p(\Omega)$. From (1.35) we have $\|u_n\| \to 0$ and therefore $Du_n \to 0$ in $L^p(\Omega)$ and $q(u_n) \to 0$. From $u_n \to u$, $Du_n \to 0$ both in $L^p(\Omega)$, we have $u \in W^{1,p}(\Omega)$ and $Du = 0$ a.e. in Ω; hence $u = C$, a constant, a.e. in Ω, which implies $u_n \to C$ in $W^{1,p}(\Omega)$. Since $\|u_n\|_{1,p} = 1$, it follows that $\|C\|_{1,p} = 1$ and thus $C \ne 0$. However, since q is semi-norm, by (i), we have $|q(u_n) - q(C)| \le q(u_n - C) \le d\|u_n - C\|_{1,p} \to 0$, and thus $q(C) = 0$, which implies $C = 0$ by (ii). We thus derive a contradiction. □

Example 1.41. Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^1$. Assume $a(x) \in C(\overline{\Omega}), \sigma(x) \in C(\partial\Omega)$ with $a \geq 0\ (\not\equiv 0),\ \sigma \geq 0\ (\not\equiv 0)$. Then the following norms are equivalent to $\|\cdot\|_{1,p}$ on $W^{1,p}(\Omega)$:

$$\|u\| = \left(\int_\Omega \sum_{i=1}^n |D_i u|^p dx + \left|\int_\Omega u dx\right|^p\right)^{1/p} \quad \text{with } q(u) = \left|\textstyle\int_\Omega u dx\right|. \tag{1.36}$$

$$\|u\| = \left(\int_\Omega \sum_{i=1}^n |D_i u|^p dx + \left|\int_{\partial\Omega} \gamma_0 u dS\right|^p\right)^{1/p} \quad \text{with } q(u) = \left|\textstyle\int_{\partial\Omega} \gamma_0 u dS\right|. \tag{1.37}$$

$$\|u\| = \left(\int_\Omega \sum_{i=1}^n |D_i u|^p dx + \int_{\partial\Omega} \sigma|\gamma_0 u|^p dS\right)^{1/p} \quad \text{with } q(u) = \left(\textstyle\int_{\partial\Omega} \sigma|\gamma_0 u|^p dS\right)^{1/p}. \tag{1.38}$$

$$\|u\| = \left(\int_\Omega \sum_{i=1}^n |D_i u|^p dx + \int_\Omega a|u|^p dx\right)^{1/p} \quad \text{with } q(u) = \left(\textstyle\int_\Omega a|u|^p dx\right)^{1/p}. \tag{1.39}$$

Clearly property (ii) of Theorem 1.40 is satisfied for each of these semi-norms $q(u)$. In order to verify condition (i), one uses the trace theorem in (1.37) and (1.38).

Theorem 1.42. (Poincaré's inequalities) *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^1$ and $1 \leq p < \infty$. Then there exist constants C_1, C_2 depending only on p, n and Ω such that*

$$\int_\Omega |u(x)|^p dx \leq C_1 \int_\Omega \sum_{i=1}^n |D_i u|^p dx \quad \forall\, u \in W_0^{1,p}(\Omega), \tag{1.40}$$

and

$$\int_\Omega |u(x) - (u)_\Omega|^p dx \leq C_2 \int_\Omega \sum_{i=1}^n |D_i u|^p dx \quad \forall\, u \in W^{1,p}(\Omega). \tag{1.41}$$

Proof. For (1.40), use the equivalent norm (1.37), while for (1.41), use the equivalent norm (1.36) for function $u - (u)_\Omega$. Note that by the Sobolev embedding, (1.40) also holds for all bounded open sets Ω. $\square$

1.8.2. Difference Quotients. For later use in elliptic regularity theory, we study the difference quotient approximations to weak derivatives.

Assume $u \in L^1_{loc}(\Omega)$. Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Define the ***i*-th difference quotient of size** h of u by

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}, \quad h \neq 0.$$

Then $D_i^h u$ is defined on $\Omega_{h,i} = \{x \in \Omega \,|\, x + he_i \in \Omega\}$. Note that

$$\Omega_h = \{x \in \Omega \,|\, \operatorname{dist}(x; \partial\Omega) > |h|\} \subset \Omega_{h,i}.$$

We have the following properties of $D_i^h u$. (Exercise!)

Lemma 1.43. *(a) If $u \in W^{1,p}(\Omega)$ then $D_i^h u \in W^{1,p}(\Omega_{h,i})$ and*

$$D(D_i^h u) = D_i^h(Du) \quad \text{on } \Omega_{h,i}.$$

(b) (**integration-by-parts**) *If either u or v has compact support $\Omega' \subset\subset \Omega$ then*

$$\int_\Omega u\, D_i^h v\, dx = -\int_\Omega v\, D_i^{-h} u\, dx \quad \forall\, 0 < |h| < \operatorname{dist}(\Omega'; \partial\Omega).$$

(c) (**product rule**) $D_i^h(\phi\, u)(x) = \phi(x)\, D_i^h u(x) + u(x + he_i)\, D_i^h \phi(x)$.

Theorem 1.44. (Difference quotient and weak derivatives)

(a) Let $u \in W^{1,p}(\Omega)$. Then $D_i^h u \in L^p(\Omega')$ for any $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega'; \partial\Omega)$. Moreover,

$$\|D_i^h u\|_{L^p(\Omega')} \le \|D_i u\|_{L^p(\Omega)}.$$

(b) Let $u \in L^p(\Omega)$, $1 < p < \infty$, and $\Omega' \subset\subset \Omega$. If there exists a constant $K > 0$ such that

$$\liminf_{h\to 0} \|D_i^h u\|_{L^p(\Omega')} \le K,$$

then the weak derivative $D_i u$ exists in Ω' and satisfies $\|D_i u\|_{L^p(\Omega')} \le K$.

Assertion (b) is false if $p = 1$. (Exercise!)

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Proof. *(a)* First assume $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ and $0 < h < \operatorname{dist}(\Omega', \partial\Omega)$. Then

$$D_i^h u(x) = \frac{1}{h}\int_0^h D_i u(x + te_i)\, dt;$$

thus, by Hölder's inequality,

$$|D_i^h u(x)|^p \le \frac{1}{h}\int_0^h |D_i u(x + te_i)|^p\, dt,$$

and hence,

$$\int_{\Omega'} |D_i^h u(x)|^p\, dx \le \frac{1}{h}\int_0^h \int_{B_t(\Omega')} |D_i u(y)|^p\, dy\, dt \le \int_\Omega |D_i u|^p\, dx,$$

where $B_t(\Omega') = \{x \in \Omega \mid \operatorname{dist}(x; \Omega') < t\} \subset B_h(\Omega') \subset \Omega$ for all $0 < t < h$. The extension of this inequality to arbitrary functions in $W^{1,p}(\Omega)$ follows by a straight-forward approximation argument. The same inequality also holds when $0 < -h < \operatorname{dist}(\Omega', \partial\Omega)$.

(b) Since $1 < p < \infty$, there exists a sequence $\{h_m\}$ tending to zero and a function $v \in L^p(\Omega')$ with $\|v\|_{p,\Omega'} \le K$ such that $D_i^{h_m} u \rightharpoonup v$ in $L^p(\Omega')$ as $m \to \infty$. This implies that for all $\phi \in C_0^\infty(\Omega')$

$$\lim_{m\to\infty} \int_{\Omega'} \phi D_i^{h_m} u\, dx = \int_{\Omega'} \phi v\, dx.$$

Now for $|h_m| < \operatorname{dist}(\operatorname{supp}\phi; \partial\Omega')$, we have

$$\int_{\Omega'} \phi D_i^{h_m} u\, dx = -\int_{\Omega'} u D_i^{-h_m} \phi\, dx.$$

Since $D_i^{h_m} u \rightharpoonup v$ in $L^p(\Omega')$ and $D_i^{-h_m}\phi \to D_i\phi$ uniformly on Ω' as $h_m \to 0$, we have

$$\int_{\Omega'} \phi v\, dx = -\int_{\Omega'} u D_i \phi\, dx,$$

which shows $v = D_i u \in L^p(\Omega')$ in the weak sense and $\|D_i u\|_{L^p(\Omega')} \le K$. $\square$

Remark 1.14. Variants of Theorem 1.44 can be valid even for domains $\Omega' \subset \Omega$ with $\partial\Omega' \cap \partial\Omega \ne \emptyset$. For example if Ω is the open half-ball $B(0,1) \cap \{x_n > 0\}$, $\Omega' = B(0,1/2) \cap \{x_n > 0\}$, and if $u \in W^{1,p}(\Omega)$, then we have the bounds on all the *tangential* difference quotients (but not the *normal* difference quotient) on the part of $x_n = 0$:

$$\|D_i^h u\|_{L^p(\Omega')} \le \|D_i u\|_{L^p(\Omega)} \quad \forall\, 1 \le i \le n-1, \;\; 0 < |h| < 1/2.$$

Also, if $u \in L^p(\Omega)$ and for some $i = 1, 2, \dots, n-1$,

$$\liminf_{h \to 0} \|D_i^h u\|_{L^p(\Omega')} \le K,$$

then $D_i u$ exists in $L^p(\Omega')$ and $\|D_i u\|_{L^p(\Omega')} \le K$.

We will need this remark for studying elliptic boundary regularity later.

1.8.3. Lipschitz Functions and $W^{1,\infty}(\Omega)$.

Theorem 1.45. *Let Ω be open bounded and $\partial\Omega \in C^1$. Then u is Lipschitz continuous in Ω if and only if $u \in W^{1,\infty}(\Omega)$.*

Proof. By Theorem 1.37, we only need to show that if u is Lipschitz continuous in Ω then $u \in W^{1,\infty}(\Omega)$. Assume u is Lipschitz in $\bar\Omega$. Define $\tilde u \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde u(x) = \min_{y \in \bar\Omega} \{u(y) + [Du]_{C^{0,1}(\Omega)} |y - x|\} \quad \forall\, x \in \mathbb{R}^n.$$

Then $\tilde u(x) = u(x)$ for all $x \in \bar\Omega$ and $\tilde u$ is Lipschitz continuous on $\mathbb{R}^n$ with $[D\tilde u]_{C^{0,1}} = [Du]_{C^{0,1}(\Omega)}$. (**Exercise!**) For the difference quotient of $\tilde u$, we have

$$\|D_i^{-h} \tilde u\|_{L^\infty(\mathbb{R}^n)} \le [Du]_{C^{0,1}(\Omega)}$$

for all $h \ne 0$. Hence $\{D_i^{-h} \tilde u\}$ is bounded in $L^2_{loc}(\mathbb{R}^n)$ and thus there exists a function $v_i \in L^2_{loc}(\mathbb{R}^n)$ such that

$$D_i^{-h_k} \tilde u \rightharpoonup v_i \quad \text{weakly in } L^2_{loc}(\mathbb{R}^n),$$

for a subsequence $\{D_i^{-h_k} \tilde u\}$ with $h_k \to 0$. Clearly $\|v_i\|_{L^\infty(\mathbb{R}^n)} \le [Du]_{C^{0,1}(\Omega)}$. We now show $v_i|_\Omega = D_i u$ weakly in Ω. Given $\phi \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_\Omega u \phi_{x_i}\, dx &= \lim_{h_k \to 0} \int_{\mathbb{R}^n} \tilde u D_i^{h_k} \phi\, dx = -\lim_{h_k \to 0} \int_{\mathbb{R}^n} (D_i^{-h_k} \tilde u) \phi\, dx \\ &= -\int_{\mathbb{R}^n} v_i \phi\, dx = -\int_\Omega v_i \phi\, dx.\end{aligned}$$

This proves $v_i|_\Omega = D_i u$ weakly in Ω. Hence $u \in W^{1,\infty}(\Omega)$. $\square$

Theorem 1.46. *Let $u \in W^{1,p}_{loc}(\Omega)$ for some $n < p \le \infty$. Then u differentiable a.e. in Ω and its gradient equals its weak gradient a.e.*

Proof. Assume $n < p < \infty$; the case $p = \infty$ follows easily. By Morrey's theorem, we assume u is Hölder continuous on $\bar\Omega$. Let Du be the weak gradient of u. Then $Du \in L^p_{loc}(\Omega)$. Hence, for a.e. $x \in \Omega$,

$$\lim_{r \to 0} \fint_{B(x,r)} |Du(x) - Du(z)|^p\, dz = 0.$$

Fix any such point x and consider function

$$v(y) = u(y) - u(x) - Du(x) \cdot (y - x).$$

Then $v \in W^{1,p}_{loc}(\Omega)$ and thus, by (1.25), for all $y \in \Omega$ with $|y-x| < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$,

$$|v(y) - v(x)| \leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Dv(z)|^p \, dz \right)^{1/p},$$

where $r = |x-y|$. This inequality gives

$$|u(y) - u(x) - Du(x)\cdot(y-x)| \leq Cr \left(\fint_{B(x,2r)} |Du(x) - Du(z)|^p \, dz \right)^{1/p} = o(|x-y|)$$

as $y \to x$. Hence, u is differentiable at x with Jacobian gradient $Du = Du(x)$. $\square$

Theorem 1.47. (Rademacher's Theorem) *Every locally Lipschitz continuous function is differentiable almost everywhere.*

1.8.4. Fourier Transform Methods. For a function $u \in L^1(\mathbb{R}^n)$, we define the **Fourier transform** of u by

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x) \, dx \quad \forall\, y \in \mathbb{R}^n,$$

and the **inverse Fourier transform** by

$$\check{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(x) \, dx \quad \forall\, y \in \mathbb{R}^n.$$

Theorem 1.48. (Plancherel's Theorem) *Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u}$, $\check{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Since $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can use this result to extend the Fourier transforms onto $L^2(\mathbb{R}^n)$. We still use the same notations for them.

Theorem 1.49. (Properties of Fourier Tranforms) *Assume $u, v \in L^2(\mathbb{R}^n)$. Then*

(i) $\int_{\mathbb{R}^n} u\bar{v} \, dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}} \, dy$,

(ii) $\widehat{D^\alpha u}(y) = (iy)^\alpha \hat{u}(y)$ *for each multiindex α such that $D^\alpha u \in L^2(\mathbb{R}^n)$,*

(iii) $u = \check{\hat{u}}$.

Next we use the Fourier transform to characterize the spaces $H^k(\mathbb{R}^n)$.

Theorem 1.50. *Let k be a nonnegative integer. Then, a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\hat{u}(y) \in L^2(\mathbb{R}^n).$$

In addition, there exists a constant C such that

$$C^{-1} \|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\,\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C \|u\|_{H^k(\mathbb{R}^n)}$$

for all $u \in H^k(\mathbb{R}^n)$.

Proof. 1. Assume $u \in H^k(\mathbb{R}^n)$. Then $D^\alpha u \in L^2(\mathbb{R}^n)$ for all multiindices α with $|\alpha| \leq k$. Hence $\widehat{D^\alpha u} = (iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ with $\|D^\alpha u\|_{L^2} = \|\widehat{D^\alpha u}\|_{L^2} = \|y^\alpha \hat{u}\|_{L^2}$. With $\alpha = (k, 0, \ldots, 0), \ldots, \alpha = (0, \ldots, 0, k)$, we have

$$\int_{\mathbb{R}^n} |y|^{2k} |\hat{u}|^2 dy \leq C \int_{\mathbb{R}^n} |D^k u|^2 dx$$

and hence
$$\int_{\mathbb{R}^n} (1+|y|^k)^2 |\hat{u}|^2 dy \le C\|u\|^2_{H^k(\mathbb{R}^n)} < \infty.$$

2. Suppose $(1+|y|^k)\hat{u} \in L^2(\mathbb{R}^n)$. Let $|\alpha| \le k$. Then
$$\|(iy)^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} |y|^{2|\alpha|}|\hat{u}|^2 dy \le C\|(1+|y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} < \infty.$$

Let $u_\alpha = ((iy)^\alpha \hat{u})^\vee \in L^2(\mathbb{R}^n)$. Then, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,
$$\int_{\mathbb{R}^n} (D^\alpha \phi)\bar{u} dx = \int_{\mathbb{R}^n} \widehat{D^\alpha \phi}\bar{\hat{u}} dy = \int_{\mathbb{R}^n} (iy)^\alpha \hat{\phi}\bar{\hat{u}} dy = (-1)^{|\alpha|}\int_{\mathbb{R}^n} \phi \overline{u_\alpha} dx.$$

This proves $u_\alpha = D^\alpha u$ (which must be real) in the weak sense. Hence $u \in H^k(\mathbb{R}^n)$. Clearly, one also has
$$\begin{aligned}\|D^\alpha u\|_{L^2(\mathbb{R}^n)} &= \|u_\alpha\|_{L^2(\mathbb{R}^n)} = \|(iy)^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^n)} \\ &\le \int_{\mathbb{R}^n} |y|^{2|\alpha|}|\hat{u}|^2 dy \le C\|(1+|y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

□

Definition 1.15. For any $s \ge 0$, define the **fractional Sobolev space** $H^s(\mathbb{R}^n)$ by
$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \,|\, (1+|y|^s)\,\hat{u} \in L^2(\mathbb{R}^n)\},$$
equipped with the norm given by by
$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1+|y|^s)\,\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

Theorem 1.51. *If $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.*

Proof. Let $s > n/2$ and $u \in H^s(\mathbb{R}^n)$. Then
$$\begin{aligned}\|\hat{u}\|_{L^1(\mathbb{R}^n)} &= \|(1+|y|^s)\hat{u}\,(1+|y|^s)^{-1}\|_{L^1(\mathbb{R}^n)} \\ &\le \|(1+|y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}\|(1+|y|^s)^{-1}\|_{L^2(\mathbb{R}^n)} \\ &\le C\,\|u\|_{H^s(\mathbb{R}^n)},\end{aligned}$$
where $C = \|(1+|y|^s)^{-1}\|_{L^2(\mathbb{R}^n)} < \infty$ since $s > \frac{n}{2}$. Therefore,
$$\|u\|_{L^\infty(\mathbb{R}^n)} = \|\check{\hat{u}}\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{(2\pi)^{n/2}}\|\hat{u}\|_{L^1(\mathbb{R}^n)} \le \frac{C}{(2\pi)^{n/2}}\,\|u\|_{H^s(\mathbb{R}^n)}.$$

□

Part II – Second-Order Linear Elliptic Equations

Lecture 16 – 2/15/19

2.1. Differential Equations in Divergence Form

2.1.1. Linear Elliptic Equations. We study the Dirichlet boundary value problem (BVP)

$$\begin{cases} Lu = f \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^n$ denotes a bounded domain, f is a given function in $L^2(\Omega)$ (or more generally, an element in the dual space of $H_0^1(\Omega)$) and L is a **second-order linear differential operator** having either the **divergence form**

$$Lu \equiv -\sum_{i,j=1}^{n} D_i\big(a_{ij}(x)D_j u\big) + \sum_{i=1}^{n} b_i(x)D_i u + c(x)u \tag{2.2}$$

or else the **nondivergence form**

$$Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u + \sum_{i=1}^{n} b_i(x)D_i u + c(x)u$$

with given real coefficients $a_{ij}(x),\ b_i(x)$ and $c(x)$. We also assume

$$a_{ij}(x) = a_{ji}(x) \quad (i, j = 1, \ldots, n).$$

Remark 2.1. If the coefficients a_{ij} are C^1 functions, then an operator in divergence form can be rewritten into nondivergence form, and vice versa. However, there are definite advantages to considering the two different representations of L separately. The divergence form is most natural for energy methods, based on integration by parts, and nondivergence form is most appropriate for maximum principle techniques. We focus on the operators in divergence form.

Definition 2.2. The differential operator L (in either form) is said to be **uniformly elliptic** in Ω if there exists a number $\theta > 0$ such that for almost every $x \in \Omega$ and every real vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta \sum_{i=1}^{n} |\xi_i|^2. \tag{2.3}$$

Or, equivalently, the symmetric matrix $A(x) = (a_{ij}(x))$ is positive definite with smallest eigenvalue $\geq \theta$ for a.e. $x \in \Omega$.

2.1.2. Weak Solutions. We assume Lu is in the divergence form (2.2) and assume $a_{ij}, b_i, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

How should we define a weak or generalized solution of the equation $Lu = f$ in Ω? Assume the derivatives appearing in Lu are all classical derivatives; then for a test function $v \in C_0^\infty(\Omega)$ we have by integration by parts

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i v + \sum_{i=1}^{n} b_i (D_i u) v + cuv \Big) dx = \int_\Omega f v \, dx.$$

By approximation we find the same identity holds for all $v \in H_0^1(\Omega)$. The left-hand side of this identity also makes sense if only $u \in H^1(\Omega)$. This motivates the definition of **weak solutions**.

Definition 2.3. A function $u \in H^1(\Omega)$ is called a **weak solution** of equation $Lu = f$ in Ω provided the following **variational formulation** holds:

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i v + (\sum_{i=1}^{n} b_i D_i u + cu) v \Big) dx = \int_\Omega f v dx \quad \forall\, v \in H_0^1(\Omega). \tag{2.4}$$

By a **weak solution** to the Dirichlet BVP (2.1) with L given by (2.2) we mean a weak solution u of $Lu = f$ in Ω that belongs to $H_0^1(\Omega)$.

Exercise 2.4. Consider the following weak formulation: Given $f \in L^2(\Omega)$. Find $u \in H^1(\Omega)$ satisfying

$$\int_\Omega Du \cdot Dv dx = \int_\Omega f v dx \quad \forall\, v \in H^1(\Omega).$$

Find the boundary value problem solved by u. What is the necessary condition for the existence of such a u?

To study the existence of weak solutions and the equations $Lu = f$ with more general right-hand side f, we need some functional analysis.

2.1.3. Some Functional Analysis.

Definition 2.5. Let X be a normed vector space with norm $\|\cdot\|$. The **dual space** of X, denoted by X^*, is the space of all linear bounded functionals $f\colon X \to \mathbb{R}$ with the **norm**

$$\|f\|_{X^*} = \sup\{\langle f, u\rangle \mid u \in X,\ \|u\| \leq 1\},$$

where $\langle\ ,\ \rangle$ is the pairing between X and X^*; namely, $\langle f, u\rangle = f(u)$ for all $f \in X^*$ and $u \in X$.

Theorem 2.1. *X^* with the given norm is always a Banach space.*

Theorem 2.2. (Riesz Representation Theorem) *Let H be a (real) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Then, for each $f \in H^*$, there exists a unique $u \in H$ such that*

$$\langle f, v\rangle = (u, v) \quad \forall\, v \in H. \tag{2.5}$$

Moreover, the map $J\colon H^* \to H$ defined by $Jf = u$ is a linear **isometry** *from H^* onto H; that is, J is one-to-one and onto and $\|f\|_{H^*} = \|Jf\|$ for all $f \in H^*$.*

Proof. Let $f \in H^*$. Clearly, the element u satisfying (2.5) is unique; we prove the existence of such a $u \in H$. If $f \equiv 0$ then let $u = Jf = 0$. Assume $f \neq 0$. Then $\langle f, x\rangle \neq 0$ for some $x \in H$. Let

$$V = \{y \in H :\ \langle f, y\rangle = 0\}.$$

Then V is a closed subspace of H and $V \neq H$ since $x \notin V$. Let

$$\mu = \inf_{y\in V} \|x - y\|.$$

The following is a **direct method of Calculus of Variations**. There exists a sequence $y_n \in V$ such that $\|y_n - x\| \to \mu$. Note that, for $n, m = 1, 2, \ldots,$

$$\|y_m - y_n\|^2 = 2(\|y_m - x\|^2 + \|y_n - x\|^2) - 4\|\frac{y_m + y_n}{2} - x\|^2.$$

Since $\frac{y_m+y_n}{2} \in V$ and thus $\|\frac{y_m+y_n}{2} - x\| \geq \mu$, it follows that

$$\|y_m - y_n\|^2 \leq 2(\|y_m - x\|^2 + \|y_n - x\|^2) - 4\mu^2.$$

Thus

$$\lim_{m,n\to\infty} \|y_m - y_n\|^2 \leq 2(\mu^2 + \mu^2) - 4\mu^2 = 0,$$

which proves that $\{y_n\}$ is Cauchy in H; hence $y_n \to y$ as $n \to \infty$ for some $y_0 \in H$. This implies $\|y_0 - x\| = \mu$. Also, as V is closed, one has $y_0 \in V$. Therefore,

$$\|x - y_0\| \leq \|x - y\| \quad \forall\, y \in V.$$

This implies that, for each $z \in V$, the quadratic function $h(t) = \|x - y_0 + tz\|^2$ has minimum at $t = 0$; hence $h'(0) = 0$, which gives

$$(x - y_0, z) = 0 \quad \forall\, z \in V.$$

Since $x \notin V$ and $y_0 \in V$, one has $x - y_0 \neq 0$; hence, let

$$u = \frac{\langle f, x\rangle}{\|x - y_0\|^2}(x - y_0) \in H,$$

so that $\langle f, u\rangle = \|u\|^2 > 0$. For each $v \in H$, let $z = v - \frac{\langle f,v\rangle}{\langle f,u\rangle}u$. Then

$$\langle f, z\rangle = \langle f, v\rangle - \frac{\langle f, v\rangle}{\langle f, u\rangle}\langle f, u\rangle = 0$$

and hence $z \in V$. Consequently $(u, z) = 0$, and thus

$$(u, v) = \frac{\langle f, v\rangle}{\langle f, u\rangle}(u, u) = \langle f, v\rangle \quad \forall\, v \in H,$$

proving (2.5). Therefore, $Jf = u$ defines a map $J\colon H^* \to H$. Clearly J is linear and one-to-one. To show J is onto, let $u \in H$ and define $f_u\colon H \to \mathbb{R}$ by $\langle f_u, v\rangle = (u, v)$ for all $v \in H$. Then $f_u \in H^*$ and $J(f_u) = u$, proving the surjectivity of J onto H. Finally, to show J is an isometry, let $f \in H^*$ and $u = Jf \in H$. Since

$$\langle f, v\rangle = (u, v) \leq \|u\|\|v\| \leq \|u\| \quad \forall\, v \in H,\ \|v\| \leq 1,$$

it follows that

$$\|f\|_{H^*} = \sup\{\langle f, v\rangle \mid v \in H,\ \|v\| \le 1\} \le \|u\| = \|Jf\|.$$

If $f = 0$ then $\|Jf\| = \|u\| = 0$. Assume $f \neq 0$. Then $u = Jf \neq 0$ and thus

$$\|f\|_{H^*} \ge \langle f, \frac{u}{\|u\|}\rangle = (u, \frac{u}{\|u\|}) = \|u\| = \|Jf\|.$$

Therefore $\|Jf\| = \|f\|_{H^*}$ for all $f \in H^*$. □

For general boundary conditions, we need to study the Dirichlet problem (2.1) with more general right-hand sides f. For this purpose, we study the **dual space** of $H_0^1(\Omega)$.

Theorem 2.3. (Characterization of $H^{-1}(\Omega)$) *Let $H^{-1}(\Omega) = (H_0^1(\Omega))^*$. Then, for each $f \in H^{-1}(\Omega)$, there exist functions $f^0, f^1, \dots, f^n$ in $L^2(\Omega)$ such that*

$$\langle f, v\rangle = \int_\Omega \Big(f^0 v + \sum_{i=1}^n f^i D_i v\Big)dx \quad \forall\, v \in H_0^1(\Omega). \tag{2.6}$$

In this case, we write $f = f^0 - \sum_{i=1}^n D_i f^i$. One also has

$$\|f\|_{H^{-1}(\Omega)} = \inf\left\{ \Big(\int_\Omega \sum_{i=0}^n |f^i|^2 dx\Big)^{1/2} \,\Big|\, f^0, f^1, \dots, f^n \in L^2(\Omega) \text{ satisfy (2.6)}\right\}. \tag{2.7}$$

Proof. Note that the inner product in $H_0^1(\Omega)$ is defined by

$$(u, v) = \int_\Omega (uv + Du\cdot Dv)dx.$$

By the **Riesz Representation Theorem**, there exists a unique $u \in H_0^1(\Omega)$ such that

$$\langle f, v\rangle = (u, v) \quad \forall\, v \in H_0^1(\Omega),$$

and $\|f\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)}$. This establishes (2.6) with the functions $f^0 = u$, $f^i = D_i u$.

To prove (2.7), let $g^0, \dots, g^n$ be any functions in $L^2(\Omega)$ such that

$$\langle f, v\rangle = \int_\Omega \Big(g^0 v + \sum_{i=1}^n g^i D_i v\Big)dx \quad \forall\, v \in H_0^1(\Omega).$$

With $v = u \in H_0^1(\Omega)$ determined above, we have

$$\int_\Omega \Big(\sum_{i=0}^n |f^i|^2\Big)dx = \int_\Omega (|Du|^2 + u^2)dx = \langle f, u\rangle = \int_\Omega \Big(g^0 f^0 + \sum_{i=1}^n g^i f^i\Big)dx.$$

This implies (**why?**)

$$\int_\Omega \Big(\sum_{i=0}^n |f^i|^2\Big)dx \le \int_\Omega \Big(\sum_{i=0}^n |g^i|^2\Big)dx.$$

Hence

$$\|f\|^2_{H^{-1}(\Omega)} = \|u\|^2_{H_0^1(\Omega)} = \int_\Omega \Big(\sum_{i=0}^n |f^i|^2\Big)dx \le \int_\Omega \Big(\sum_{i=0}^n |g^i|^2\Big)dx,$$

which proves (2.7). The proof also shows that the infimum in (2.7) is in fact a minimum. □

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Definition 2.6. Let X denote a real vector space. A map $B : X \times X \to \mathbb{R}$ is called a **bilinear form** if

$$B[\alpha u + \beta v, w] = \alpha B[u, w] + \beta B[v, w],$$

$$B[w, \alpha u + \beta v] = \alpha B[w, u] + \beta B[w, v]$$

for all $u, v, w \in X$ and all $\alpha, \beta \in \mathbb{R}$.

2.1.4. Weak Solutions for General Right-hand Side f.

Definition 2.7. (i) The **bilinear form** $B[u, v]$ associated with the divergence form operator L given by (2.2) is defined by

$$B[u, v] \equiv \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i v + (\sum_{i=1}^n b_i D_i u + cu) v \Big) dx \tag{2.8}$$

for all $u, v \in H^1(\Omega)$.

(ii) Let $f = f^0 - \sum_{i=1}^n D_i f^i \in H^{-1}(\Omega)$, where $f^0, f^1, \ldots, f^n \in L^2(\Omega)$. We say that $u \in H^1(\Omega)$ is a **weak solution** of $Lu = f$ in Ω provided $B[u, v] = \langle f, v \rangle$ for all $v \in H_0^1(\Omega)$; that is,

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i v + (\sum_{i=1}^n b_i D_i u + cu) v \Big) dx = \int_\Omega \Big(f^0 v + \sum_{i=1}^n f^i D_i v \Big)\, dx \quad \forall\, v \in H_0^1(\Omega).$$

(iii) A **weak solution** u to the Dirichlet problem

$$\begin{cases} Lu = f^0 - \sum_{i=1}^n D_i f^i \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases} \tag{2.9}$$

with L given by (2.2) is a weak solution u of $Lu = f$ in Ω that belongs to $H_0^1(\Omega)$.

Remark 2.8. For general boundary condition $u = g$ on $\partial\Omega$, we assume $g = \gamma_0(w)$ for some $w \in H^1(\Omega)$. In this case, let $\tilde{u} = u - w$; then the Dirichlet problem

$$\begin{cases} Lu = f \ \text{ in } \ \Omega, \\ u = g \ \text{ on } \ \partial\Omega \end{cases}$$

is equivalent to the Dirichlet problem with zero boundary condition

$$\begin{cases} L\tilde{u} = \tilde{f} \ \text{ in } \ \Omega, \\ \tilde{u} = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\tilde{f} = f - Lw \in H^{-1}(\Omega)$. Therefore, for general boundary value problems it is necessary to study (2.9) with right-hand side in $H^{-1}(\Omega)$.

2.1.5. General Linear PDE Systems in Divergence Form. For N unknown functions, $u^1, \cdots, u^N$, we write $u = (u^1, \cdots, u^N)$ and define that $u \in X(\Omega; \mathbb{R}^N)$ if each $u^k \in X(\Omega)$, where X is a symbol of any function spaces we have learned; for instance, $X = W^{1,p}$, C_0^∞, $C^{k,\gamma}$, etc. If $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ then we use Du to denote the $N \times n$ Jacobi matrix

$$Du = (D_i u^k = \partial u^k / \partial x_i)_{1 \le k \le N, 1 \le i \le n}.$$

A general second-order linear operator Lu of **PDE system in divergence form** is given by

$$Lu = -\operatorname{div} A(x, u, Du) + b(x, u, Du), \tag{2.10}$$

where $A(x, s, Du) = (A_i^k(x, u, Du))$, $1 \le i \le n$, $1 \le k \le N$, and $b(x, s, Du) = (b^k(x, u, Du))$, $1 \le k \le N$, are linear operators of u and Du given by

$$\begin{aligned} A_i^k(x, u, Du) &= \sum_{1\le l\le N,\, 1\le j\le n} a_{ij}^{kl}(x)\, D_j u^l + \sum_{1\le l\le N} d_i^{kl}(x)\, u^l, \\ b^k(x, u, Du) &= \sum_{1\le j\le n,\, 1\le l\le N} b_j^{kl}(x)\, D_j u^l + \sum_{1\le l\le N} c^{kl}(x)\, u^l, \end{aligned} \tag{2.11}$$

with given coefficient functions $a_{ij}^{kl}(x), b_i^{kl}(x), c^{kl}(x)$ and $d_i^{kl}(x)$.

Remark 2.9. Even when $N = 1$ the form (2.10) with (2.11) is more general than (2.2) because of the term $d(x)u$ inside $A(x, u, Du)$, which cannot be included into the term $b(x) \cdot Du$ if d is not smooth.

Definition 2.10. (i) The bilinear form associated with Lu given by (2.10) and (2.11) is defined by

$$B[u, v] \equiv \int_\Omega \Big(a_{ij}^{kl} D_j u^l D_i v^k + d_i^{kl} u^l D_i v^k + b_j^{kl} v^k D_j u^l + c^{kl} u^l v^k \Big)\, dx \tag{2.12}$$

for all $u, v \in H^1(\Omega; \mathbb{R}^N)$; here the **conventional summation notation is used** (i.e., *repeated indices are summed up*).

(ii) Let $F = (f^1, \dots, f^N)$ with $f^k \in H^{-1}(\Omega)$ for each $k = 1, 2, \dots, N$. A function $u \in H^1(\Omega; \mathbb{R}^N)$ is called a **weak solution** of $Lu = F$ in Ω provided

$$B[u, v] = \langle F, v\rangle \quad \forall\, v \in H_0^1(\Omega; \mathbb{R}^N),$$

where $\langle F, v\rangle$ is the pairing between $H_0^1(\Omega; \mathbb{R}^N)$ and its dual $H^{-1}(\Omega; \mathbb{R}^n)$ given by

$$\langle F, v\rangle = \sum_{k=1}^N \langle f^k, v^k\rangle \quad \text{for all } v = (v^k) \in H_0^1(\Omega; \mathbb{R}^N).$$

(iii) A **weak solution** u to the Dirichlet BVP

$$\begin{cases} Lu = F \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases} \tag{2.13}$$

is a weak solution u of $Lu = F$ in Ω that belongs to $H_0^1(\Omega; \mathbb{R}^N)$.

Ellipticity Conditions for Systems.

Definition 2.11. Let Lu be defined by (2.10) and (2.11).

(i) L is said to satisfy the (uniform, strong) **Legendre ellipticity condition** if there exists a $\theta > 0$ such that, for almost every $x \in \Omega$, it holds

$$\sum_{i,j=1}^n \sum_{k,l=1}^N a_{ij}^{kl}(x)\, \eta_i^k\, \eta_j^l \ge \theta\, |\eta|^2 \quad \text{for all } N \times n \text{ matrix } \eta = (\eta_i^k). \tag{2.14}$$

(ii) L is said to satisfy the (uniform, strong) **Legendre-Hadamard condition** if for almost every $x \in \Omega$, it holds

$$\sum_{i,j=1}^{n}\sum_{k,l=1}^{N} a_{ij}^{kl}(x)\, q^k q^l p_i p_j \geq \theta\, |p|^2 |q|^2 \quad \forall\, p \in \mathbb{R}^n,\ q \in \mathbb{R}^N. \tag{2.15}$$

Remark 2.12. The Legendre condition (2.14) implies the Legendre-Hadamard condition (2.15), and they are equivalent if $N = 1$ or $n = 1$.

However, if $N > 1$ and $n > 1$, then the Legendre-Hadamard condition does not imply the Legendre ellipticity condition. For example, let $n = N = 2$ and $\varepsilon > 0$. Define constants a_{ij}^{kl} (*not uniquely*) by

$$\sum_{i,j,k,l=1}^{2} a_{ij}^{kl}\, \xi_i^k\, \xi_j^l \equiv \det \xi + \varepsilon|\xi|^2.$$

Use such a_{ij}^{kl}'s to define an operator Lu as above. Show that Lu satisfies the Legendre-Hadamard condition holds for all $\varepsilon > 0$, but satisfies the Legendre condition if and only if $\varepsilon > 1/2$.

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2.2. Existence of Weak Solutions

2.2.1. Contraction Mapping Theorem.

Definition 2.13. Let X be a normed vector space. A map $T\colon X \to X$ (not necessarily linear) is called a **contraction** if there exists a number $\theta \in [0, 1)$ such that

$$\|T(u) - T(v)\| \leq \theta \|u - v\| \quad \forall\, u, v \in X. \tag{2.16}$$

Theorem 2.4. (Contraction Mapping Theorem) *Let X be a Banach space and $T\colon X \to X$ be a contraction. Then T has a unique* **fixed point** *in X; that is, there exists a unique $x \in X$ such that $T(x) = x$.*

Proof. Assume T satisfies (2.16). The fixed point of T must be unique, for if $T(x) = x$ and $T(y) = y$ then $\|x - y\| = \|T(x) - T(y)\| \leq \theta\|x - y\|$ and thus $\|x - y\| = 0$ as $0 \leq \theta < 1$. We now prove the existence of a fixed point. Fix any $x_0 \in X$. Let

$$x_1 = T(x_0),\ x_2 = T(x_1),\ \cdots, x_n = T(x_{n-1}), \cdots$$

Then $\{x_n\}$ is a sequence in X satisfying

$$\|x_n - x_{n-1}\| = \|T(x_{n-1}) - T(x_{n-2})\| \leq \theta\|x_{n-1} - x_{n-2}\| \leq \cdots \leq \theta^{n-1}\|x_1 - x_0\|.$$

Hence, for all $m > n \geq 0$,

$$\|x_m - x_n\| \leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \leq \|x_1 - x_0\| \sum_{i=n}^{m-1} \theta^i \leq \|x_1 - x_0\| \sum_{i=n}^{\infty} \theta^i = \|x_1 - x_0\| \frac{\theta^n}{1-\theta}.$$

This shows that $\{x_n\}$ is Cauchy in X; thus, $x_n \to x$ for some $x \in X$. Hence

$$\|x - T(x)\| \leq \|x - x_n\| + \|T(x_{n-1}) - T(x)\| \leq \|x_n - x\| + \theta\|x_{n-1} - x\| \to 0,$$

which shows that x is a fixed point of T. $\square$

2.2.2. Lax-Milgram Theorem. Let H denote a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We denote by $\langle,\rangle$ the pairing between H and its dual H^*.

The following theorem generalizes the **Riesz Representation Theorem**.

Theorem 2.5. (Lax-Milgram Theorem) *Let $B\colon H\to H$ be a bilinear form. Assume*

(i) *B is* **bounded***; i.e., $|B[u,v]|\le\alpha\|u\|\|v\|$, and*

(ii) *B is* **strongly positive***; i.e., $B[u,u]\ge\beta\|u\|^2$,*

where α,β are positive constants. (Strong positivity is also called the **coercivity** for B.) *Let $f\in H^*$. Then there exists a unique element $u\in H$ such that*

$$B[u,v]=\langle f,v\rangle,\quad \forall\, v\in H. \tag{2.17}$$

Moreover, the solution u satisfies $\|u\|\le\frac{1}{\beta}\|f\|$.

Proof. For each fixed $u\in H$, the functional $v\mapsto B[u,v]$ is in H^*, and hence by the **Riesz Representation Theorem**, there exists a unique element $w=Au\in H$ such that

$$B[u,v]=(w,v)\quad\forall\, v\in H.$$

It can be easily shown that $A:H\to H$ is linear. From (i), $\|Au\|^2=B[u,Au]\le\alpha\|u\|\|Au\|$, and hence $\|Au\|\le\alpha\|u\|$ for all $u\in H$; that is, A is bounded. Furthermore, by (ii), $\beta\|u\|^2\le B[u,u]=(Au,u)\le\|Au\|\|u\|$ and hence $\|Au\|\ge\beta\|u\|$ for all $u\in H$. By the Riesz Representation Theorem again, we have a unique $w_0\in H$ such that $\langle f,v\rangle=(w_0,v)$ for all $v\in H$ and $\|f\|=\|w_0\|$.

We show that the equation $Au=w_0$ has a (unique) solution. There are many different proofs for this, and here we use the **Contraction Mapping Theorem**. Note that the solution u to equation $Au=w_0$ is equivalent to the fixed-point of the map $T\colon H\to H$ defined by $T(v)=v-tAv+tw_0$ ($v\in H$) for any fixed $t>0$. For all $v,w\in H$ we have $\|T(v)-T(w)\|=\|(I-tA)(v-w)\|$. We compute that, for all $u\in H$,

$$\|(I-tA)u\|^2=\|u\|^2+t^2\|Au\|^2-2t(Au,u)\le\|u\|^2(1+t^2\alpha^2-2\beta t).$$

Now let $0<t<\min\{\frac{1}{2\beta},\frac{2\beta}{\alpha^2}\}$, so that $\theta=1+t^2\alpha^2-2\beta t\in(0,1)$. Hence the map $T\colon H\to H$ is a contraction on H and thus has a unique fixed point u; this fixed point u solves $Au=w_0$. Therefore

$$B[u,v]=(Au,v)=(w_0,v)=\langle f,v\rangle\quad\forall\, v\in H;$$

hence u is a solution of (2.17); clearly such a solution must be unique, by (ii). Moreover, we have $\|f\|=\|w_0\|=\|Au\|\ge\beta\|u\|$ and hence $\|u\|\le\frac{1}{\beta}\|f\|$. The proof is complete. □

Remark 2.14. If B is also symmetric; that is, $B[u,v]=B[v,u]$ for all $u,v\in H$, then $((u,v))\equiv B[u,v]$ is an inner product which makes H an equivalent Hilbert space. In this case, the Lax-Milgram Theorem is just the Riesz Representation Theorem. In general, *the Lax-Milgram Theorem is primarily significant in that it does not require the symmetry of $B[u,v]$.*

2.2.3. Energy Estimates. Let $B[u,v]$ be the bilinear form defined by (2.8) or (2.12) above. From the Hölder inequality,

$$\left|\int_\Omega fg\,dx\right|\le\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)},$$

it is easy to see that $B[u,v]$ satisfies the boundedness:

$$|B[u,v]|\le\alpha\|u\|\|v\|$$

for all u, v in the respective Hilbert spaces $H = H_0^1(\Omega)$ or $H = H_0^1(\Omega; \mathbb{R}^N)$.

However, the strong positivity (or *coercivity*) for B is not always guaranteed, but involves estimating the quadratic form $B[u,u]$; such estimates are usually called **energy estimates** or **Gårding's estimates**.

LECTURE 19 – 2/25/19

Gårding's Estimates for (2.8).

Theorem 2.6. *Assume the ellipticity condition* (2.3) *holds. Then, there are constants $\beta > 0$ and $\gamma \geq 0$ such that*

$$B[u,u] \geq \beta \|u\|^2 - \gamma \|u\|^2_{L^2(\Omega)} \quad \forall\, u \in H_0^1(\Omega). \tag{2.18}$$

Proof. Note that, by the ellipticity,

$$B[u,u] \geq \theta \int_\Omega \sum_{i=1}^n |D_i u|^2 dx + \int_\Omega (\sum_{i=1}^n b_i D_i u + cu) u\, dx.$$

Let $m = \max\{\|b_i\|_{L^\infty(\Omega)} \,|\, 1 \leq i \leq n\}$ and $k_0 = \|c\|_{L^\infty(\Omega)}$. Then

$$\begin{aligned} |(b_i D_i u, u)_2| &\leq m \|D_i u\|_2 \|u\|_2 \\ &\leq (m/2)(\varepsilon \|D_i u\|_2^2 + (1/\varepsilon)\|u\|_2^2) \end{aligned}$$

where we have used the Cauchy inequality with ε: $|\alpha\beta| \leq (\varepsilon/2)\alpha^2 + (1/2\varepsilon)\beta^2$. Combining the estimates we find

$$B[u,u] \geq (\theta - m\varepsilon/2)\|Du\|^2_{L^2(\Omega)} - (k_0 + mn/2\varepsilon)\|u\|^2_{L^2(\Omega)}.$$

By choosing $\varepsilon > 0$ sufficiently small so that $\theta - m\varepsilon/2 > 0$ we arrive at the desired inequality, using the Poincare inequality: $\|u\|_{H^1(\Omega)} \leq C\|Du\|_{L^2(\Omega)}$ for all $u \in H_0^1(\Omega)$. □

Theorem 2.7. (First Existence Theorem for weak solutions) *Let $\beta > 0$, $\gamma \geq 0$ be the constants in* (2.18)*. Then, for each $\lambda \geq \gamma$ and for each $f \in H^{-1}(\Omega)$, the Dirichlet boundary value problem*

$$\begin{cases} Lu + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.19}$$

has a unique weak solution $u \in H_0^1(\Omega)$, which also satisfies

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\beta}\|f\|_{H^{-1}(\Omega)}.$$

Proof. Let $\lambda \geq \gamma$ and define

$$B^\lambda[u,v] \equiv B[u,v] + \lambda(u,v)_2 \quad \forall\, u, v \in H.$$

Then $B^\lambda[u,v]$ is the bilinear form associated with differential operator $Lu + \lambda u$. Moreover, $B^\lambda[u,v]$ satisfies the boundedness and coercivity, with $B^\lambda[u,u] \geq \beta\|u\|^2$. Thus the result follows from the Lax-Milgram Theorem. □

Remark 2.15. For elliptic operators Lu with $b_i(x) = 0$ and $c(x) \geq 0$, we have $\gamma = 0$ in (2.18) and hence Theorem 2.7 holds with $\gamma = 0$. This includes the special case of the Laplace operator $Lu = -\Delta u$.

Example 2.8. Consider the **Neumann boundary** value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

A function $u \in H^1(\Omega)$ is said to be a **weak solution** to (2.20) if

$$\int_\Omega Du \cdot Dv \, dx = \int_\Omega fv \, dx \quad \forall\, v \in H^1(\Omega). \tag{2.21}$$

Obviously, taking $v \equiv 1 \in H^1(\Omega)$, a necessary condition to have a weak solution is $\int_\Omega f(x)\,dx = 0$. We show that this is also sufficient for the existence of a weak solution. Note that, if u is a weak solution, then $u + c$, for all constants c, is also a weak solution. Therefore, to fix the constants, we consider the vector space

$$H = \left\{ u \in H^1(\Omega) \,\Big|\, \int_\Omega u(x)\,dx = 0 \right\}$$

equipped with inner product

$$(u, v)_H = \int_\Omega Du \cdot Dv \, dx.$$

By the equivalent norm theorem or **Poincaré's inequality**, it follows that H with this inner product, is indeed a Hilbert space, and $(f, u)_{L^2(\Omega)}$ is a bounded linear functional on H:

$$|(f, u)_{L^2(\Omega)}| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \|v\|_H.$$

Hence the Riesz Representation Theorem implies that there exists a unique $u \in H$ such that

$$(u, w)_H = (f, w)_{L^2(\Omega)}, \quad \forall\, w \in H. \tag{2.22}$$

It follows that u is a weak solution to the Neumann problem since for any $v \in H^1(\Omega)$ we take $w = v - c \in H$, where $c = \frac{1}{|\Omega|}\int_\Omega v dx$, in (2.22) and obtain (2.21) using $\int_\Omega f dx = 0$.

Example 2.9. Let us consider the nonhomogeneous Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where $f \in L^2(\Omega)$ and φ is the trace of a function $w \in H^1(\Omega)$.

Note that it is not sufficient to just require that $\varphi \in L^2(\partial\Omega)$ since the trace operator is not onto. If, for example, $\varphi \in C^1(\partial\Omega)$, then φ has a C^1 extension to $\bar{\Omega}$, which is the desired w.

Definition 2.16. A function $u \in H^1(\Omega)$ is called a **weak solution** of (2.23) if $u - w \in H_0^1(\Omega)$ and if

$$\int_\Omega Du \cdot Dv dx = \int_\Omega fv dx \quad \forall\, v \in H_0^1(\Omega).$$

Let u be a weak solution of (2.23) and set $\tilde{u} = u - w$. Then $\tilde{u} \in H_0^1(\Omega)$ satisfies

$$\int_\Omega D\tilde{u} \cdot Dv dx = \int_\Omega (fv - Dw \cdot Dv) dx \quad \forall\, v \in H_0^1(\Omega). \tag{2.24}$$

Therefore, $\tilde u = u - w \in H^1_0(\Omega)$ is a weak solution of the problem

$$\begin{cases} -\Delta \tilde u = \tilde f & \text{in } \Omega, \\ \tilde u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.25}$$

where $\tilde f = f + \operatorname{div}(Dw) = f + \Delta w \in H^{-1}(\Omega)$. The Lax-Milgram theorem yields the existence of a unique $\tilde u \in H^1_0(\Omega)$ of (2.25). Thus, (2.23) has a unique weak solution $u \in H^1(\Omega)$.

LECTURE 20 – 2/27/19

Gårding's Estimates for (2.12). Assume $\Omega \subset \mathbb{R}^n$ is a bounded open set. Let $H = H^1_0(\Omega; \mathbb{R}^N)$ be equipped with the equivalent inner product

$$(u, v)_H = \sum_{i,j=1}^n \sum_{k,l=1}^N \int_\Omega D_j u^l \, D_i v^k \, dx \quad \forall\, u, v \in H.$$

and let $\|u\|_H$ be the associated equivalent norm. Define the bilinear form associated with the leading term of L by

$$A[u, v] = \sum_{i,j=1}^n \sum_{k,l=1}^N \int_\Omega a^{kl}_{ij}(x) \, D_j u^l \, D_i v^k \, dx.$$

Theorem 2.10. *Assume that* either *coefficients a^{kl}_{ij} satisfy the* Legendre condition or a^{kl}_{ij} *are all constants and satisfy the* Legendre-Hadamard condition. *Then*

$$A[u, u] \ge \theta \, \|u\|^2_H \quad \forall\, u \in H. \tag{2.26}$$

Proof. The conclusion in the first case follows easily from the pointwise inequality by Legendre condition. We prove the second case when a^{kl}_{ij} are constants and satisfy the Legendre-Hadamard condition

$$\sum_{i,j=1}^n \sum_{k,l=1}^N a^{kl}_{ij} \, q^k q^l p_i p_j \ge \theta \, |p|^2 |q|^2 \quad \forall p \in \mathbb{R}^n, \; q \in \mathbb{R}^N.$$

We prove

$$A[u, u] = \sum_{i,j=1}^n \sum_{k,l=1}^N \int_\Omega a^{kl}_{ij} \, D_j u^l D_i u^k \, dx \ge \theta \int_\Omega |Du|^2 \, dx \quad \forall\, u \in C^\infty_0(\Omega; \mathbb{R}^N);$$

then the estimate (2.26) follows by approximation. Let $u \in C^\infty_0(\Omega; \mathbb{R}^N)$. We extend u onto $\mathbb{R}^n$ by zero outside Ω and thus consider u as functions in $C^\infty_0(\mathbb{R}^n; \mathbb{R}^N)$. Consider the **Fourier transforms** of u,

$$\hat u(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\, y \cdot x} \, u(x) \, dx \quad (y \in \mathbb{R}^n).$$

Then, for any u, $v \in C^\infty_0(\mathbb{R}^n; \mathbb{R}^N)$,

$$\int_{\mathbb{R}^n} u(x) \cdot v(x) \, dx = \int_{\mathbb{R}^n} \hat u(y) \cdot \overline{\hat v(y)} \, dy, \quad \widehat{D_j u^k}(y) = i \, y_j \widehat{u^k}(y);$$

the last identity can also be written as $\widehat{Du}(y) = i \, \hat u(y) \otimes y$. Now, using these identities, *with an abuse of the index i and the imaginary number i,* we have

$$\int_{\mathbb{R}^n} a^{kl}_{ij} \, D_i u^k(x) \, D_j u^l(x) \, dx = \int_{\mathbb{R}^n} a^{kl}_{ij} \, \widehat{D_i u^k}(y) \, \overline{\widehat{D_j u^l}(y)} \, dy$$

$$= \int_{\mathbb{R}^n} a_{ij}^{kl}\, y_i\, y_j\, \widehat{u^k}(y)\, \overline{\widehat{u^l}(y)}\, dy = \mathrm{Re}\left(\int_{\mathbb{R}^n} a_{ij}^{kl}\, y_i\, y_j\, \widehat{u^k}(y)\, \overline{\widehat{u^l}(y)}\, dy\right).$$

Write $\hat{u}(y) = \eta + i\xi$ with $\eta,\, \xi \in \mathbb{R}^N$. Then

$$\mathrm{Re}\left(\widehat{u^k}(y)\, \overline{\widehat{u^l}(y)}\right) = \eta^k\eta^l + \xi^k\xi^l.$$

Therefore, by the Legendre-Hadamard condition,

$$\mathrm{Re}\sum_{i,j=1}^{n}\sum_{k,l=1}^{N}\left(a_{ij}^{kl}\, y_i\, y_j\, \widehat{u^k}(y)\, \overline{\widehat{u^l}(y)}\right) \ge \theta\, |y|^2\, (|\eta|^2 + |\xi|^2) = \theta\, |y|^2\, |\hat{u}(y)|^2.$$

Hence,

$$\begin{aligned} A[u,u] &= \sum_{i,j=1}^{n}\sum_{k,l=1}^{N}\int_{\mathbb{R}^n} a_{ij}^{kl}\, D_i u^k(x)\, D_j u^l(x)\, dx \\ &= \mathrm{Re}\sum_{i,j=1}^{n}\sum_{k,l=1}^{N}\left(\int_{\mathbb{R}^n} a_{ij}^{kl}\, y_i\, y_j\, \widehat{u^k}(y)\, \overline{\widehat{u^l}(y)}\, dy\right) \\ &\ge \theta\int_{\mathbb{R}^n} |y|^2\, |\hat{u}(y)|^2\, dy = \theta\int_{\mathbb{R}^n} |i\hat{u}(y)\otimes y|^2\, dy \\ &= \theta\int_{\mathbb{R}^n} |\widehat{Du}(y)|^2\, dy = \theta\int_{\mathbb{R}^n} |Du(x)|^2\, dx. \end{aligned}$$

The proof is complete. □

Theorem 2.11. (Gårding's Estimate for Systems) *Let* $B[u,v]$ *be defined by* (2.12). *Assume that* either

$$\begin{cases} a_{ij}^{kl} \in L^\infty(\Omega) \text{ satisfy the Legendre condition,} \\ b_i^{kl},\, c^{kl},\, d_i^{kl} \in L^\infty(\Omega), \end{cases} \tag{2.27}$$

or

$$\begin{cases} a_{ij}^{kl} \in C(\bar{\Omega}) \text{ satisfy the Legendre-Hadamard condition,} \\ b_i^{kl},\, c^{kl},\, d_i^{kl} \in L^\infty(\Omega). \end{cases} \tag{2.28}$$

Then there exist constants $\beta > 0$ *and* $\gamma \ge 0$ *such that*

$$B[u,u] \ge \beta\|u\|_H^2 - \gamma\|u\|_{L^2}^2 \quad \forall\, u \in H = H_0^1(\Omega;\mathbb{R}^N). \tag{2.29}$$

Proof. We only prove the case (2.28); the proof for case (2.27) is similar and much easier.

1. By uniform continuity of a_{ij}^{kl} on $\bar{\Omega}$, there exists $\epsilon > 0$ such that

$$|a_{ij}^{kl}(x) - a_{ij}^{kl}(y)| \le \frac{\theta}{2} \quad \forall\, x,\, y \in \bar{\Omega},\ |x-y| \le \epsilon.$$

We claim

$$\int_\Omega a_{ij}^{kl}(x)\, D_i u^k D_j u^l\, dx \ge \frac{\theta}{2}\int_\Omega |Du(x)|^2\, dx = \frac{\theta}{2}\sum_{\substack{1\le i\le n \\ 1\le k\le N}}\int_\Omega |D_i u^k(x)|^2\, dx \tag{2.30}$$

for all test functions $u \in C_0^\infty(\Omega;\mathbb{R}^N)$ with $\mathrm{diam}(\mathrm{supp}\, u) \le \epsilon$. To prove this, we use a **freezing coefficient method**. Fix any point $x_0 \in \mathrm{supp}\, u$. Then, by Theorem 2.10,

$$\int_\Omega a_{ij}^{kl}(x)\, D_i u^k\, D_j u^l\, dx = \int_\Omega a_{ij}^{kl}(x_0)\, D_i u^k\, D_j u^l\, dx + \int_{\mathrm{supp}\, u}\left(a_{ij}^{kl}(x) - a_{ij}^{kl}(x_0)\right) D_i u^k\, D_j u^l\, dx$$

$$\geq \theta \int_\Omega |Du(x)|^2\,dx - \frac{\theta}{2}\int_\Omega |Du(x)|^2\,dx,$$

which proves (2.30).

2. Now assume $u \in C_0^\infty(\Omega;\mathbb{R}^N)$, with arbitrary compact support. We cover $\bar\Omega$ by finitely many open balls $\{B_{\epsilon/4}(x^m)\}$ with $x^m \in \Omega$ and $m = 1, 2, ..., M$. For each m, let $\zeta_m \in C_0^\infty(B_{\epsilon/2}(x^m))$ with $\zeta_m(x) = 1$ for $x \in B_{\epsilon/4}(x^m)$. Since every $x \in \bar\Omega$ belongs to a ball $B_{\epsilon/4}(x^m)$ and thus $\zeta_m(x) = 1$, it follows that $\sum_{j=1}^M \zeta_j^2(x) \geq 1$ for all $x \in \bar\Omega$. Define

$$\varphi_m(x) = \frac{\zeta_m(x)}{\left(\sum_{j=1}^M \zeta_j^2(x)\right)^{1/2}}, \quad m = 1, 2, ..., M.$$

Then $\varphi_m \in C_0^\infty(B_{\epsilon/2}(x^m))$ and $\sum_{m=1}^M \varphi_m^2(x) = 1$ for all $x \in \bar\Omega$. (This is a special case of the **partition of unity**.) We thus have

$$\begin{aligned}
a_{ij}^{kl}(x)\,D_iu^k\,D_ju^l &= \sum_{m=1}^M \left(a_{ij}^{kl}(x)\,\varphi_m^2\,D_iu^k\,D_ju^l\right)\\
&= \sum_{m=1}^M a_{ij}^{kl}(x)\,D_i(\varphi_m\,u^k)\,D_j(\varphi_m\,u^l)\\
&\quad - \sum_{m=1}^M a_{ij}^{kl}(x)\,\left(\varphi_m\,u^l\,D_i\varphi_m\,D_iu^k + \varphi_m\,u^k\,D_i\varphi_m\,D_ju^l + u^k\,u^l\,D_i\varphi_m\,D_j\varphi_m\right).
\end{aligned} \tag{2.31}$$

Since $\varphi_m u \in C_0^\infty(\Omega \cap B_{\epsilon/2}(x^m);\mathbb{R}^N)$ and $\operatorname{diam}(\Omega \cap B_{\epsilon/2}(x^m)) \leq \epsilon$, we have by (2.30)

$$\begin{aligned}
\int_\Omega a_{ij}^{kl}(x)\,D_i(\varphi_m\,u^k)D_j(\varphi_m\,u^l)\,dx &\geq \frac{\theta}{2}\sum_{\substack{1\leq i\leq n\\ 1\leq k\leq N}}\int_\Omega |D_i(\varphi_m\,u^k)|^2\,dx\\
&= \frac{\theta}{2}\sum_{\substack{1\leq i\leq n\\ 1\leq k\leq N}}\int_\Omega \left(\varphi_m^2\,|D_iu^k|^2 + |D_i\varphi_m|^2\,|u^k|^2 + 2\varphi_m\,u^k\,D_i\varphi_m\,D_iu^k\right)dx\\
&\geq \frac{\theta}{2}\sum_{\substack{1\leq i\leq n\\ 1\leq k\leq N}}\int_\Omega \left(\varphi_m^2\,|D_iu^k|^2\,dx + 2\varphi_m\,u^k\,D_i\varphi_m\,D_iu^k\right)dx\\
&\geq \frac{\theta}{4}\sum_{\substack{1\leq i\leq n\\ 1\leq k\leq N}}\int_\Omega \varphi_m^2\,|D_iu^k|^2\,dx - C\|u\|_{L^2(\Omega)}^2 = \frac{\theta}{4}\int_\Omega \varphi_m^2\,|Du|^2\,dx - C\|u\|_{L^2(\Omega)}^2,
\end{aligned}$$

where we have used the Cauchy inequality with ϵ. Then by (2.31) and the fact that $\sum_{m=1}^M \varphi_m^2 = 1$ on Ω,

$$\int_\Omega a_{ij}^{kl}(x)\,D_iu^k\,D_ju^l\,dx \geq \frac{\theta}{4}\int_\Omega |Du|^2\,dx - CM\,\|u\|_{L^2(\Omega)}^2 - C_1\,\|u\|_{L^2(\Omega)}\,\|Du\|_{L^2(\Omega)}.$$

The terms in $B[u,u]$ involving b, c and d can all be estimated by

$$C_2(\|u\|_{L^2(\Omega)}\,\|Du\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^2).$$

Finally, using the Cauchy inequality with ϵ again, we have

$$B[u,u] \geq \frac{\theta}{8}\|u\|_{H_0^1(\Omega)}^2 - C_3\|u\|_{L^2(\Omega)}^2 \quad \forall\, u \in C_0^\infty(\Omega;\mathbb{R}^N)$$

and, by density, the estimate holds for all $u \in H_0^1(\Omega;\mathbb{R}^N)$. This completes the proof. $\square$

Lecture 21 – 3/1/19

Theorem 2.12. (First Existence Theorem for weak solutions of systems) *Under the hypotheses of Theorem 2.11, let $\beta > 0$, $\gamma \geq 0$ be the constants in* (2.29)*. Then, for each $\lambda \geq \gamma$ and each $F \in H^{-1}(\Omega; \mathbb{R}^N)$, the Dirichlet problem*

$$\begin{cases} Lu + \lambda u = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.32}$$

has a unique weak solution u in $H_0^1(\Omega; \mathbb{R}^N)$. Moreover, the solution u satisfies

$$\|u\|_{H^1} \leq \frac{1}{\beta} \|F\|_{H^{-1}}.$$

Proof. Note that the bilinear form $B^\lambda[u, v] = B[u, v] + \lambda\,(u, v)_{L^2}$ satisfies the condition of the Lax-Milgram theorem on $H = H_0^1(\Omega; \mathbb{R}^N)$ for all $\lambda \geq \gamma$; thus, the result follows from the Lax-Milgram theorem. □

Fix $\lambda \geq \gamma$. Let $F \in L^2(\Omega; \mathbb{R}^N)$. Define $\tilde{F} \in H^{-1}(\Omega; \mathbb{R}^N)$ by $\langle \tilde{F}, v \rangle = (F, v)_{L^2}$ for all $v \in H_0^1(\Omega; \mathbb{R}^N)$. Then $\|\tilde{F}\|_{H^{-1}} \leq \|F\|_{L^2}$. Let $u = \mathcal{K}F \in H_0^1(\Omega; \mathbb{R}^N)$ be the unique weak solution to $Lu + \lambda u = \tilde{F}$ in Ω; that is, formally, $u = \mathcal{K}F = (L + \lambda I)^{-1}F$. In this way, we defined an operator $\mathcal{K} \colon L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$. Note that $\mathcal{K}$ is linear and maps $L^2(\Omega; \mathbb{R}^N)$ into $H_0^1(\Omega; \mathbb{R}^N)$ with

$$\|\mathcal{K}F\|_{H^1} \leq \frac{1}{\beta}\|\tilde{F}\|_{H^{-1}} \leq \frac{1}{\beta}\|F\|_{L^2}. \tag{2.33}$$

Recall the following definition.

Definition 2.17. Let X, Y be two Banach spaces. A linear operator $T \colon X \to Y$ is called a **compact operator** if $\|Tu\|_Y \leq C\|u\|_X$ for all $u \in X$ and for each bounded sequence $\{u_i\}$ in X there exists a subsequence $\{u_{i_k}\}$ such that $\{Tu_{i_k}\}$ converges in Y.

Corollary 2.13. *Given $\lambda \geq \gamma$ as in Theorem 2.12, the operator $\mathcal{K} = (L+\lambda I)^{-1} \colon L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$ defined above is a compact linear operator.*

Proof. By (2.33), $\mathcal{K}$ is a bounded linear operator from $L^2(\Omega; \mathbb{R}^N)$ into $H_0^1(\Omega; \mathbb{R}^N)$. By the **Rellich-Kondrachov Theorem**, $H_0^1(\Omega; \mathbb{R}^N)$ is compactly embedded in $L^2(\Omega; \mathbb{R}^N)$ and hence, as a linear operator from $L^2(\Omega; \mathbb{R}^N)$ to $L^2(\Omega; \mathbb{R}^N)$, $\mathcal{K}$ is compact. □

2.2.4. More Functional Analysis.

Let H be a (real) Hilbert space with inner product $(\cdot\,,\,\cdot)$ and norm $\|\cdot\|$.

Definition 2.18. Let $T \colon H \to H$ be a bounded linear operator on H.

(1) We define the **nullspace** or **kernel** of T to be $\mathcal{N}(T) = \{x \in H \mid Tx = 0\}$, and define the **range** of T to be $\mathcal{R}(T) = \{Tx \mid x \in H\}$.

(2) We define the Hilbert **adjoint operator** of T to be the operator $T^* \colon H \to H$ by

$$(Tx, y) = (x, T^*y) \quad \forall\, x, y \in H.$$

T is called **symmetric** if $T^* = T$.

Theorem 2.14. *Let $T \colon H \to H$ be a bounded linear operator on H. Then $T^* \colon H \to H$ is linear and bounded with $\|T^*\| = \|T\|$. Moreover, if T is compact, then T^* is also compact.*

Proof. 1. By definition, T^* is linear. Note also that

$$\|T^*y\|^2 = (T^*y, T^*y) = (TT^*y, y) \le \|TT^*y\|\|y\| \le \|T\|\|T^*y\|\|y\| \quad \forall\, y \in H,$$

which gives $\|T^*y\| \le \|T\|\|y\|$ and hence $\|T^*\| \le \|T\|$. The other direction follows similarly or by the identity $(T^*)^* = T$.

2. Assume T is compact. Let $\{x_n\}$ be any bounded sequence in H. Then $\{T^*x_n\}$ is also bounded in H; thus, by the compactness of T, there exists a subsequence $\{T^*x_{n_k}\}$ such that $\{TT^*x_{n_k}\}$ is a Cauchy sequence in H. Note that

$$\begin{aligned}\|T^*x_{n_k} - T^*x_{n_j}\|^2 &= (T^*(x_{n_k} - x_{n_j}), T^*(x_{n_k} - x_{n_j})) = (TT^*(x_{n_k} - x_{n_j}), x_{n_k} - x_{n_j}) \\ &\le \|TT^*x_{n_k} - TT^*x_{n_j}\|\|x_{n_k} - x_{n_j}\| \le C\|TT^*x_{n_k} - TT^*x_{n_j}\|;\end{aligned}$$

thus $\{T^*x_{n_k}\}$ is a Cauchy sequence in H. This proves the compactness of T^*. □

For a subspace V of H we denote by $V^\perp$ the **orthogonal space** of V defined by $V^\perp = \{x \in H \mid (x, y) = 0 \ \ \forall\, y \in V\}$.

Lemma 2.15. *Let V be a subspace of H. Then for each $u \in H$ there exist unique elements $v \in \bar V$ and $w \in V^\perp$ such that $u = v + w$. The operators $P_{\bar V}\colon H \to \bar V$ and $P_{V^\perp}\colon H \to V^\perp$ defined by $P_{\bar V}u = v$ and $P_{V^\perp}u = w$ are called the* **orthogonal projections** *onto $\bar V$ and $V^\perp$, respectively.*

Proof. Let $\mu = \operatorname{dist}(u, V)$. Assume $v_k \in V$ and $\|v_k - u\| \to \mu$. As in the proof of the Riesz Representation Theorem, for $n, m = 1, 2, \ldots,$

$$\|v_m - v_n\|^2 = 2(\|v_m - u\|^2 + \|v_n - u\|^2) - 4\|\frac{v_m + v_n}{2} - u\|^2 \le 2(\|v_m - u\|^2 + \|v_n - u\|^2) - 4\mu^2.$$

This proves that $\{v_k\}$ is a Cauchy sequence in H. So assume $v_k \to v$; then $v \in \bar V$. Also, for all $z \in V$, $h(t) = \|u - v + tz\|^2$ assumes the minimum at $t = 0$; this implies $h'(0) = 2(u - v, z) = 0$, which holds for all $z \in V$, and hence $w = u - v \in V^\perp$. This proves the existence of $v \in \bar V$ and $w \in V^\perp$. If v' and w' are other elements satisfying this property then $v - v' = w' - w \in \bar V \cap (V^\perp) = \{0\}$. This proves the uniqueness of v, w. □

Lemma 2.16. *Let $A\colon H \to H$ be a linear bounded operator. Then $\overline{\mathcal{R}(A)} = (\mathcal{N}(A^*))^\perp$.*

Proof. Given any $y \in \mathcal{R}(A)$ and $z \in \mathcal{N}(A^*)$, let $y = Ax$ for some $x \in H$. Then

$$(y, z) = (Ax, z) = (x, A^*z) = 0.$$

Thus $\mathcal{R}(A) \subseteq (\mathcal{N}(A^*)^\perp$. Since $(\mathcal{N}(A^*)^\perp$ is closed, we have $\overline{\mathcal{R}(A)} \subseteq (\mathcal{N}(A^*))^\perp$. To show the opposite inclusion, let $u \in (\mathcal{N}(A^*))^\perp$. By the lemma above, $u = v + w$ for some $v \in \overline{\mathcal{R}(A)}$ and $w \in (\mathcal{R}(A))^\perp$. Since $w \in (\mathcal{R}(A))^\perp$, we have $(x, A^*w) = (Ax, w) = 0$ for all $x \in H$, and thus $A^*w = 0$; that is, $w \in \mathcal{N}(A^*)$ and thus $(u, w) = 0$, which gives $(w, w) = 0$, $w = 0$ and thus $u = v \in \overline{\mathcal{R}(A)}$. □

Lecture 22 – 3/11/19

Theorem 2.17. (Fredholm Alternative) *Let $T\colon H \to H$ be a compact linear operator on H. Then*

(i) $\dim \mathcal{N}(I - T) = \dim \mathcal{N}(I - T^*) < \infty$.

(ii) $\mathcal{R}(I - T) = (\mathcal{N}(I - T^*))^\perp$.

Remark 2.19. (i) Note that (i) and (ii) of the theorem imply that

$$\mathcal{N}(I-T)=\{0\} \iff \mathcal{R}(I-T)=H; \tag{2.34}$$

that is, $I-T$ is onto if and only if it is one-to-one. Therefore, **either** equation $(I-T)u=f$ has a unique solution $u\in H$ for each $f]\in H$, **or else**, $(I-T)u=0$ has a nontrivial solution $u\in H$ (in this case $\dim\mathcal{N}(I-T)$ is also finite); this dichotomy is the **Fredholm alternative**.

(ii) Moreover, by (i) and (ii) of the theorem, given $f\in H$, the equation $(I-T)u=f$ is solvable for u if and only if there exist functions $g_1,\dots,g_d$ in H such that

$$(f,g_i)=0 \quad \forall\, i=1,2,\dots,d, \tag{2.35}$$

where $d=\dim\mathcal{N}(I-T^*)<\infty$ and $g_1,\dots,g_d$ are linearly independent in $\mathcal{N}(I-T^*)$.

Proof of Theorem 2.17. 1. Suppose $\dim\mathcal{N}(I-T)$ is not finite; then there exists a sequence $\{u_k\}$ in $\mathcal{N}(I-T)$ such that $(u_k,u_j)=\delta_{kj}$ for all k,j. Since $Tu_k=u_k$, it follows that for all $k\neq j$,

$$\|Tu_k-Tu_j\|^2=\|u_k-u_j\|^2=\|u_k\|^2+\|u_j\|^2-2(u_k,u_j)=2,$$

and hence $\{Tu_k\}$ cannot have a subsequence that is Cauchy in H; this contradicts the compactness of T. So $\dim\mathcal{N}(I-T)<\infty$.

2. We claim there exists a constant $\delta>0$ such that

$$\|u-Tu\|\ge\delta\|u\| \quad \forall\, u\in(\mathcal{N}(I-T))^\perp. \tag{2.36}$$

Suppose this is not true; then there exists a sequence $\{u_k\}$in $(\mathcal{N}(I-T))^\perp$ with $\|u_k\|=1$ but $\|u_k-Tu_k\|<1/k$ for all k. Since T is compact, there exists a subsequence $\{Tu_{k_i}\}$ converging to $y\in H$ as $k_i\to\infty$. Then $\|u_{k_i}-y\|\le\|u_{k_i}-Tu_{k_i}\|+\|Tu_{k_i}-y\|\to 0$; that is, $u_{k_i}\to y$. Hence $\|y\|=1$ and $Tu_{k_i}\to Ty$, which implies $Ty=y$ and thus $y\in\mathcal{N}(I-T)$. Since $u_k\in(\mathcal{N}(I-T))^\perp$, one has $(y,u_{k_i})=0$, which implies $(y,y)=0$, a contradiction to $\|y\|=1$.

3. Let $A=I-T$. Since $A^*=I-T^*$, (ii) will follow from Lemma 2.16 if we show that $\mathcal{R}(A)$ is closed. So let $v_k\in\mathcal{R}(A)$ and $v_k\to v\in H$. Let $v_k=Au_k$, and $u_k=x_k+y_k$, where $x_k\in\mathcal{N}(A)$ and $y_k\in(\mathcal{N}(A))^\perp$. Then $v_k=Ay_k$. By (2.36),

$$\|y_k-y_j\|\le\frac{1}{\delta}\|Ay_k-Ay_j\|=\frac{1}{\delta}\|v_k-v_j\|,$$

and thus $\{y_k\}$ is Cauchy in $(\mathcal{N}(A))^\perp$. Hence $y_k\to y$ for some $y\in(\mathcal{N}(A))^\perp$. This implies $v_k=Ay_k\to Ay$ and thus $v=Ay\in\mathcal{R}(A)$. This proves the closedness of $\mathcal{R}(A)$ and hence completes the proof of (ii).

4. Next we assert

$$\dim\mathcal{N}(I-T)\ge\dim(\mathcal{R}(I-T))^\perp.$$

Again write $A=I-T$. Suppose instead $\dim\mathcal{N}(A)<\dim(\mathcal{R}(A))^\perp$. Then there exists a bounded linear operator $P\colon\mathcal{N}(A)\to(\mathcal{R}(A))^\perp$ that is one-to-one but not onto. Define $Q=P\Pi\colon H\to H$, where $\Pi\colon H\to\mathcal{N}(A)$ is the orthogonal projection onto $\mathcal{N}(A)$. Then Q is compact since $\dim\mathcal{R}(Q)<\infty$. We claim $\mathcal{N}(I-T-Q)=\{0\}$. Indeed if $Tu+Qu=u$ then $Au=Qu\in(\mathcal{R}(A))^\perp$ and hence $Qu=Au=0$, which implies $u\in\mathcal{N}(A)$ and $Qu=Pu=0$; thus $u=0$ as P is one-to-one. Now claim $H_1=(I-T-Q)(H)=H$. If not, suppose $H_1\neq H$. Since $I-T-Q$ is one-to-one, $H_2=(I-T-Q)(H_1)\neq H_1$, $\cdots$,

$H_k = (I-T-Q)(H_{k-1}) \neq H_{k-1}$ for all $k = 2, 3, \cdots$. Choose $u_k \in H_k$, $\|u_k\| = 1$ and $u_k \in H_{k+1}^{\perp}$. Let $K = T + Q$. Then K is compact (**why?**) and, for $l > k$, from

$$Ku_k - Ku_l = -(u_k - Ku_k) + (u_l - Ku_l) + (u_k - u_l),$$

it follows that $\|Ku_k - Ku_l\|^2 \geq 1$; this contradicts the compactness of K. Thus $I - T - Q$ is also onto. However, take an element $v \in (\mathcal{R}(A))^{\perp}$ but $v \notin \mathcal{R}(P)$. Then equation

$$v = (I-T-Q)u = Au - Qu$$

has no solution $u \in H$ as, otherwise, one would have $Au = v + Qu \in (\mathcal{R}(A))^{\perp}$ and thus $Au = 0$ and $v = -Qu \in \mathcal{R}(Q) = \mathcal{R}(P)$, a contradiction.

5. Finally, we prove $\dim \mathcal{N}(I-T) = \dim (I-T^*)$. Note that, by (ii), $(\mathcal{R}(I-T))^{\perp} = \mathcal{N}(I-T^*)$ and thus, by Step 4,

$$\dim \mathcal{N}(I-T) \geq \dim(\mathcal{R}(I-T))^{\perp} = \dim \mathcal{N}(I-T^*).$$

The opposite inequality follows by using the identity $(T^*)^* = T$. □

2.2.5. Adjoint Bilinear Form and Adjoint Operator. We study the general linear system Lu whose bilinear form $B[u,v]$ is defined by (2.12) above on $H = H_0^1(\Omega; \mathbb{R}^N)$, where $N \geq 1$.

Definition 2.20. The **adjoint bilinear form** B^* of B is defined by

$$B^*[u,v] = B[v,u] \quad \forall\, u,\, v \in H = H_0^1(\Omega; \mathbb{R}^N).$$

This bilinear form $B^*[u,v]$ is associated to the **formal adjoint** of Lu of the form

$$L^*u = -\operatorname{div} A^*(x,u,Du) + b^*(x,u,Du), \tag{2.37}$$

with $A^*(x,u,Du) = (\tilde{A}_i^k)$ and $b^*(x,u,Du) = (\tilde{b}^k)$ given by

$$\begin{aligned} \tilde{A}_i^k(x,u,Du) &= \sum_{1\leq l\leq N,\, 1\leq j\leq n} \tilde{a}_{ij}^{kl}(x)\, D_j u^l + \sum_{l=1}^N \tilde{d}_i^{kl}(x)\, u^l, \\ \tilde{b}^k(x,u,Du) &= \sum_{1\leq j\leq n,\, 1\leq l\leq N} \tilde{b}_j^{kl}(x)\, D_j u^l + \sum_{l=1}^N \tilde{c}^{kl}(x)\, u^l, \end{aligned} \tag{2.38}$$

where

$$\tilde{a}_{ij}^{kl} = a_{ji}^{lk}, \quad \tilde{d}_i^{kl} = b_i^{lk}, \quad \tilde{b}_j^{kl} = d_i^{lk}, \quad \tilde{c}^{kl} = c^{lk} \quad (1 \leq i, j \leq n,\ 1 \leq k, l \leq N).$$

Note that the Legendre or Legendre-Hadamard condition for L^*u is the same as that of Lu, and also that $B^*[u,u] = B[u,u]$.

Remark 2.21. Suppose $B[u,v]$ satisfies the Gårding's estimate in Theorem 2.11. Let $\lambda \geq \gamma$ and $\mathcal{K} = (L + \lambda I)^{-1} \colon L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$. Then $\mathcal{K}^* = (L^* + \lambda I)^{-1}$.

Theorem 2.18. (Second Existence Theorem for weak solutions) *Assume the conditions of Theorem* 2.11 *hold.*

(i) *Precisely one of the following statements holds:*
either

$$\begin{cases} \text{for each } F \in L^2(\Omega; \mathbb{R}^N) \text{ there exists a unique} \\ \text{weak solution } u \in H_0^1(\Omega; \mathbb{R}^N) \text{ of } Lu = F, \end{cases} \tag{2.39}$$

or else

$$\text{there exists a weak solution } u \neq 0 \text{ in } H_0^1(\Omega; \mathbb{R}^N) \text{ of } Lu = 0. \tag{2.40}$$

(ii) *Furthermore, should case* (2.40) *hold, the dimension of the subspace* $\mathcal{N} \subset H_0^1(\Omega;\mathbb{R}^N)$ *of weak solutions of* $Lu = 0$ *is finite and equals the dimension of the subspace* $\mathcal{N}^* \subset H_0^1(\Omega;\mathbb{R}^N)$ *of weak solutions of adjoint problem* $L^*u = 0$.

(iii) *Finally, the problem* $Lu = F$ *has a weak solution if and only if*

$$(F, v)_{L^2(\Omega;\mathbb{R}^N)} = 0 \quad \forall\, v \in \mathcal{N}^*.$$

The dichotomy (2.39), (2.40) *is called the* **Fredholm alternative**.

Proof. Let $\mathcal{K} = (L+\lambda I)^{-1} \colon L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ be an operator defined as in Corollary 2.13 with a fixed number $\lambda > \gamma \geq 0$. Then $\mathcal{K} \colon L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ is a compact linear operator; moreover, $\mathcal{K}^* = (L^* + \lambda I)^{-1} \colon L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ is also a compact linear operator.

Given $F \in L^2(\Omega;\mathbb{R}^N)$, a function $u \in H_0^1(\Omega;\mathbb{R}^N)$ is a weak solution of $Lu = F$ if and only if $Lu + \lambda u = F + \lambda u$, which is equivalent to $u = \mathcal{K}(F + \lambda u) = \mathcal{K}F + \lambda\mathcal{K}u$; therefore,

$$Lu = F \iff (I - \lambda\mathcal{K})u = \mathcal{K}F. \tag{2.41}$$

In particular, $Lu = 0$, i.e., $u \in \mathcal{N}$, if and only if $u \in \mathcal{N}(I - \lambda\mathcal{K})$; thus $\mathcal{N} = \mathcal{N}(I - \lambda\mathcal{K})$. Similarly, $\mathcal{N}^* = \mathcal{N}(I - \lambda\mathcal{K}^*)$. Also, by (2.41), $Lu = F$ has solution u if and only if $\mathcal{K}F \in \mathcal{R}(I - \lambda\mathcal{K})$.

Since $\lambda\mathcal{K} \colon L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ is compact, by the **Fredholm alternative**,

$$\mathcal{N}(I - \lambda\mathcal{K}) = \{0\} \iff \mathcal{R}(I - \lambda\mathcal{K}) = L^2(\Omega;\mathbb{R}^N).$$

Therefore, if $\mathcal{N} = \mathcal{N}(I - \lambda\mathcal{K}) = \{0\}$ then equation $(I - \lambda\mathcal{K})u = \mathcal{K}F$ has unique solution for each $F \in L^2(\Omega;\mathbb{R}^N)$; thus $Lu = F$ has a unique solution for each $F \in L^2(\Omega;\mathbb{R}^N)$. This is the case (2.39). Or else, if $\mathcal{N} = \mathcal{N}(I - \lambda\mathcal{K}) \neq \{0\}$, then

$$\dim\mathcal{N} = \dim\mathcal{N}(I - \lambda\mathcal{K}) = \dim\mathcal{N}(I - \lambda\mathcal{K}^*) = \dim\mathcal{N}^* < \infty.$$

This is the case (2.40), which also proves (ii).

Finally, $Lu = F$ has a weak solution if and only if $\mathcal{K}F \in \mathcal{R}(I - \lambda\mathcal{K}) = (\mathcal{N}(I - \lambda\mathcal{K}^*))^\perp = (\mathcal{N}^*)^\perp$. Note that if $v \in \mathcal{N}^*$ then $v = \lambda\mathcal{K}^*v$ and so

$$(F, v) = (F, \lambda\mathcal{K}^*v) = \lambda(\mathcal{K}F, v).$$

Since $\lambda > 0$, it follows that $\mathcal{K}F \in (\mathcal{N}^*)^\perp$ if and only if $F \in (\mathcal{N}^*)^\perp$. Therefore, $Lu = F$ has a weak solution if and only if $F \in (\mathcal{N}^*)^\perp$. This proves (iii). □

LECTURE 23 – 3/13/19

2.3. Regularity

We now turn to the question as to whether a weak solution u of the PDE

$$Lu = f \quad \text{in } \Omega$$

is smooth or not. This is the **regularity** problem for weak solutions.

Although the following regularity theory holds for general second-order linear differential systems in divergence form, we will instead focus only on second-order linear differential equations for single unknown function of the divergence form

$$Lu \equiv -\sum_{i,j=1}^n D_i\left(a_{ij}(x)D_ju\right) + \sum_{i=1}^n b_i(x)D_iu + c(x)u \tag{2.42}$$

2.3.1. Interior H^2 Regularity.

Theorem 2.19. (Interior H^2-regularity) *Let L be uniformly elliptic with $a_{ij} \in C^1(\Omega)$, b_i and $c \in L^\infty(\Omega)$. Let $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω, then for any $\Omega' \subset\subset \Omega$ we have $u \in H^2(\Omega')$ and*

$$\|u\|_{H^2(\Omega')} \leq C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right), \tag{2.43}$$

where the constant C depends only on n, Ω', Ω and the coefficients of L.

Proof. Set $q = f - \sum_{i=1}^n b_i D_i u - cu$. Since u is a weak solution of $Lu = f$ in Ω, it follows that

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i \varphi dx = \int_\Omega q\varphi dx \quad \forall\, \varphi \in H_0^1(\Omega),\ \ \operatorname{supp}\varphi \subset\subset \Omega. \tag{2.44}$$

In the following, we select different types of test functions φ.

Step 1: (Interior H^1-estimate). Take any $\Omega'' \subset\subset \Omega$. Choose a cutoff function $\zeta \in C_0^\infty(\Omega)$ with $0 \leq \zeta \leq 1$ and $\zeta|_{\Omega''} = 1$. We take $\varphi = \zeta^2 u$ in (2.44) to obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u(\zeta^2 D_i u + 2\zeta u D_i \zeta) dx = \int_\Omega q\zeta^2 u dx$$

and hence

$$\int_\Omega \zeta^2 \sum_{i,j=1}^n a_{ij} D_j u D_i u dx = \int_\Omega (-2u \sum_{i,j=1}^n a_{ij}(\zeta D_j u) D_i \zeta + q\zeta^2 u) dx.$$

Inside $q\zeta^2 u$ we also group the term $\zeta D_i u$. Then use the ellipticity condition and the **Cauchy's inequality with** ε to obtain

$$\theta \int_\Omega \zeta^2 |Du|^2 dx \leq \varepsilon \int_\Omega \zeta^2 |Du|^2 dx + C_\varepsilon \int_\Omega (f^2 + u^2) dx.$$

Thus, taking $0 < \varepsilon < \theta$, we deduce the so-called **Caccioppoli inequality**:

$$\int_\Omega \zeta^2 |Du|^2 dx \leq C \int_\Omega (f^2 + u^2) dx.$$

This proves

$$\|u\|_{H^1(\Omega'')} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{2.45}$$

where the constant C depends on Ω''.

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Step 2: (**Difference Quotient Method**). Take $\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega'' \subset\subset \Omega$. Let $v \in H_0^1(\Omega)$ be any function with $\operatorname{supp} v \subset\subset \Omega_1$. Let

$$\delta = \frac{1}{2} \min\Big\{ \operatorname{dist}(\operatorname{supp} v, \partial\Omega_1), \operatorname{dist}(\Omega_1, \partial\Omega_2), \operatorname{dist}(\Omega_2, \partial\Omega'') \Big\} > 0.$$

For $0 < |h| < \delta$, we choose the test function $\varphi = D_k^{-h} v$ in (2.44) and obtain, using integration by parts for difference quotient,

$$\int_\Omega [D_k^h (\sum_{i,j=1}^n a_{ij} D_j u)] D_i v dx = -\int_\Omega q D_k^{-h} v dx.$$

Notice that the integrals are in fact over domain Ω_1. Henceforth, we omit the $\sum$ sign. Using the definition of q and the equality

$$D_k^h(a_{ij}D_ju) = a_{ij}^h D_k^h D_j u + D_j u D_k^h a_{ij},$$

where $a_{ij}^h(x) = a_{ij}(x + he_k)$, we obtain

$$\begin{aligned}\int_\Omega a_{ij}^h D_j D_k^h u D_i v dx &= -\int_\Omega \left(D_k^h a_{ij} D_j u D_i v + q D_k^{-h} v\right) dx \\ &\le C\left(\|u\|_{H^1(\Omega_1)} + \|f\|_{L^2(\Omega_1)}\right)\|Dv\|_{L^2(\Omega_2)},\end{aligned}$$

where we have used the identity $D_j D_k^h u = D_k^h D_j u$. Take $\eta \in C_0^\infty(\Omega_1)$ such that $\eta(x) = 1$ for $x \in \Omega'$ and choose $v = \eta^2 D_k^h u$. Then

$$\begin{aligned}&\int_\Omega \eta^2 a_{ij}^h D_j D_k^h u D_i D_k^h u dx \le -2\int_\Omega \eta a_{ij}^h D_j D_k^h u (D_i\eta) D_k^h u dx \\ &+C\left(\|u\|_{H^1(\Omega_1)} + \|f\|_{L^2(\Omega_1)}\right)\left(\|\eta D D_k^h u\|_{L^2(\Omega_2)} + 2\|D_k^h u D\eta\|_{L^2(\Omega_2)}\right).\end{aligned}$$

Using the ellipticity condition and the Cauchy inequality with ε, we obtain

$$\frac{\theta}{2}\int_\Omega |\eta D_k^h Du|^2 dx \le C\int_\Omega |D\eta|^2 |D_k^h u|^2 dx + C\left(\|u\|^2_{H^1(\Omega'')} + \|f\|^2_{L^2(\Omega'')}\right).$$

Hence

$$\|\eta D_k^h Du\|^2_{L^2(\Omega)} \le C\left(\|u\|^2_{H^1(\Omega'')} + \|f\|^2_{L^2(\Omega'')}\right).$$

Since $\eta = 1$ on Ω', we derive that $D_k Du \in L^2(\Omega')$, with

$$\|D_k^h Du\|^2_{L^2(\Omega')} \le C\left(\|u\|^2_{H^1(\Omega'')} + \|f\|^2_{L^2(\Omega'')}\right).$$

This shows that the weak derivatives $D_k Du$ in $L^2(\Omega')$ with

$$\|D_k Du\|^2_{L^2(\Omega')} \le C\left(\|u\|^2_{H^1(\Omega'')} + \|f\|^2_{L^2(\Omega'')}\right)$$

for all $k = 1, \ldots, n$. Therefore, $u \in H^2(\Omega')$ and

$$\|u\|_{H^2(\Omega')} \le C\left(\|u\|_{H^1(\Omega'')} + \|f\|_{L^2(\Omega'')}\right), \tag{2.46}$$

where C depends on Ω'. Combining with (2.45) it follows u satisfies (2.43). □

Remark 2.22. (i) The result holds if the coefficients a_{ij} are only (*locally*) Lipschitz continuous in Ω, since the proof above only uses the fact that $D_k^h a_{ij}$ is bounded.

(ii) The proof shows that $D_k Du \in L^2(\Omega')$ as long as the function $\varphi = D_k^{-h}(\eta^2 D_k^h u)$ is a function in $H^1(\Omega)$ with compact support in Ω even when $\Omega' \cap \partial\Omega \neq \emptyset$. This is used in the boundary regularity theory later.

By using an induction argument, we can also get higher regularity for the solution.

Theorem 2.20. (Higher interior regularity) *Let L be uniformly elliptic, with $a_{ij} \in C^{k+1}(\Omega)$, $b_i, c \in C^k(\Omega)$, and $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω, then for any $\Omega' \subset\subset \Omega$ we have $u \in H^{k+2}(\Omega')$ and*

$$\|u\|_{H^{k+2}(\Omega')} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)}\right) \tag{2.47}$$

where the constant C depends only on n, Ω', Ω and the coefficients of L.

Proof. We use induction on k. The estimate (2.47) with $k=0$ has been already proved. Suppose we have proved the theorem for some $k \in \{0,1,\cdots\}$. Now assume $a_{ij} \in C^{k+2}(\Omega)$, $b_i, c \in C^{k+1}(\Omega)$, $f \in H^{k+1}(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of $Lu=f$ in Ω. Then, by the induction assumption, $u \in H^{k+2}_{loc}(\Omega)$, with the estimate (2.47). We want to show $u \in H^{k+3}_{loc}(\Omega)$. Fix $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and a multiindex α with $|\alpha| = k+1$. Let

$$\tilde{u} = D^\alpha u \in H^1(\Omega'').$$

Given any $\tilde{v} \in C_0^\infty(\Omega'')$, let $\varphi = (-1)^{|\alpha|} D^\alpha \tilde{v}$ be put into the identity $B[u,\varphi] = (f,\varphi)_{L^2(\Omega)}$ and perform some elementary integration by parts, and eventually we discover

$$B[\tilde{u},\tilde{v}] = (\tilde{f},\tilde{v})_{L^2(\Omega)},$$

where

$$\tilde{f} := D^\alpha f - \sum_{\beta\le\alpha,\beta\neq\alpha} \binom{\alpha}{\beta}\left[-\sum_{i,j=1}^n (D^{\alpha-\beta}a_{ij}D^\beta u_{x_j})_{x_i} + \sum_{i=1}^n D^{\alpha-\beta}b_i D^\beta u_{x_i} + D^{\alpha-\beta}cD^\beta u\right].$$

That is, $\tilde{u} \in H^1(\Omega'')$ is a weak solution of $L\tilde{u} = \tilde{f}$ in Ω''. (This is equivalent to differentiating the equation $Lu=f$ with D^α-operator.) We have $\tilde{f} \in L^2(\Omega'')$, with, in light of the induction assumption on the $H^{k+2}(\Omega'')$-estimate of u,

$$\|\tilde{f}\|_{L^2(\Omega'')} \le C(\|f\|_{H^{k+1}(\Omega'')} + \|u\|_{H^{k+2}(\Omega'')}) \le C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Therefore, by Theorem 2.19, $\tilde{u} \in H^2(\Omega')$, with the estimate

$$\|\tilde{u}\|_{H^2(\Omega')} \le C(\|\tilde{f}\|_{L^2(\Omega'')} + \|\tilde{u}\|_{L^2(\Omega'')}) \le C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

This exactly proves $u \in H^{k+3}(\Omega')$ and the corresponding estimate (2.47) with $k+1$. ☐

Theorem 2.21. (Infinite interior smoothness) *Let L be uniformly elliptic and a_{ij}, b_i, c and f be all in $C^\infty(\Omega)$. Then a weak solution $u \in H^1(\Omega)$ of $Lu=f$ in Ω belongs to $C^\infty(\Omega)$.*

Proof. By Theorem 2.20, $u \in H^k_{loc}(\Omega)$ for all $k=1,2,\ldots$. By the general **Sobolev embedding theorem**, it follows that $u \in C^m(\Omega)$ for each $m=1,2,\ldots$. ☐

LECTURE 25 – 3/18/19

2.3.2. Boundary Regularity.

We now study the regularity up to the boundary.

We first prove the following result concerning the cut-off functions for balls.

Lemma 2.22. *There exists a constant $C>0$ such that for all $0<s<t<\infty$ and $a \in \mathbb{R}^n$ there exists a cut-off function $\zeta \in C_0^\infty(B(a,t))$ such that*

$$0 \le \zeta(x) \le 1, \quad \zeta|_{B(a,s)} \equiv 1, \quad \|D\zeta\|_{L^\infty} \le \frac{C}{t-s}.$$

Proof. Let $\rho \in C^\infty(\mathbb{R})$ be such that $0 \le \rho \le 1$, $\rho = 1$ on $(-\infty,0]$ and $\rho = 0$ on $[\frac{1}{2},\infty)$. Then the function

$$\zeta(x) = \rho\Big(\frac{|x-a|-s}{t-s}\Big) \quad \forall\, x \in \mathbb{R}^n$$

satisfies the requirements. ☐

Theorem 2.23. (Boundary H^2-regularity for half balls) *Let $\Omega = B(0,r) \cap \{x_n > 0\}$ and L be a uniform elliptic operator on Ω with $a_{ij} \in C^1(\bar{\Omega})$. Suppose $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ in Ω such that $\gamma_0(u) = 0$ on $\partial\Omega \cap \{x_n = 0\}$. Let $0 < s < r$ and $\Omega_s = B(0,s) \cap \{x_n > 0\}$. Then $u \in H^2(\Omega_s)$ with*

$$\|u\|_{H^2(\Omega_s)} \le C_s(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Proof. 1. Select s' such that $0 < s < s' < r$ and set $\Omega' = B(0,s') \cap \{x_n > 0\}$. Let $\zeta \in C_0^\infty(B(0,r))$ be a cut-off function with

$$0 \le \zeta \le 1, \quad \zeta|_{B(0,s')} \equiv 1.$$

So $\zeta \equiv 1$ on Ω' and $\zeta = 0$ near the curved part of $\partial\Omega$; hence $\varphi = \zeta^2 u \in H_0^1(\Omega)$. Use this φ as a test function, and we obtain, as in Step 1 in the proof of Theorem 2.19, that

$$\|u\|_{H^1(\Omega')} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2.48}$$

2. Let $\zeta \in C_0^\infty(B(0,r))$ be a cut-off function with

$$0 \le \zeta \le 1, \quad \zeta|_{B(0,s)} \equiv 1$$

and fix $k \in \{1, 2, \cdots, n-1\}$. For $h > 0$ sufficiently small, let $\varphi = D_k^{-h}(\zeta^2 D_k^h u)$. Note that if $x \in \Omega$ and $h > 0$ is sufficiently small then

$$\varphi(x) = \frac{\zeta^2(x - he_k)[u(x) - u(x - he_k)] - \zeta^2(x)[u(x + he_k) - u(x)]}{h^2}.$$

Since $u = 0$ along $\{x_n = 0\}$ and $\zeta = 0$ near the curved portion of $\partial\Omega$, we see $\varphi \in H_0^1(\Omega)$. Then we use this φ as a test function in (2.44) as we did in the Step 2 in the proof of Theorem 2.19 and use (2.48) to obtain

$$D_k Du \in L^2(\Omega_s) \quad (k = 1, 2, \cdots, n-1)$$

with the estimate

$$\sum_{k=1}^{n-1}\sum_{l=1}^{n} \|D_{kl}u\|_{L^2(\Omega_s)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2.49}$$

This proves that $D_{kl}u \in L^2(\Omega_s)$ for all k, l except $k = l = n$.

3. We must estimate $\|D_{nn}u\|_{L^2(\Omega_s)}$. Since $a_{ij} \in C^1$, the interior H^2-regularity implies that the equation $Lu = f$ is satisfied almost everywhere in Ω; namely

$$-\sum_{i,j=1}^{n} a_{ij} D_{ij}u + \sum_{i=1}^{n} \tilde{b}_i D_i u + cu = f \quad a.e. \ \Omega, \tag{2.50}$$

where $\tilde{b}_i = b_i - \sum_{j=1}^n D_j(a_{ji}) \in L^\infty(\Omega)$. (In this case, *we say $u \in H^2_{loc}(\Omega)$ is a* **strong solution** *of $Lu = f$.*) From the ellipticity condition, we have $a_{nn}(x) \ge \theta > 0$ and thus we can actually solve $D_{nn}u$ from (2.50) in terms of $D_{ij}u$ and $D_i u$ with $i + j < 2n, \ i, j = 1, 2, \cdots, n$, which yields the pointwise estimate

$$|D_{nn}u| \le C\Big(\sum_{i,j=1, i+j<2n}^{n} |D_{ij}u| + |Du| + |u| + |f|\Big) \quad a.e. \ \Omega.$$

Therefore, by (2.48) and (2.49), we have $\|D_{nn}u\|_{L^2(\Omega_s)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$ and thus

$$\|u\|_{H^2(\Omega_s)} \le C_s(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

□

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Theorem 2.24. (Global H^2-regularity) *Assume in addition to the assumptions of Theorem 2.19 that $a_{ij} \in C^1(\bar{\Omega})$ and $\partial\Omega \in C^2$. Let $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution to $Lu = f$ in Ω, then $u \in H^2(\Omega)$, and*

$$\|u\|_{H^2(\Omega)} \le C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \tag{2.51}$$

where the constant C depends only on $n, \|a_{ij}\|_{C^1(\bar{\Omega})}, \|b_i\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}$ and $\partial\Omega$.

Proof. 1. We first establish the global $H^1(\Omega)$-estimate. Let $q = f - \sum_{i=1}^n b_i D_i u - cu$. Take $\varphi = u$ as test function in (2.44) to obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i u dx = \int_\Omega q u dx$$

and hence the ellipticity condition and the **Cauchy's inequality with** ε give

$$\theta \int_\Omega |Du|^2 dx \le \varepsilon \int_\Omega |Du|^2 dx + C_\varepsilon \int_\Omega (f^2 + u^2) dx.$$

Thus, taking $0 < \varepsilon < \theta$, we deduce $\int_\Omega |Du|^2 dx \le C \int_\Omega (f^2 + u^2) dx$. This proves

$$\|u\|_{H^1(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2.52}$$

2. Since $\partial\Omega$ is C^2, at each point $x^0 \in \partial\Omega$, we have a small ball $B(x^0, r)$ and a C^2 map $y = \Phi(x)$, with $\Phi(x^0) = 0$, that maps $B(x^0, r)$ *bijectively* onto a domain in the y space such that

$$\Phi(\Omega \cap B(x^0, r)) \subset \{y \in \mathbb{R}^n \mid y_n > 0\}.$$

Assume the inverse of this map is $x = \Psi(y)$. Then both Ψ and Φ are C^2. Choose $s > 0$ so small that the half-ball $V := B(0, s) \cap \{y_n > 0\}$ lies in $\Phi(\Omega \cap B(x^0, r))$. Set $V' = B(0, s/2) \cap \{y_n > 0\}$. Finally define

$$v(y) = u(\Psi(y)) \quad (y \in V).$$

Then $v \in H^1(V)$ and $v = 0$ on $\partial V \cap \{y_n = 0\}$ (in the sense of trace). Moreover, $u(x) = v(\Phi(x))$ and hence

$$D_j u(x) = u_{x_j}(x) = \sum_{k=1}^n v_{y_k}(\Phi(x)) \Phi^k_{x_j}(x) \ \ (j = 1, 2, \cdots, n).$$

3. We show that v is a weak solution of a linear PDE $Mv = g$ in V. To find this PDE, let $I(y) = \det \frac{\partial \Psi(y)}{\partial y}$ be the Jacobi matrix of $x = \Psi(y)$; since $I(y) \ne 0$ and $\Psi \in C^2$, we have $|I|, |I|^{-1} \in C^1(\bar{V})$. Let $\zeta \in H^1(V)$ with $\operatorname{supp} \zeta \subset\subset V$ and let $\varphi = \zeta / |I|$. Then $\varphi \in H^1(V)$ with $\operatorname{supp} \varphi \subset\subset V$. Let $w(x) = \varphi(\Phi(x))$ for $x \in \Omega' = \Psi(V)$. Then $w \in H^1(\Omega')$ and $\operatorname{supp} w \subset\subset \Omega'$. We use the weak formulation of $Lu = f$: $B[u, w] = (f, w)_{L^2(\Omega)}$ and the change of variable $x = \Psi(y)$ to compute

$$(f, w)_{L^2(\Omega)} = \int_{\Omega'} f(x) w(x) dx = \int_V f(\Psi(y)) \varphi(y) |I(y)| dy := (g, \zeta)_{L^2(V)}, \tag{2.53}$$

where for $g(y) = f(\Psi(y))$. We compute

$$B[u, w] = \int_{\Omega'} \big(a_{ij}(x) u_{x_j}(x) w_{x_i}(x) + b_i(x) u_{x_i} w(x) + c(x) u(x) w(x)\big)\, dx$$

$$= \int_{\Omega'} \Big(a_{ij}(x) v_{y_k}(\Phi(x)) \Phi^k_{x_j}(x) \varphi_{y_l}(\Phi(x)) \Phi^l_{x_i}(x) + b_i(x) v_{y_k}(\Phi(x)) \Phi^k_{x_i}(x) w(x) + c(x) u(x) w(x)\Big)\, dx$$

$$= \int_V \Big(a_{ij}(\Psi(y)) v_{y_k}(y) \Phi^k_{x_j}(\Psi(y)) \varphi_{y_l}(y) \Phi^l_{x_i}(\Psi(y))$$
$$+ b_i(\Psi(y)) v_{y_k}(y) \Phi^k_{x_i}(\Psi(y)) \varphi(y) + c(\Psi(y)) v(y) \varphi(y) \Big) |I(y)| dy.$$

Since $\varphi_{y_l}|I| = \zeta_{y_l} - \frac{|I|_{y_l}}{|I|}\zeta$, we have

$$B[u,w] = \int_V \Big(\tilde{a}_{lk}(y) v_{y_k}(y) \zeta_{y_l}(y) + \tilde{b}_k(y) v_{y_k}(y) \zeta(y) + \tilde{c}(y) v(y) \zeta(y) \Big) dy := \tilde{B}[v,\zeta], \tag{2.54}$$

where $\tilde{c}(y) = c(\Psi(y))$,

$$\tilde{a}_{lk}(y) = \sum_{i,j=1}^n a_{ij}(\Psi(y)) \Phi^k_{x_j}(\Psi(y)) \Phi^l_{x_i}(\Psi(y)) \tag{2.55}$$

for $k,l = 1,2,\cdots,n$, and

$$\tilde{b}_k(y) = \sum_{i=1}^n b_i(\Psi(y)) \Phi^k_{x_i}(\Psi(y)) - \sum_{i,j,l=1}^n a_{ij}(\Psi(y)) \Phi^k_{x_j}(\Psi(y)) \Phi^l_{x_i}(\Psi(y)) \frac{|I|_{y_l}}{|I|}$$

for $k = 1,2,\cdots,n$. By (2.53), (2.54), it follows that

$$\tilde{B}[v,\zeta] = (g,\zeta)_{L^2(V)} \text{ for all } \zeta \in H^1(V) \text{ with } \operatorname{supp}\zeta \subset\subset V;$$

hence, $v \in H^1(V)$ is a weak solution of $Mv = g$ in V, where

$$Mv := -\sum_{k,l=1}^n D_{y_l}(\tilde{a}_{lk}(y) D_{y_k} v) + \sum_{k=1}^n \tilde{b}_k(y) D_{y_k} v + \tilde{c}(y) v.$$

4. We easily have that $\tilde{a}_{lk} \in C^1(\bar{V})$, $\tilde{b}_k$, $\tilde{c} \in L^\infty(V)$. We now check that the operator M is uniformly elliptic in V. Indeed, if $y \in V$ and $\xi \in \mathbb{R}^n$, then, again with $x = \Psi(y)$,

$$\sum_{k,l=1}^n \tilde{a}_{lk}(y) \xi_l \xi_k = \sum_{i,j=1}^n \sum_{k,l=1}^n a_{ij}(x) \Phi^k_{x_j} \Phi^l_{x_i} \xi_l \xi_k = \sum_{i,j=1}^n a_{ij}(x) \eta_j(x) \eta_i(x) \geq \theta |\eta(x)|^2,$$

where $\eta(x) = (\eta_1(x), \cdots, \eta_n(x))$, with

$$\eta_j(x) = \sum_{k=1}^n \Phi^k_{x_j}(x) \xi_k \quad (j = 1,2,\cdots,n).$$

That is, $\eta(x) = \xi D\Phi(x)$. Hence $\xi = \eta(x) D\Psi(y)$ with $y = \Phi(x)$. So $|\xi| \leq C|\eta(x)|$ for some constant C. This shows that

$$\sum_{k,l=1}^n \tilde{a}_{kl}(y) \xi_k \xi_l \geq \theta |\eta(x)|^2 \geq \theta' |\xi|^2$$

for some constant $\theta' > 0$ and all $y \in V$ and $\xi \in \mathbb{R}^n$. By the result proved in Step 1, we have

$$\|v\|_{H^2(V')} \leq C(\|g\|_{L^2(V)} + \|v\|_{L^2(V)}).$$

Consequently, with $O' = \Psi(V')$, using (2.52) and the fact Φ, Ψ are of C^2, we deduce

$$\|u\|_{H^2(O')} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2.56}$$

Note that $x^0 \in \Psi(B(0,s/2)) := G'$, which is an open set containing open set O'.

5. Since $\partial\Omega$ is compact, there exist finitely many open sets $O_i' \subset G_i'$ $(i = 1, 2, \cdots, k)$ such that $\partial\Omega \subset \cup_{i=1}^k G_i'$. Then there exists a $\delta > 0$ such that

$$F := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq \delta\} \subset \bigcup_{i=1}^{k} O_i'.$$

Then $U = (\Omega \setminus F) \subset\subset \Omega$. By (2.56), we have

$$\|u\|_{H^2(F)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

By the interior H^2-regularity,

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Combining these two estimates, we deduce (2.51). □

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Theorem 2.25. (Higher global regularity) *Let L be uniformly elliptic, with*

$$a_{ij} \in C^{k+1}(\bar{\Omega}),\ b_i, c \in C^k(\bar{\Omega}),\ f \in H^k(\Omega),\ \partial\Omega \in C^{k+2}. \tag{2.57}$$

Then a weak solution u of $Lu = f$ satisfying $u \in H_0^1(\Omega)$ belongs to $H^{k+2}(\Omega)$, and

$$\|u\|_{H^{k+2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)}), \tag{2.58}$$

where the constant C is independent of u and f.

Proof. 1. As above, we first investigate the special case

$$\Omega = B(0, r) \cap \{x_n > 0\}$$

for some $r > 0$. Set $\Omega_t = B(0, t) \cap \{x_n > 0\}$ for each $0 < t < r$. We intend to show by induction on k that under (2.57) whenever $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ satisfying $\gamma_0(u) = 0$ on $\partial\Omega \cap \{x_n = 0\}$, we have $u \in H^{k+2}(\Omega_t)$ and

$$\|u\|_{H^{k+2}(\Omega_t)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)}). \tag{2.59}$$

The case $k = 0$ has been proved in Theorem 2.23. Suppose this is proved with some k. Now assume $a_{ij} \in C^{k+2}(\bar{\Omega})$, $b_i, c \in C^{k+1}(\bar{\Omega})$, $f \in H^{k+1}(\Omega)$, and u is a weak solution of $Lu = f$ in Ω satisfying $\gamma_0(u) = 0$ on $\partial\Omega \cap \{x_n = 0\}$. Fix any $0 < t < s < r$. By induction assumption, $u \in H^{k+2}(\Omega_s)$, with

$$\|u\|_{H^{k+2}(\Omega_s)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)}). \tag{2.60}$$

Furthermore, according to the interior regularity, $u \in H^{k+3}_{loc}(\Omega)$.

2. Let α be any multiindex with $|\alpha| = k + 1$ and $\alpha_n = 0$. Then $\tilde{u} := D^\alpha u \in H^1(\Omega)$ and vanishes along $\{x_n = 0\}$. (For example, this can be shown by induction on $|\alpha|$ using the difference quotient operator D_j^h.) Furthermore, as in the proof of the interior higher regularity theorem, $\tilde{u}$ is a weak solution of $L\tilde{u} = \tilde{f}$ in Ω, where, as above,

$$\tilde{f} := D^\alpha f - \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \left[-\sum_{i,j=1}^{n} (D^{\alpha-\beta} a_{ij} D^\beta u_{x_j})_{x_i} + \sum_{i=1}^{n} D^{\alpha-\beta} b_i D^\beta u_{x_i} + D^{\alpha-\beta} c D^\beta u \right].$$

By (2.60), this $\tilde{f}$ belongs to $L^2(\Omega_s)$ and

$$\|\tilde{f}\|_{L^2(\Omega_s)} \leq C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Consequently, $\tilde{u} \in H^2(\Omega_t)$, with

$$\|\tilde{u}\|_{H^2(\Omega_t)} \le C(\|\tilde{f}\|_{L^2(\Omega_s)} + \|\tilde{u}\|_{L^2(\Omega_s)}) \le C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

This proves

$$\|D^\beta u\|_{L^2(\Omega_t)} \le C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}) \tag{2.61}$$

for all β with $|\beta| = k+3$ and $\beta_n = 0, 1, 2$.

3. We need to extend (2.61) to all β with $|\beta| = k+3$. Fix k, we prove (2.61) for all β by induction on $j = 0, 1, \cdots, k+2$ with $\beta_n \le j$. We have already shown it for $j = 0, 1, 2$. Assume we have shown it for j. Now assume β with $|\beta| = k+3$ and $\beta_n = j+1$. Let us write $\beta = \gamma + \delta$, for $\delta = (0, \cdots, 0, 2)$ and so $|\gamma| = k+1$. Since $u \in H^{k+3}_{loc}(\Omega)$ and $Lu = f$ in Ω, we have $D^\gamma Lu = D^\gamma f$ a.e. in Ω. Now

$$D^\gamma Lu = a_{nn} D^\beta u + R,$$

where R is the sum of terms involving at most j derivatives of u with respect to x_n and at most $k+3$ derivatives in all. Since $a_{nn} \ge \theta$, we can solve $D^\beta u$ in terms of R and $D^\gamma f$; hence,

$$\|D^\beta u\|_{L^2(\Omega_t)} \le C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

By induction, we deduce (2.61), which proves

$$\|u\|_{H^{k+3}(\Omega_t)} \le C(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

This estimate in turn completes the induction process on k, begun in step 2. This proves (2.59).

4. As above, we cover the domain $\bar{\Omega}$ by finitely many small balls and use the method of flattening the boundary to eventually deduce (2.58). Note that the condition $\partial\Omega \in C^{k+2}$ is needed for flattening the boundary to obtain an elliptic equation of divergence form with leading coefficients $\tilde{a}_{lk} \in C^{k+1}$; see (2.55) above. □

Corollary 2.26. *Under the assumption of Theorem 2.25, if $Lu = 0$ has only the trivial weak solution $u \equiv 0$ in $H^1_0(\Omega)$, then for each $f \in H^k(\Omega)$ there exists a unique weak solution $u \in H^1_0(\Omega) \cap H^{k+2}(\Omega)$ of $Lu = f$ in Ω such that*

$$\|u\|_{H^{k+2}(\Omega)} \le C\|f\|_{H^k(\Omega)}, \tag{2.62}$$

where C is independent of u and f.

Proof. The existence of unique weak solution $u \in H^1_0(\Omega)$ of $Lu = f$ in Ω for each given $f \in L^2(\Omega)$ follows from the **Fredholm alternative**; moreover, by the previous theorem, $u \in H^{k+2}(\Omega)$ if $f \in H^k(\Omega)$. To prove (2.62), in view of (2.58), it suffices to show that

$$\|u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

whenever $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f$ in Ω. Suppose this inequality is false; then there exist functions $u_n \in H^1_0(\Omega)$ and $f_n = Lu_n \in L^2(\Omega)$ for which $\|u_n\|_{L^2} = 1$ and $\|f_n\|_{L^2} \to 0$. By (2.58) we have $\|u_n\|_{H^2} \le C$. By compact embedding, there exist $u \in H^2(\Omega)$ and a subsequence $\{u_{n_k}\}$ converging to u in $H^1(\Omega)$; thus we also have $u \in H^1_0(\Omega)$ and $\|u\|_{L^2} = 1$. Moreover, note that

$$B[u_{n_k}, v] = \int_\Omega f_{n_k} v\, dx \quad \forall\, v \in H^1_0(\Omega).$$

Taking the limit, we see that $B[u, v] = 0$ for all $v \in H^1_0(\Omega)$ and thus $u \in H^1_0(\Omega)$ is a weak solution of $Lu = 0$; hence $u \equiv 0$ by assumption, a contradiction to $\|u\|_{L^2} = 1$. □

Finally, we iterate the higher regularity theorem to obtain

Theorem 2.27. (Infinite global smoothness) *Let L be uniformly elliptic, with a_{ij}, b_i, c, $f \in C^\infty(\bar{\Omega})$, and $\partial\Omega \in C^\infty$. Then a weak solution $u \in H_0^1(\Omega)$ of $Lu = f$ in Ω belongs to $C^\infty(\bar{\Omega})$.*

Lecture 28 – 3/25/19

2.4. Maximum Principles

(*Some of the material may have already been covered in MTH 847.*)

2.4.1. Elliptic Operators in Non-divergence Form. Consider the second-order linear differential operator in non-divergence form

$$Lu(x) = -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_iu(x) + c(x)u(x),$$

where $D_iu = u_{x_i}$, $D_{ij}u = u_{x_ix_j}$ and $a_{ij}(x)$, $b_i(x)$, $c(x)$ are given functions in an open set Ω in $\mathbb{R}^n$ for all $i, j = 1, 2, \cdots, n$. With loss of generality, we assume $a_{ij}(x) = a_{ji}(x)$ for all i, j.

Definition 2.23. The operator L is called **elliptic** in Ω if there exists $\lambda(x) > 0$ $(x \in \Omega)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)\sum_{i=1}^{n} \xi_i^2 \quad \forall\, x \in \Omega,\;\; \xi \in \mathbb{R}^n.$$

As above, if $\lambda(x) \geq \theta > 0$ in Ω, we say that L is **uniformly elliptic** in Ω.

So, if L is elliptic in Ω, then for each $x \in \Omega$ the symmetry matrix $(a_{ij}(x))$ is *positive definite*, with all eigenvalues $\geq \lambda(x)$.

Lemma 2.28. *If $A = (a_{ij})$ is an $n \times n$ symmetric nonnegative definite matrix then there exists an $n \times n$ matrix $B = (b_{ij})$ such that $A = B^TB$, i.e.,*

$$a_{ij} = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n).$$

Proof. Use the **diagonalization** of A. (**Exercise.**) □

2.4.2. Weak Maximum Principle.

Lemma 2.29. *Let L be elliptic in Ω and $u \in C^2(\Omega)$ satisfy $Lu < 0$ in Ω. If $c(x) \geq 0$, then u cannot attain a nonnegative maximum in Ω. If $c(x) \equiv 0$ then u cannot attain a maximum in Ω.*

Proof. Let $Lu < 0$ in Ω. Suppose $u(x_0)$ is maximum for some $x_0 \in \Omega$. Then, by the derivative test, $D_ju(x_0) = 0$ for each $j = 1, 2, \cdots, n$, and

$$\frac{d^2u(x_0 + t\xi)}{dt^2}\Big|_{t=0} = \sum_{i,j=1}^{n} D_{ij}u(x_0)\xi_i\xi_j \leq 0$$

for all $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$. By the lemma above, we write

$$a_{ij}(x_0) = \sum_{k=1}^{n} b_{ki}b_{kj} \quad (i, j = 1, 2, \cdots, n),$$

where $B = (b_{ij})$ is an $n \times n$ matrix. Hence

$$\sum_{i,j=1}^{n} a_{ij}(x_0) D_{ij} u(x_0) = \sum_{k=1}^{n} \sum_{i,j=1}^{n} D_{ij} u(x_0) b_{ki} b_{kj} \le 0,$$

which implies that $Lu(x_0) \ge c(x_0)u(x_0) \ge 0$ either when $c \ge 0$ and $u(x_0) \ge 0$ or when $c \equiv 0$. This is a contradiction. ☐

Theorem 2.30 (**Weak maximum principle with** $c = 0$). *Let Ω be bounded open in $\mathbb{R}^n$ and L be elliptic in Ω and*

$$|b_i(x)|/\lambda(x) \le M \quad (x \in \Omega,\ i = 1, 2, \cdots, n) \tag{2.63}$$

for some constant $M > 0$. Let $c \equiv 0$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \le 0$ in Ω. Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. Let $\alpha > 0$ and $v(x) = e^{\alpha x_1}$. Then

$$Lv(x) = (-a_{11}(x)\alpha^2 + b_1(x)\alpha) e^{\alpha x_1} = \alpha a_{11}(x) \left[-\alpha + \frac{b_1(x)}{a_{11}(x)} \right] e^{\alpha x_1} < 0$$

if $\alpha > M + 1$ because $\frac{|b_1(x)|}{a_{11}(x)} \le \frac{|b_1(x)|}{\lambda(x)} \le M$. Then consider the function $w(x) = u(x) + \varepsilon v(x)$ for $\varepsilon > 0$. Then $Lw = Lu + \varepsilon Lv < 0$ in Ω. So by Lemma 2.29, for all $x \in \bar{\Omega}$,

$$u(x) + \varepsilon v(x) \le \max_{\partial\Omega}(u + \varepsilon v) \le \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} v.$$

Letting $\varepsilon \to 0^+$ proves the theorem. ☐

Remark 2.24. (a) The weak maximum principle still holds if $(a_{ij}(x))$ is nonnegative definite, i.e., $\lambda(x) \ge 0$ in Ω, but satisfies $\frac{|b_k(x)|}{a_{kk}(x)} \le M$ for some $k = 1, 2, \cdots, n$. (In this case use $v = e^{\alpha x_k}$.)

(b) If Ω is unbounded but bounded in a slab $|x_1| < N$, then the proof is still valid if the maximum is changed supremum.

Theorem 2.31 (**Weak maximum principle with** $c \ge 0$). *Let Ω be bounded open in $\mathbb{R}^n$ and L be elliptic in Ω satisfying* (2.63). *Let $c(x) \ge 0$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then*

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+ \quad \textit{if } Lu \le 0 \textit{ in } \Omega,$$

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u| \quad \textit{if } Lu = 0 \textit{ in } \Omega,$$

where $u^+(x) = \max\{u(x), 0\}$.

Proof. 1. Let $Lu \le 0$ in Ω. Let $\Omega^+ = \{x \in \Omega \mid u(x) > 0\}$. If Ω^+ is empty then the result is trivial. Assume $\Omega^+ \ne \emptyset$; then $L_0 u \equiv Lu - c(x)u(x) \le 0$ in Ω^+. Note that $\partial(\Omega^+) = [\Omega \cap \partial\Omega^+] \cup [\partial\Omega^+ \cap \partial\Omega]$, from which we easily see that $\max_{\partial(\Omega^+)} u \le \max_{\partial\Omega} u^+$; hence, by Theorem 2.30,

$$\max_{\bar{\Omega}} u = \max_{\overline{\Omega^+}} u = \max_{\partial(\Omega^+)} u \le \max_{\partial\Omega} u^+.$$

2. Let $Lu = 0$. We apply Step 1 to u and $-u$ to complete the proof. ☐

Remark 2.25. The weak maximum principle for $Lu \le 0$ can not be replaced by $\max_{\bar\Omega} u = \max_{\partial\Omega} u$. In fact, for any $u \in C^2(\bar\Omega)$ satisfying

$$0 > \max_{\bar\Omega} u > \max_{\partial\Omega} u,$$

if we choose a constant $\theta > -\|Lu\|_{L^\infty(\Omega)}/\max_{\bar\Omega} u > 0$, then $\tilde L u = Lu + \theta u \le 0$ in Ω. But the zero-th order coefficient of $\tilde L$ is $c(x,t) + \theta > 0$.

The weak maximum principle easily implies the following uniqueness result for Dirichlet problems.

Theorem 2.32 (Uniqueness of solutions). *Let Ω be bounded open in $\mathbb{R}^n$ and the linear operator L with $c(x) \ge 0$ be elliptic in Ω and satisfy* (2.63). *Then, given any functions f and g, the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\bar\Omega)$.

Remark 2.26. The uniqueness result fails if $c(x) < 0$ in Ω. For example, if $n = 1$, then function $u(x) = \sin x$ solves the elliptic problem $Lu \equiv -u'' - u = 0$ in $\Omega = (0, \pi)$ with $u(0) = u(\pi) = 0$; but $u \not\equiv 0$.

2.4.3. Strong Maximum Principle.

Theorem 2.33 (Hopf's Lemma). *Let L be uniformly elliptic with bounded coefficients in a ball B and let $u \in C^2(B) \cap C^1(\bar B)$ satisfy $Lu \le 0$ in B. Assume $x^0 \in \partial B$ such that $u(x) < u(x^0)$ for all $x \in B$.*

(a) *If $c \equiv 0$ in B, then $\frac{\partial u}{\partial \nu}(x^0) > 0$, where ν is outer unit normal to ∂B.*

(b) *If $c(x) \ge 0$ in B, then the same conclusion holds provided $u(x^0) \ge 0$.*

(c) *If $u(x^0) = 0$, then the same conclusion holds no matter what sign of $c(x)$ is.*

Proof. 1. Without loss of generality, assume $B = B(0, R)$. Consider function

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$

Let $\tilde L u \equiv Lu - c(x)u + c^+(x)u$, where $c^+(x) = \max\{c(x), 0\}$. This operator has the zero-th order term $c^+ \ge 0$ and hence the weak maximum principle applies to $\tilde L$. We compute

$$\begin{aligned} \tilde L v(x) &= \left[-4 \sum_{i,j=1}^n a_{ij}(x)\alpha^2 x_i x_j + 2\alpha \sum_{i=1}^n (a_{ii}(x) - b_i(x)x_i) \right] e^{-\alpha|x|^2} + c^+(x)v(x) \\ &\le \left[-4\theta\alpha^2|x|^2 + 2\alpha\, \mathrm{tr}(a_{ij}(x)) + 2\alpha|b(x)||x| + c^+(x) \right] e^{-\alpha|x|^2} < 0 \end{aligned}$$

on $\frac{R}{2} \le |x| \le R$ if $\alpha > 0$ is fixed and sufficiently large.

2. For any $\varepsilon > 0$, consider function $w_\varepsilon(x) = u(x) - u(x^0) + \varepsilon v(x)$. Then

$$\tilde L w_\varepsilon(x) = \varepsilon \tilde L v(x) + Lu(x) + (c^+(x) - c(x))u(x) - c^+(x)u(x^0) \le 0$$

on $\frac{R}{2} \le |x| \le R$ in all cases of (a), (b) and (c).

3. By assumption, $u(x) < u(x^0)$ on $|x| = \frac{R}{2}$; hence there exists $\varepsilon > 0$ such that $w_\varepsilon(x) < 0$ on $|x| = \frac{R}{2}$. In addition, since $v|_{\partial B} = 0$, we have $w_\varepsilon(x) = u(x) - u(x^0) \le 0$ on $|x| = R$. Hence

the weak maximum principle implies that $w_\varepsilon(x) \le 0$ for all $\frac{R}{2} \le |x| \le R$. But $w_\varepsilon(x^0) = 0$; this implies

$$0 \le \frac{\partial w_\varepsilon}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \varepsilon \frac{\partial v}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - 2\varepsilon R\alpha e^{-\alpha R^2}.$$

Therefore

$$\frac{\partial u}{\partial \nu}(x^0) \ge 2\varepsilon R\alpha e^{-\alpha R^2} > 0,$$

as desired. □

Theorem 2.34 (**Strong maximum principle).** *Let Ω be bounded, open and connected in $\mathbb{R}^n$ and L be uniformly elliptic with bounded coefficients in Ω and let $u \in C^2(\Omega)$ satisfy $Lu \le 0$ in Ω.*

(a) *If $c(x) \ge 0$, then u cannot attain a nonnegative maximum in Ω unless u is constant.*

(b) *If $c \equiv 0$, then u cannot attain a maximum in Ω unless u is constant.*

Proof. Assume $c(x) \ge 0$ in Ω and u attains the maximum M at some point in Ω; also assume $M \ge 0$ if $c(x) \ge 0$. Suppose that u is not constant in Ω. Then, both of the following sets,

$$\Omega^- = \{x \in \Omega \,|\, u(x) < M\}; \quad \Omega_0 = \{x \in \Omega \,|\, u(x) = M\},$$

are nonempty, with Ω^- open and $\Omega_0 \ne \Omega$ relatively closed in Ω. Since Ω is connected, Ω_0 can not be open. Assume $x^0 \in \Omega_0$ is not an interior point of Ω_0; so, there exists a sequence $\{x^k\}$ not in Ω_0 but converging to x^0. Hence, for a ball $B(x^0, r) \subset\subset \Omega$ and an integer $N \in \mathbb{N}$, we have that $x^k \in B(x^0, r/2)$ for all $k \ge N$. Fix $k = N$ and let

$$S = \{\rho > 0 \mid B(x^N, \rho) \subset \Omega^-\}.$$

Then $S \subset \mathbb{R}$ is nonempty and bounded above by $r/2$. Let $\bar\rho = \sup S$; then $0 < \bar\rho \le r/2$ and hence $B(x^N, \bar\rho) \subset B(x^0, r) \subset\subset \Omega$. So $B(x^N, \bar\rho) \subset \Omega^-$, and also $\Omega_0 \cap \partial B(x^N, \bar\rho) \ne \emptyset$. So let $y \in \Omega_0 \cap \partial B(x^N, \bar\rho)$ and then $u(x) < u(y)$ for all $x \in B(x^N, \bar\rho)$. Then **Hopf's Lemma** above, applied to the ball $B(x^N, \bar\rho)$ at point $y \in \partial B(x^N, \bar\rho)$, implies that $\frac{\partial u}{\partial \nu}(y) > 0$, where ν is the outer normal of $\partial B(x^N, \bar\rho)$ at y. This contradicts the fact that $Du(y) = 0$, as u has a maximum at $y \in \Omega_0 \subset \Omega$. □

2.4.4. Maximum Principle for Weak Solutions. We study a **maximum principle** for weak solutions of second-order linear differential equations in divergence form. Let

$$Lu \equiv -\sum_{i,j=1}^{n} D_i\left(a_{ij}(x) D_j u\right) + \sum_{i=1}^{n} b_i(x) D_i u + c(x) u, \tag{2.64}$$

with the associated bilinear $B[u, v]$ defined as above.

Definition 2.27. (i) Given $f \in L^2(\Omega)$, we say that $u \in H^1(\Omega)$ is a **weak sub-solution** of $Lu = f$ and write $Lu \le f$ in Ω provided

$$B[u, v] \le (f, v)_{L^2(\Omega)} \quad \forall\, v \in H_0^1(\Omega),\;\; v(x) \ge 0 \;\; a.e.\ \Omega.$$

Similarly, we define a **weak super-solution** of $Lu = f$ and write $Lu \ge f$ in Ω.

(ii) Given functions $u, v \in H^1(\Omega)$, we say that $u \le v$ on $\partial\Omega$ if $(u - v)^+ \in H_0^1(\Omega)$; we say $u \ge v$ on $\partial\Omega$ if $-u \le -v$ on $\partial\Omega$; that is, if $(v - u)^+ \in H_0^1(\Omega)$. We define

$$\sup_{\partial\Omega} u = \inf\{\rho \in \mathbb{R} \mid u \le \rho \;\text{ on } \partial\Omega\} = \inf\{\rho \in \mathbb{R} \mid (u - \rho)^+ \in H_0^1(\Omega)\},$$

and

$$\inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u) = \sup\{\rho \in \mathbb{R} \mid u \geq \rho \text{ on } \partial\Omega\}.$$

(iii) Recall that for $u \in L^1_{loc}(E)$, where $E \subset \mathbb{R}^n$ is measurable,

$$\operatorname*{ess\,sup}_E u = \inf\{\rho \in \mathbb{R} \mid u \leq \rho \ a.e.\ E\} = \inf\{\rho \in \mathbb{R} \mid (u-\rho)^+ = 0 \ a.e.\ E\}.$$

Note that

$$\sup_{\partial\Omega} u \leq \operatorname*{ess\,sup}_{\Omega} u \quad \forall\, u \in H^1(\Omega).$$

Remark 2.28. (i) Note that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω if and only if u is both a weak sub-solution and a weak super-solution of $Lu = f$ in Ω.

(ii) If $u \leq v$ and $v \leq w$ on $\partial\Omega$ then $u \leq w$ on $\partial\Omega$. (See the **Exercise** below.)

Exercise 2.29. Suppose $v \in H^1_0(\Omega)$ and $v(x) \geq 0$ a.e. in Ω. Assume $u \in H^1(\Omega)$ and $|u(x)| \leq v(x)$ a.e. in Ω. Show that $u \in H^1_0(\Omega)$.
Hint: Show $u^+ \in H^1_0(\Omega)$. Let $v_m \in C^\infty_0(\Omega)$ and $v_m \to v$ in $H^1(\Omega)$. Then $f_m = \min\{u^+, v_m^+\} \in H^1_0(\Omega)$ and $f_m \to u^+$ in $H^1(\Omega)$.

Lemma 2.35. *Let Ω be a bounded domain with $\partial\Omega \in C^1$ and $u, v \in H^1(\Omega)$. Then, $u \leq v$ on $\partial\Omega$ if and only if $\gamma_0(u) \leq \gamma_0(v)$ a.e. on $\partial\Omega$. Furthermore,*

$$\sup_{\partial\Omega} u = \operatorname*{ess\,sup}_{\partial\Omega} \gamma_0(u). \tag{2.65}$$

Proof. Let $u \in H^1(\Omega)$ and let $u_m \in C^\infty(\bar{\Omega})$ be such that $u_m \to u$ in $H^1(\Omega)$. Then $u_m^+ \to u^+$ in $H^1(\Omega)$. Thus $u_m|_{\partial\Omega} \to \gamma_0(u)$, $u_m^+|_{\partial\Omega} \to \gamma_0(u^+)$ and $u_m^+|_{\partial\Omega} \to (\gamma_0(u))^+$, all strongly in $L^2(\partial\Omega)$. This proves $\gamma_0(u^+) = (\gamma_0(u))^+$ a.e. on $\partial\Omega$. Therefore, given $u, v \in H^1(\Omega)$, it follows that $u \leq v$ on $\partial\Omega \iff (u-v)^+ \in H^1_0(\Omega) \iff \gamma_0((u-v)^+) = (\gamma_0(u) - \gamma_0(v))^+ = 0 \iff \gamma_0(u) \leq \gamma_0(v)$.

The identity (2.65) follows easily as $u \leq \rho$ on $\partial\Omega \iff \gamma_0(u) \leq \rho$ a.e. on $\partial\Omega$. □

Theorem 2.36. (Maximum Principle for weak sub-solutions) *Let $\Omega \subset \mathbb{R}^n$ be bounded open and L be uniformly elliptic in Ω with $c(x) \geq 0$ a.e. in Ω. Suppose $u \in H^1(\Omega)$ is a weak subsolution of $Lu = 0$ in Ω. Then*

$$\operatorname*{ess\,sup}_{\Omega} u \leq \sup_{\partial\Omega} u^+.$$

Proof. Suppose, for the contrary, that $\sup_{\partial\Omega} u^+ < \operatorname{ess\,sup}_\Omega u$. Let k be any number such that $\sup_{\partial\Omega} u^+ < k < \operatorname{ess\,sup}_\Omega u$, and define $v^k = (u-k)^+$. Then $k > 0$, $v^k \in H^1(\Omega)$, $v^k \geq 0$ and $uv^k \geq 0$, both a.e. in Ω, and

$$Dv^k = \begin{cases} Du & \text{on } \{u > k\}, \\ 0 & \text{on } \{u \leq k\}. \end{cases}$$

Since $k > \sup_{\partial\Omega} u^+$, we have $u^+ \leq k$ on $\partial\Omega$, i.e., $(u^+ - k)^+ \in H^1_0(\Omega)$ and thus $v^k = (u-k)^+ = (u^+ - k)^+ \in H^1_0(\Omega)$. As $u \in H^1(\Omega)$ is a weak subsolution of $Lu = 0$ in Ω, we have $B[u, v^k] \leq 0$, which, combined with the ellipticity condition and $c(x)uv^k \geq 0$ a.e. in Ω, gives

$$\begin{aligned} \theta \int_\Omega |Dv^k|^2 &\leq \int_\Omega a_{ij} D_j u D_i v^k \leq -\int_\Omega b_j v^k D_j u - \int_\Omega cuv^k \\ &\leq -\int_\Omega b_j v^k D_j u \leq C \int_{\Omega_k} |v^k||Dv^k| \leq C\|v^k\|_{L^2(\Omega_k)} \|Dv^k\|_{L^2(\Omega)}, \end{aligned}$$

where $\Omega_k = \{x \in \Omega \,|\, u(x) > k,\ Du(x) \neq 0\}$ and $C > 0$ is independent of k. Hence, with $C_1 = C/\theta > 0$,

$$\|Dv^k\|_{L^2(\Omega)} \leq C_1 \|v^k\|_{L^2(\Omega_k)}. \tag{2.66}$$

Now let $p = 2^* = \frac{2n}{n-2}$ if $n > 2$ and let p be any number larger than 2 if $n = 1, 2$. Then, by (2.66) and the **Gagliardo-Nirenberg-Morrey-Poincaré-Sobolev** and **Hölder** inequalities,

$$\|v^k\|_{L^p(\Omega)} \leq C_p \|Dv^k\|_{L^2(\Omega)} \leq C_p C_1 \|v^k\|_{L^2(\Omega_k)} \leq C_p C_1 |\Omega_k|^{\frac{1}{2}-\frac{1}{p}} \|v^k\|_{L^p(\Omega)},$$

where $C_p > 0$ is a constant independent of k. Since $k < \operatorname{ess\,sup}_\Omega u$, we have $\|v^k\|_{L^p(\Omega)} > 0$ and hence the previous inequality gives

$$|\Omega_k| \geq \mu > 0,$$

where $\mu > 0$ is independent of k. Let $l = \operatorname{ess\,sup}_\Omega u$ and N be an integer such that $l - \frac{1}{N} > \sup_{\partial\Omega} u^+$. Define

$$S = \cap_{j=N}^{\infty} \Omega_{l-\frac{1}{j}}.$$

Then S is measurable. Since $\{\Omega_k\}$ is decreasing as k increases, it follows that

$$|S| = \lim_{k \to l^-} |\Omega_k| \geq \mu > 0.$$

Note that one has $Du \neq 0$ and $u \geq l$ on S; hence $Du \neq 0$ and $u = l$ a.e. on S, which is impossible since $Du = 0$ a.e. on any level set $\{u = \textit{constant}\}$ of u, the constant including possibly ∞. (**Exercise!**) This completes the proof. $\square$

Remark 2.30. The maximum principle for weak subsolutions is independent of the first-order coefficients b_i, as in the case of the maximum principle for classical $C^2(\Omega) \cap C(\bar{\Omega})$-subsolutions of non-divergence elliptic operators.

Applying Theorem 2.36 to both u and $-u$, it follows that the elliptic equation $Lu = 0$ has the unique trivial solution $u = 0$ in $H_0^1(\Omega)$ if $c \geq 0$ in Ω. Hence, by the **Fredholm alternative**, for such an operator L, the equation $Lu = f$ has a unique weak solution $u \in H_0^1(\Omega)$ for each $f \in L^2(\Omega)$.

Lecture 29 – 3/27/19

2.5. Eigenvalues and Eigenfunctions

We come back to the general linear system of divergence form Lu defined by (2.10), whose bilinear form $B[u, v]$ is defined by (2.12) above on $H = H_0^1(\Omega; \mathbb{R}^N)$, where $N \geq 1$.

Definition 2.31. A real number $\lambda \in \mathbb{R}$ is called a (Dirichlet) **eigenvalue** of operator L if the BVP problem

$$\begin{cases} Lu - \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.67}$$

has *nontrivial* weak solutions in $H_0^1(\Omega; \mathbb{R}^N)$; these nontrivial solutions are called the **eigenfunctions** corresponding to eigenvalue λ.

The eigenvalues and eigenfunctions of elliptic equations can be studied by the spectral theory of compact operators.

2.5.1. Some Functional Analysis – Spectral Theory of Compact Operators.

Definition 2.32. Let $T\colon H \to H$ be a bounded linear operator on a Hilbert space H.

(1) We define the **resolvent set** of T to be
$$\rho(T) = \{\lambda \in \mathbb{R} \mid (T - \lambda I)\colon H \to H \text{ is one-to-one and onto}\}$$
and define the **spetrum** of T to be the set $\sigma(T) = \mathbb{R} \setminus \rho(T)$.

(2) If $\mathcal{N}(T-\lambda I) \neq \{0\}$, then λ is called an **eigenvalue** of T; in this case, any nonzero element in $\mathcal{N}(T - \lambda I)$ is called an **eigenvector** of T corresponding to λ.

Remark 2.33. By the **Closed Graph Theorem**, $\lambda \in \rho(T)$ if and only if $(T-\lambda I)^{-1}$ exists and is bounded.

Theorem 2.37. (Spectrum of Compact Operator) *Let H be an infinite dimensional Hilbert space and $T\colon H \to H$ be linear and compact. Then*

(i) $0 \in \sigma(T)$.

(ii) λ *is an eigenvalue of* T *if* $\lambda \in \sigma(T) \setminus \{0\}$.

(iii) $\sigma(T) \setminus \{0\}$ *is either finite or a sequence converging to 0.*

Proof. 1. Suppose $0 \in \rho(T)$. Then T^{-1} exists and is bounded on H; thus $I = T \circ T^{-1}\colon H \to H$ is compact, which implies each bounded sequence in H has a convergent subsequence. This is clearly false if $\dim H = \infty$.

2. Assume $\lambda \in \sigma(T), \lambda \neq 0$. Suppose $\mathcal{N}(\lambda I - T) = \{0\}$. Then, by the **Fredholm alternative**, $\mathcal{R}(\lambda I - T) = H$ and hence $\lambda \in \rho(T)$, a contradiction. Consequently $\mathcal{N}(\lambda I - T) \neq \{0\}$ and thus λ is an eigenvalue of T.

3. Assume $S = \sigma(T) \setminus \{0\}$ is infinite. We show that the only limit point of S is 0. Once this is proved it follows that S consists of a sequence converging to 0. Assume $\eta \in \mathbb{R}$, $\eta_k \in S$, $\eta_j \neq \eta_k$ $(j \neq k)$, and $\{\eta_k\} \to \eta$; we are to prove $\eta = 0$. For each k let $w_k \in H$ be such that $w_k \neq 0$ and $Tw_k = \eta_k w_k$. Let $H_k = \operatorname{span}\{w_1, w_2, \ldots, w_k\}$. Use induction, we see that $\{w_1, \ldots, w_k\}$ is linearly independent. Hence H_k is a proper subspace of H_{k+1}. Note that $(T - \eta_k I)(H_k) \subset H_{k-1} \subset H_k$. Thus choose $u_k \in H_k$ such that $u_k \in (H_{k-1})^\perp$ and $\|u_k\| = 1$. Then if $k > j$
$$\left\| \frac{Tu_k}{\eta_k} - \frac{Tu_j}{\eta_j} \right\| = \left\| \frac{Tu_k - \eta_k u_k}{\eta_k} - \frac{Tu_j - \eta_j u_j}{\eta_j} + u_k - u_j \right\| \geq 1,$$
since $Tu_k - \eta_k u_k, Tu_j - \eta_j, u_j \in H_{k-1}$. If $\eta_k \to \eta \neq 0$, then we obtain a contradiction to the compactness of T. $\square$

2.5.2. Eigenvalues of Elliptic Operators.

Theorem 2.38. (Third Existence Theorem for weak solutions) *Assume the conditions of Theorem* 2.11 *hold.*

(i) *There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the problem $Lu - \lambda u = F$ has a unique weak solution in $H_0^2(\Omega; \mathbb{R}^N)$ for each $F \in L^2(\Omega; \mathbb{R}^N)$ if and only if $\lambda \notin \Sigma$.*

(ii) *If Σ is infinite, then $\Sigma = \{\lambda_k\}$, with $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, and $\lambda_k \to \infty$.*

Proof. By the **Fredholm alternative**, the problem $Lu - \lambda u = F$ has a unique weak solution in $H_0^2(\Omega; \mathbb{R}^N)$ for each $F \in L^2(\Omega; \mathbb{R}^N)$ if and only if $Lu - \lambda u = 0$ has only trivial weak solution $u = 0$ in $H_0^1(\Omega; \mathbb{R}^N)$. Let $\gamma \geq 0$ be the number in the **Gårding's estimate** (2.29). If $\lambda \leq -\gamma$, then $Lu - \lambda u = 0$ has only the trivial weak solution $u = 0$ since the bilinear form $B_\lambda[u, v]$ for $Lu - \lambda u$ satisfies

$$B_\lambda[u, u] = B[u, u] - \lambda \|u\|_{L^2}^2 \geq \beta \|u\|_{H^1}^2 - (\lambda + \gamma)\|u\|_{L^2}^2 \geq \beta \|u\|_{H^1}^2.$$

Assume $\lambda > -\gamma$. Then equation $Lu - \lambda u = 0$ is equivalent to $(L + \gamma I)u = (\gamma + \lambda)u$; that is, $(I - (\gamma + \lambda)\mathcal{K})u = 0$, where $\mathcal{K} = (L + \gamma I)^{-1} \colon L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$ is the compact operator defined before. Hence, in this case, equation $Lu - \lambda u = 0$ has only trivial weak solution $u = 0$ if and only if $\mathcal{N}(I - (\gamma + \lambda)\mathcal{K}) = \{0\}$; since $\lambda + \gamma > 0$, this last condition is equivalent to $\frac{1}{\gamma + \lambda} \notin \sigma(\mathcal{K})$. Let

$$\Sigma = \left\{ \lambda > -\gamma \;\middle|\; \frac{1}{\gamma + \lambda} \in \sigma(\mathcal{K}) \right\}.$$

Therefore, we have proved that the problem $Lu - \lambda u = F$ has a unique weak solution in $H_0^2(\Omega; \mathbb{R}^N)$ for each $F \in L^2(\Omega; \mathbb{R}^N)$ if and only if $\lambda \notin \Sigma$. Since $\sigma(\mathcal{K})$ is at most an infinite sequence converging to 0, it follows that Σ is at most an infinite sequence converging to $+\infty$. □

Remark 2.34. The set Σ in the theorem is exactly the set of (real) eigenvalues of L.

Lecture 30 – 3/29/19

2.5.3. More Functional Analysis – Spectral Theory of Symmetric Compact Operators. Let H be a real Hilbert space, with inner product $(\,,\,)$ and norm $\|\cdot\|$, and let $S \colon H \to H$ be linear, bounded and symmetric. Define

$$m = \inf_{u \in H,\, \|u\|=1} (Su, u), \quad M = \sup_{u \in H,\, \|u\|=1} (Su, u).$$

Lemma 2.39. *We have $m, M \in \sigma(S) \subset [m, M]$.*

Proof. 1. Clearly $-\|S\| \leq m \leq M \leq \|S\|$. Let $\eta > M$ and consider the bilinear form $B[u, v] = (\eta u - Su, v)$. Then B is bounded on H and

$$B[u, u] = (\eta u - Su, u) = \eta \|u\|^2 - (Su, u) \geq (\eta - M)\|u\|^2.$$

Hence, by the **Lax-Milgram Theorem**, for each $w \in H$, there exists a unique $u \in H$ such that $B[u, v] = (w, v)$ for all $v \in H$; that is, $\eta u - Su = w$. Moreover, $\|u\| \leq \frac{1}{\eta - M}\|w\|$. So $(\eta I - S)^{-1} \colon H \to H$ exists and $\|(\eta I - S)^{-1} w\| \leq \frac{1}{\eta - M}\|w\|$; thus $\eta \in \rho(S)$ and hence $\sigma(S) \subset (-\infty, M]$. Similarly, $\sigma(S) \subset [m, \infty)$. This proves $\sigma(S) \subset [m, M]$.

2. Let $B[u, v] = (Mu - Su, v)$. Then $B[u, v]$ is symmetric and

$$0 \leq B[u, u] \leq (M - m)\|u\|^2 \quad \forall\, u \in H.$$

This implies $h(t) = B[u + tv, u + tv] \geq 0$ for all t; hence

$$B[u, v]^2 \leq B[u, u]\, B[v, v] \quad \forall\, u, v \in H.$$

Setting $v = Mu - Su$ yields

$$\|Mu - Su\|^4 = B[u, v]^2 \leq B[u, u]\, B[v, v] \leq (M - m) B[u, u] \|v\|^2,$$

which gives

$$\|Mu - Su\|^2 \leq (M - m)(Mu - Su, u) \quad \forall\, u \in H.$$

3. We prove $M \in \sigma(S)$. Let $u_k \in H$ be such that $\|u_k\| = 1$ and $(Su_k, u_k) \to M$. Then $(Mu_k - Su_k, u_k) \to 0$ and hence by the inequality above,

$$\|(MI - S)u_k\|^2 = \|Mu_k - Su_k\|^2 \leq (M - m)(Mu_k - Su_k, u_k) \to 0.$$

So $(MI - S)u_k \to 0$. If $M \in \rho(S)$, then $u_k = (MI - S)^{-1}(Mu_k - Su_k) \to 0$, a contradiction to $\|u_k\| = 1$. Therefore $M \in \sigma(S)$. Similarly, $m \in \sigma(S)$. □

Theorem 2.40. (Spectral Property for Compact Symmetric Operators) *Let H be a separable Hilbert space and $S\colon H \to H$ be compact and symmetric. Then there exists a countable orthonormal basis of H consisting of eigenvectors of S.*

Proof. 1. Let $\sigma(S) \setminus \{0\} = \{\eta_k\}_{k=1}^{\infty}$ and $\eta_0 = 0$. Write $H_k = \mathcal{N}(S - \eta_k I)$ for $k = 0, 1, \ldots$. According to the **Fredholm alternative**, $0 < \dim H_k < \infty$ for $k = 1, 2, \ldots$. If $k \neq l$ and $u \in H_k, v \in H_l$, then $\eta_k(u, v) = (Su_k, v) = (u_k, Sv) = \eta_l(u, v)$, which implies $(u, v) = 0$ and thus $H_k \perp H_l$ if $k \neq l$.

2. Let $\tilde{H}$ be the smallest subspace of H containing all H_k's for $k = 0, 1, \ldots$; namely,

$$\tilde{H} = \Big\{ \sum_{k=0}^{m} a_k u_k \mid m \in \{0, 1, \ldots\},\ a_k \in \mathbb{R},\ u_k \in H_k \ \ \forall\, k = 0, 1, \ldots, m \Big\}.$$

Then $S(\tilde{H}) \subset \tilde{H}$. Furthermore, if $u \in \tilde{H}^{\perp}$ and $v \in \tilde{H}$, then $(Su, v) = (u, Sv) = 0$; hence $S(\tilde{H}^{\perp}) \subset \tilde{H}^{\perp}$.

3. Consider the operator $\tilde{S} = S|_{\tilde{H}^{\perp}}\colon \tilde{H}^{\perp} \to \tilde{H}^{\perp}$. Then $\tilde{S}$ is compact and symmetric; moreover, any nonzero eigenvalue of $\tilde{S}$ would be a nonzero eigenvalue of S and all eigenvectors of S are in $\tilde{H}$ not in $\tilde{H}^{\perp}$. Hence $\tilde{S}$ has no nonzero eigenvalues; thus $\sigma(\tilde{S}) = \{0\}$. By Lemma 2.39, this implies $(\tilde{S}u, u) = 0$ for all $u \in \tilde{H}^{\perp}$. Hence, for all $u, v \in \tilde{H}^{\perp}$,

$$0 = (\tilde{S}(u + v), u + v) = (\tilde{S}u, u) + (\tilde{S}v, v) + 2(\tilde{S}u, v) = 2(\tilde{S}u, v);$$

that is, $(\tilde{S}u, v) = 0$ for all $u, v \in \tilde{H}^{\perp}$; this shows that $\tilde{S} \equiv 0$. Thus $\tilde{H}^{\perp} = \mathcal{N}(\tilde{S}) \subset \mathcal{N}(S) = H_0 \subset \tilde{H}$, which implies that $\tilde{H}^{\perp} = \{0\}$; hence $\tilde{H}$ is dense in H.

4. Choose an orthonormal basis for each H_k ($k = 0, 1, 2, \ldots$), noting that since H is separable, if $\dim H_0 > 0$ (in this case H_0 is the eigenspace of S for eigenvalue 0), then H_0 has an at most countable orthonormal basis. Each of other H_k's ($k \neq 0$) is finite dimensional. We thus obtain a countable orthonormal basis of H consisting of eigenvectors of S. □

2.5.4. Eigenvalue Problems for Symmetric Elliptic Operators. In what follows, we assume that the bilinear form $B[u, v]$ of L is **symmetric** on $H_0^1(\Omega; \mathbb{R}^N)$, that is,

$$B[u, v] = B[v, u] \quad \forall\ u, v \in H_0^1(\Omega; \mathbb{R}^N). \tag{2.68}$$

In this case, Lu is **symmetric** or **self-adjoint**: $L^*u = Lu$. We also assume the **Gårding's inequality**:

$$B[u, u] \geq \beta \|u\|_{H_0^1}^2 - \gamma \|u\|_{L^2}^2, \quad \forall\ u \in H_0^1(\Omega; \mathbb{R}^N), \tag{2.69}$$

where $\beta > 0$ and $\gamma \in \mathbb{R}$ are constants; see Theorem 2.11 for sufficient conditions.

For each $F \in L^2(\Omega; \mathbb{R}^N)$, define $u = \mathcal{K}F$ to be the unique weak solution in $H_0^1(\Omega; \mathbb{R}^N)$ of the BVP

$$Lu + \gamma u = F \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

By Theorem 2.12 and Corollary 2.13, $\mathcal{K} = (L + \gamma I)^{-1}$ is a compact linear operator on $L^2(\Omega; \mathbb{R}^N)$. We easily verify the following result.

Lemma 2.41. $\mathcal{K}\colon L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ *is symmetric and positive; that is,*

$$(\mathcal{K}F, G)_{L^2} = (\mathcal{K}G, F)_{L^2}, \quad (\mathcal{K}F, F)_{L^2} \geq 0, \quad \forall\, F,\, G \in L^2(\Omega;\mathbb{R}^N).$$

Theorem 2.42. (Eigenvalue Theorem) *Assume* (2.68) *and* (2.69).

(i) *The eigenvalues of L consist of a countable set $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$, where*

$$-\gamma < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

are listed repeatedly the same times as the multiplicity, and $\lambda_k \to \infty$. Furthermore, let w_k be an eigenfunction to λ_k satisfying $\|w_k\|_{L^2(\Omega;\mathbb{R}^N)} = 1$. Then $\{w_k\}_{k=1}^{\infty}$ forms an **orthonormal basis** *of $L^2(\Omega;\mathbb{R}^N)$.*

(ii) *The first (smallest) eigenvalue λ_1 is called the (Dirichlet)* **principal eigenvalue** *of L and is characterized by the* **Rayleigh's formula**

$$\lambda_1 = \min_{\substack{u \in H_0^1(\Omega;\mathbb{R}^N) \\ \|u\|_{L^2(\Omega;\mathbb{R}^N)} = 1}} B[u,u] = \min_{\substack{u \in H_0^1(\Omega;\mathbb{R}^N) \\ u \neq 0}} \frac{B[u,u]}{\|u\|^2_{L^2(\Omega;\mathbb{R}^N)}}. \tag{2.70}$$

Moreover, if $u \in H_0^1(\Omega;\mathbb{R}^N)$, $u \not\equiv 0$, then u is an eigenfunction corresponding to λ_1 if and only if $B[u,u] = \lambda_1 \|u\|^2_{L^2(\Omega;\mathbb{R}^N)}$.

Proof. 1. Let $\mathcal{K} = (L + \gamma I)^{-1}$ be the symmetric positive compact operator defined above. We see that λ is an eigenvalue of L if and only if equation $(I - (\lambda+\gamma)\mathcal{K})u = 0$ has nontrivial solutions $u \in L^2(\Omega;\mathbb{R}^N)$; this exactly asserts that

$$\lambda \text{ is an eigenvalue of } L \text{ if and only if } \tfrac{1}{\lambda+\gamma} \text{ is an eigenvalue of operator } \mathcal{K}. \tag{2.71}$$

Moreover, u is an eigenfunction of L corresponding to eigenvalue λ if and only if u is an eigenvector of $\mathcal{K}$ corresponding to eigenvalue $\frac{1}{\lambda+\gamma}$. Therefore, (i) follows from Theorem 2.40.

2. We now prove the second statement. If u is an eigenfunction corresponding to λ_1 with $\|u\|_{L^2(\Omega;\mathbb{R}^N)} = 1$, then easily $B[u,u] = \lambda_1(u,u)_{L^2} = \lambda_1\|u\|^2_{L^2} = \lambda_1$. We now assume

$$u \in H_0^1(\Omega;\mathbb{R}^N), \;\; \|u\|_{L^2(\Omega;\mathbb{R}^N)} = 1.$$

Let $\{w_k\}$ be the orthonormal basis of $L^2(\Omega;\mathbb{R}^N)$ consisting of eigenfunctions as given in (i). Then $B[w_k, w_l] = \lambda_k (w_k, w_l)_{L^2} = \lambda_k \delta_{kl}$. Set $\tilde{w}_k = (\lambda_k + \gamma)^{-1/2} w_k$, and consider the inner product on $H = H_0^1(\Omega;\mathbb{R}^N)$ defined by

$$((u,v)) := B_\gamma[u,v] \equiv B[u,v] + \gamma(u,v)_{L^2} \quad (u, v \in H).$$

Then $((u,u))^{1/2}$ defines an equivalent norm on H and $((\tilde{w}_k, \tilde{w}_l)) = \delta_{kl}$. Let $d_k = (u, w_k)_{L^2}$. We have

$$\sum_{k=1}^{\infty} d_k^2 = \|u\|^2_{L^2} = 1, \quad u = \sum_{k=1}^{\infty} d_k w_k = \sum_{k=1}^{\infty} \tilde{d}_k \tilde{w}_k \tag{2.72}$$

with $\tilde{d}_k = d_k \sqrt{\lambda_k + \gamma}$, where the series for u are in the norm-convergence in $L^2(\Omega;\mathbb{R}^N)$.

3. We claim that the series for u converges also in the equivalent norm $((u,u))^{1/2}$ on H. Indeed, for $m = 1, 2, \cdots$, define

$$u_m = \sum_{k=1}^{m} d_k w_k = \sum_{k=1}^{m} \tilde{d}_k \tilde{w}_k \in H.$$

From $((\tilde{w}_k, u)) = B[\tilde{w}_k, u] + \gamma(\tilde{w}_k, u)_{L^2} = (\lambda_k + \gamma)(\tilde{w}_k, u)_{L^2} = \tilde{d}_k$, we have

$$((u_m, u)) = \sum_{k=1}^{m} \tilde{d}_k^2 = ((u_m, u_m)) \quad (m = 1, 2, \cdots).$$

This implies $((u_m, u_m)) \le ((u, u))$ for all $m = 1, 2, \cdots$. Hence, $\{u_m\}$ is bounded in H and so, by a subsequence, $u_m \rightharpoonup \tilde{u}$ in H as $m \to \infty$ (here we use the notion of **weak convergence**). Since $u_m \to u$ in L^2, we must have $\tilde{u} = u$ and so

$$((u, u)) \le \liminf_{m\to\infty} ((u_m, u_m)),$$

which, combined with $((u - u_m, u - u_m)) = ((u, u)) + ((u_m, u_m)) - 2((u, u_m)) = ((u, u)) - ((u_m, u_m))$, implies that $u_m \to u$ in H, and the claim is proved.

4. Now, by (2.72), we have

$$B[u, u] = \sum_{k=1}^{\infty} d_k B[w_k, u] = \sum_{k=1}^{\infty} d_k^2 \lambda_k \ge \sum_{k=1}^{\infty} d_k^2 \lambda_1 = \lambda_1.$$

Hence (2.70) is proved. Moreover, if in addition $B[u, u] = \lambda_1$, then we have

$$\sum_{k=1}^{\infty} (\lambda_k - \lambda_1) d_k^2 = 0; \text{ so } d_k = 0 \text{ if } \lambda_k > \lambda_1.$$

Assume λ_1 has multiplicity m, with $Lw_k = \lambda_1 w_k$ $(k = 1, 2, \cdots, m)$. Then $u = \sum_{k=1}^{m} d_k w_k$, and so $Lu = \lambda_1 u$; that is, u is an eigenfunction corresponding to λ_1. □

2.5.5. The Scalar Case $N = 1$. We consider a special scalar symmetric elliptic operator Lu given by

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u) + c(x)u,$$

where the uniform ellipticity condition is satisfied, $\partial\Omega$ is smooth, and a_{ij}, c are smooth functions on $\bar{\Omega}$ satisfying

$$a_{ij}(x) = a_{ji}(x), \ \ c(x) \ge 0 \quad (x \in \bar{\Omega}).$$

Theorem 2.43. *The principal eigenvalue of L is positive; that is $\lambda_1 > 0$. Let w_1 be an eigenfunction corresponding to the principal eigenvalue λ_1 of L. Then, either $w_1(x) > 0$ for all $x \in \Omega$ or $w_1(x) < 0$ for all $x \in \Omega$. Moreover, the eigenspace corresponding to λ_1 is one-dimensional; that is,* **the principal eigenvalue λ_1 is simple.**

Proof. 1. Since in this case the bilinear form B is positive: $B[u, u] \ge \theta \|u\|^2_{H_0^1(\Omega)} \ge \sigma \|u\|^2_{L^2(\Omega)}$, we have $\lambda_1 > 0$. Let w_1 be an eigenfunction corresponding to λ_1 with $\|w_1\|_{L^2(\Omega)} = 1$. Then $w_1^{\pm} \in H_0^1(\Omega)$, $w_1 = w_1^+ + w_1^-$, $\|w_1^+\|^2_{L^2(\Omega)} + \|w_1^-\|^2_{L^2(\Omega)} = \|w_1\|^2_{L^2(\Omega)} = 1$, and

$$Dw_1^+ = \chi_{\{w_1 > 0\}} Dw_1, \quad Dw_1^- = \chi_{\{w_1 < 0\}} Dw_1.$$

Hence $B[w_1^+, w_1^-] = 0$, and thus

$$\lambda_1 = B[w_1, w_1] = B[w_1^+, w_1^+] + B[w_1^-, w_1^-] \ge \lambda_1 \|w_1^+\|^2_{L^2(\Omega)} + \lambda_1 \|w_1^-\|^2_{L^2(\Omega)} = \lambda_1.$$

So the inequality must be equality, which yields that

$$B[w_1^+, w_1^+] = \lambda_1 \|w_1^+\|^2_{L^2(\Omega)}, \quad B[w_1^-, w_1^-] = \lambda_1 \|w_1^-\|^2_{L^2(\Omega)}.$$

Therefore, $u = w_1^{\pm}$ is a $H_0^1(\Omega)$-solution to the equation $Lu = \lambda_1 u$ in Ω. Since the coefficients of L and Ω are smooth on $\bar{\Omega}$, $u = w_1^{\pm}$ is smooth on $\bar{\Omega}$. (See Theorem 2.27.) Since $Lw_1^- =$

$\lambda_1 w_1^- \leq 0$ in Ω, by the **Strong Maximum Principle**, either $w_1^- \equiv 0$ or else $w_1^- < 0$ in Ω; similarly, either $w_1^+ \equiv 0$ or else $w_1^+ > 0$ in Ω. This proves that either $w_1 < 0$ in Ω or else $w_1 > 0$ in Ω.

2. To prove the eigenspace of λ_1 is one-dimensional, let w be another eigenfunction. Let $t \in \mathbb{R}$ be such that $\int_\Omega (w - tw_1)dx = 0$. Since $u = w - tw_1$ is also a solution to $Lu = \lambda_1 u$, it follows that either $u \equiv 0$, $u > 0$, or $u < 0$, in Ω; however, in the last two cases, $\int_\Omega udx \neq 0$. Hence $u \equiv 0$; namely, $w(x) = tw_1(x)$ for all $x \in \Omega$. $\square$

Chapter 3

Part III – The Calculus of Variations

Lecture 31 – 4/1/19

3.1. Variational Problems

3.1.1. Basic Ideas. This chapter will discuss certain methods for solving the boundary value problem for some partial differential equations; these problems, in an abstract form, can be written as

$$\mathcal{A}[u] = 0. \tag{3.1}$$

There is, of course, no general theory for solving such problems. The **Calculus of Variations** identifies an important class of problems which can be solved using relatively simple techniques motivated from the elementary *Calculus.* This is the class of **variational problems**, where the operator $\mathcal{A}[u]$ can be formulated as the **first variation** ("derivative") of an appropriate "energy" functional $I(u)$ on a Banach space X; that is, $\mathcal{A}[u] = I'(u)$. In this way, $\mathcal{A}\colon X \to X^*$ and equation $\mathcal{A}[u] = 0$ can be formulated as

$$\langle I'(u), v\rangle = 0, \quad \forall\, v \in X.$$

The advantage of this new formulation is that solving problem (3.1) (at least *weakly*) is equivalent to finding the **critical points** of I on X. The **minimization method** for a variational problem is to solve the problem by finding the minimizers of the related energy functional.

We should also mention that many physical laws in applications arise directly as variational principles. However, although powerful, *not all* PDE problems can be formulated as a variational problem; there are other important (non-variational) methods for studying PDEs, and we shall not study them in this chapter.

3.1.2. Multiple Integral Functionals, First Variation and the Euler-Lagrange Equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary $\partial\Omega$. For a function $u\colon \Omega \to \mathbb{R}^N$ (we say u is scalar if $N = 1$ and u is vector if $N \geq 2$), let $u = (u^1, u^2, \cdots, u^N)$ and use

$$Du = (D_i u^k) = (\partial_{x_i} u^k) \quad (k = 1, 2, \cdots, N;\ i = 1, 2, \cdots, n)$$

to denote the Jacobi matrix of u; for each $x \in \Omega$, $Du(x) \in \mathbb{M}^{N\times n}$.

Given a function $L\colon \Omega \times \mathbb{R}^N \times \mathbb{M}^{N\times n} \to \mathbb{R}$, consider the multiple integral functional

$$I(u) = \int_\Omega L(x, u(x), Du(x))\, dx. \tag{3.2}$$

The function $L(x, s, \xi)$ is usually called the **Lagrangian** of the functional I. (Here we are using different notation from the textbook and doing the general cases including both the scalar and the system cases.)

Suppose $L(x, s, \xi)$ is continuous in (x, s, ξ) and smooth in (s, ξ). Assume u is a nice (say, $u \in C^1(\bar{\Omega}; \mathbb{R}^N)$) minimizer of $I(u)$ with its own boundary data; that is,

$$I(u) \le I(u + t\varphi)$$

for all $t \in \mathbb{R}$ and $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$. Then by taking derivative of $I(u + t\,\varphi)$ at $t = 0$ we see that u satisfies

$$\int_\Omega \Big(L_{\xi_i^k}(x, u, Du)\, D_i\varphi^k + L_{s^k}(x, u, Du)\, \varphi^k \Big)\, dx = 0 \tag{3.3}$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$. (Summation notation is used here.)

The left-hand side of (3.3) is called the **first variation** of I at u (in the direction of φ), and is denoted by $\langle I'(u), \varphi\rangle$. Since (3.3) holds for all test functions φ, we conclude after integration by parts that u solves the system of nonlinear PDEs:

$$-\sum_{i=1}^n D_i(A_i^k(x, u, Du)) + b^k(x, u, Du) = 0 \quad (k = 1, 2, \cdots, N), \tag{3.4}$$

where functions $A_i^k(x, s, \xi)$ and $b^k(x, s, \xi)$ are defined by

$$A_i^k(x, s, \xi) = L_{\xi_i^k}(x, s, \xi), \quad b^k(x, s, \xi) = L_{s^k}(x, s, \xi). \tag{3.5}$$

Definition 3.1. The coupled system (3.4) of quasilinear PDE in divergence form is called the **Euler-Lagrange equation** associated with the integral functional $I(u)$. We often write the Euler-Lagrange PDE system (3.4) as

$$-\operatorname{div} A(x, u, Du) + b(x, u, Du) = 0 \quad \text{in } \Omega,$$

with matrix function $A = (A_i^k)$ and vector function $b = (b^k)$ given by (3.5).

In summary, any smooth minimizer of $I(u)$ is a solution of the Euler-Lagrange equation associated with $I(u)$, and thus we may try to solve the PDEs of the type (3.4) by searching for minimizers or general critical points of functional $I(u)$. This is the method of calculus of variations or **variational method** for PDE. The fundamental issues of this method are whether minimizers exist and are smooth enough to be a solution of the PDE. These issues lead to the existence and regularity theories that we will discuss separately.

Example 3.1. **(Generalized Dirichlet's principle).** Take

$$I(u) = \int_\Omega \Big(\frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} - uf \Big)\, dx,$$

where $a_{ij} = a_{ji}\, (i, j = 1, 2, \cdots, n)$ and $f\colon \Omega \to \mathbb{R}$ are given functions. Then the Euler-Lagrange equation is the divergence form PDE

$$-\sum_{i,j=1}^n (a_{ij}(x) u_{x_i})_{x_j} = f \quad \text{in } \Omega.$$

EXAMPLE 3.2. **(Nonlinear Poisson equations).** Assume : $\mathbb{R} \to \mathbb{R}$ is a smooth function and $F(z) = \int_0^z f(s)\,ds$. Consider

$$I(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 - F(u)\right) dx.$$

Then the Euler-Lagrange equation is the nonlinear Poisson equation

$$-\Delta u = f(u) \quad \text{in } \Omega.$$

EXAMPLE 3.3. **(Minimal surfaces).** Let

$$I(u) = \int_\Omega (1 + |Du|^2)^{1/2} dx$$

be the area of the graph of $u\colon \Omega \to \mathbb{R}$. The associated Euler-Lagrange equation is

$$\operatorname{div} \frac{Du}{(1+|Du|^2)^{1/2}} = 0 \quad \text{in } \Omega,$$

which is called the **minimal surface equation**. The left side of the equation represents n-times the **mean curvature** of the graph of u. Thus a minimal surface has zero mean curvature.

3.1.3. Second Variation and Legendre-Hadamard Conditions. If L, u are sufficiently smooth (e.g. of class C^2) then, at the minimizer u, for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$, we have

$$\langle I''(u)\varphi, \varphi\rangle := \frac{d^2}{dt^2} I(u + t\varphi)\Big|_{t=0} \geq 0,$$

which gives

$$\begin{aligned}\langle I''(u)\varphi, \varphi\rangle = \int_\Omega \Big(& L_{\xi_i^k \xi_j^l}(x,u,Du)\, D_i\varphi^k D_j\varphi^l + 2L_{\xi_i^k s^l}(x,u,Du)\, \varphi^l D_i\varphi^k \\ & + L_{s^k s^l}(x,u,Du)\, \varphi^k\varphi^l \Big)\, dx \geq 0 \qquad \forall\, \varphi \in C_0^\infty(\Omega;\mathbb{R}^N).\end{aligned} \tag{3.6}$$

The quantity $\langle I''(u)\varphi, \varphi\rangle$ is called the **second variation** of I at u (in direction φ).

We can extract useful information from (3.6). Note that a routine approximation argument shows that (3.6) is also valid for all Lipschitz functions φ vanishing on $\partial\Omega$. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be the periodic **zig-zag function** of period 1 with $\rho(t) = t$ if $0 \leq t \leq \frac{1}{2}$ and $\rho(t) = 1 - t$ if $\frac{1}{2} \leq t \leq 1$. Given $p \in \mathbb{R}^n$, $q \in \mathbb{R}^N$, $\epsilon > 0$ and $\zeta \in C_0^\infty(\Omega)$, define

$$\varphi(x) = \epsilon\, \rho(\frac{x\cdot p}{\epsilon})\zeta(x) q, \quad \forall\, x \in \Omega.$$

Note that $D_i\varphi^k(x) = \rho'(\frac{x\cdot p}{\epsilon}) p_i q^k \zeta + O(\epsilon)$ as $\epsilon \to 0^+$. Substitute this φ into (3.6) and let $\epsilon \to 0^+$ to obtain

$$\int_\Omega \Big(\sum_{i,j=1}^n \sum_{k,l=1}^N L_{\xi_i^k \xi_j^l}(x,u,Du)\, p_i p_j q^k q^l \Big) \zeta^2\, dx \geq 0.$$

Since this holds for all $\zeta \in C_0^\infty(\Omega)$, we deduce

$$\sum_{i,j=1}^n \sum_{k,l=1}^N L_{\xi_i^k \xi_j^l}(x,u,Du)\, p_i p_j q^k q^l \geq 0, \quad \forall\, x \in \Omega,\ p \in \mathbb{R}^n,\ q \in \mathbb{R}^N. \tag{3.7}$$

This necessary condition is called the (weak) **Legendre-Hadamard condition** for L at the minimum point u.

Lecture 32 – 4/3/19

3.2. Existence of Minimizers

3.2.1. Some Definitions in Nonlinear Functional Analysis.

Definition 3.2. Let X be a Banach space and X^* be its dual space.

(1) A sequence u_ν in X is said to **weakly converge** to an element $u \in X$ if
$$\langle f, u_\nu \rangle \to \langle f, u \rangle \quad \forall\, f \in X^*.$$
(2) A set $\mathcal{C} \subset X$ is said to be (*sequentially*) **weakly closed** provided $u \in \mathcal{C}$ whenever $\{u_\nu\} \subset \mathcal{C},\ u_\nu \rightharpoonup u$.

(3) A function $I\colon X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is said to be (*sequentially*) **weakly lower semicontinuous** (**w.l.s.c.**) on X provided
$$I(u) \le \liminf_{\nu\to\infty} I(u_\nu) \quad \text{whenever } u_\nu \rightharpoonup u \text{ in } X.$$
(4) A function $I\colon X \to \bar{\mathbb{R}}$ is said to be **coercive** on an unbounded set $\mathcal{C} \subseteq X$ provided $I(u) \to \infty$ as $\|u\| \to \infty$ in $\mathcal{C}$.

(5) A function $I\colon X \to \mathbb{R}$ is said to be **Gateaux-differentiable** at $u \in X$ if for all $v \in X$ the function $h(t) = I(u + tv)$ is differentiable at $t = 0$. In this case, we define $\langle I'(u), v\rangle = h'(0)$ to be the **Gateaux** or **directional derivative** of I at u in direction v.

(6) A function $I\colon X \to \mathbb{R}$ is said to be **Fréchet-differentiable** or simply **differentiable** at $u \in X$ if there exists an element $f \in X^*$ such that
$$\lim_{v\in X,\, \|v\|\to 0} \frac{I(u+v) - I(u) - \langle f, v\rangle}{\|v\|} = 0.$$
In this case, we define $I'(u) = f$ to be the **Fréchet derivative** of I at u. We say I is C^1 **on** X provided that $I'(u)$ is defined for all $u \in X$ and $I'\colon X \to X^*$ is continuous.

3.2.2. The Direct Method of the Calculus of Variations. We study a general method for proving the existence of minimizers of a function defined on a Banach space. This method is called the **direct method of the calculus of variations**.

Theorem 3.4. (Direct Method of the Calculus of Variations) *Let X be a reflexive Banach space, $\mathcal{C} \subseteq X$ be a nonempty weakly closed set, and $I\colon X \to \bar{\mathbb{R}}$ be w.l.s.c. and coercive on $\mathcal{C}$ if $\mathcal{C}$ is unbounded. Assume $\inf_{u\in\mathcal{C}} I(u) < \infty$. Then there is at least one $u_0 \in \mathcal{C}$ such that $I(u_0) = \inf_{u\in\mathcal{C}} I(u)$; such a function $u_0 \in \mathcal{C}$ is called a* **minimizer** *of I on $\mathcal{C}$.*

Proof. First of all, take a sequence $\{u_\nu\}$, called a **minimizing sequence**, such that
$$\lim_{\nu\to\infty} I(u_\nu) = \inf_{u\in\mathcal{C}} I(u) < \infty.$$
Then the coercivity condition implies that $\{u_\nu\}$ must be bounded in X. Since X is reflexive, there exists a subsequence of $\{u_\nu\}$, denoted by $\{u_{\nu_j}\}$, and $u_0 \in X$ such that $u_{\nu_j} \rightharpoonup u_0$ *weakly* in X. The weak closedness of $\mathcal{C}$ implies $u_0 \in \mathcal{C}$. Now the w.l.s.c. of I implies
$$I(u_0) \le \liminf_{j\to\infty} I(u_{\nu_j}) = \inf_{u\in\mathcal{C}} I(u).$$
This proves that u_0 is a minimizer of I on $\mathcal{C}$. The procedure presented in this proof is called a **direct method proof**. □

3.2.3. The Coecivity and Lower Semicontinuity. We now study the multiple integral functionals of the type

$$I(u) = \int_\Omega L(x, u, Du)\, dx$$

on Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$, where Ω is bounded open in $\mathbb{R}^n$. We discuss some conditions on the Lagrangian L in order to use the direct method on functional I.

Dirichlet classes. Let $1 \le p < \infty$ and $\varphi \in W^{1,p}(\Omega;\mathbb{R}^N)$. We define the **Dirichlet class**

$$\mathcal{D}_\varphi = \{u \in W^{1,p}(\Omega;\mathbb{R}^N) \,|\, u - \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)\}.$$

Note that if $\partial\Omega \in C^1$ then using the trace operator γ_0 it is easy to show that $\mathcal{D}_\varphi$ is weakly closed in $W^{1,p}(\Omega;\mathbb{R}^N)$. For general open sets Ω, the proof of weak closedness of $\mathcal{D}_\varphi$ needs the **Mazur's lemma** which asserts that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit; we will not discuss this result but will freely use the weak closedness of $\mathcal{D}_\varphi$ for all Ω's in the following. Also, by the **Poincaré's inequality**, we have

$$\|Du\|_{L^p(\Omega)} \le \|u\|_{W^{1,p}(\Omega)} \le C(\|Du\|_{L^p(\Omega)} + \|\varphi\|_{W^{1,p}(\Omega)}) \quad \forall\, u \in \mathcal{D}_\varphi. \tag{3.8}$$

Coercivity. Assume $L(x,s,\xi)$ is continuous on (s,ξ) and measurable on x, and

$$L(x,s,\xi) \ge \alpha|\xi|^p - \beta(x) \quad \forall\, x \in \Omega,\ s \in \mathbb{R}^N,\ \xi \in \mathbb{M}^{N\times n}, \tag{3.9}$$

where $\alpha > 0$ is a constant and $\beta \in L^1(\Omega)$ is a function. Then $I\colon W^{1,p}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ is well-defined and

$$I(w) \ge \alpha\|Dw\|^p_{L^p(\Omega)} - \|\beta\|_{L^1(\Omega)} \quad \forall\, w \in W^{1,p}(\Omega;\mathbb{R}^N).$$

Hence, by (3.8), we have for some constants $\delta > 0$ and $\gamma \in \mathbb{R}$,

$$I(w) \ge \delta\|w\|^p_{W^{1,p}(\Omega)} - \gamma \quad \forall\, w \in \mathcal{D}_\varphi. \tag{3.10}$$

This implies the coercivity of I on the Dirichlet class $\mathcal{D}_\varphi$. Note that condition (3.10) can hold under some conditions weaker than (3.9); we will not discuss such conditions.

Weak lower semicontinuity. The necessary and sufficient condition for weak lower semicontinuity of $I(u)$ on a Sobolev space (especially for general systems) is a difficult problem involving **Morrey's quasiconvexity**, which we will not study in this course. Instead, we prove a semicontinuity result for certain Lagrangians $L(x,s,\xi)$ that are convex in ξ. Another lower semicontinuity theorem for polyconvex functionals will be proved later.

Recall that a function $L(x,s,\xi)$ is said to be **convex** in $\xi \in \mathbb{M}^{N\times n}$ if

$$L(x,s,t\xi + (1-t)\eta) \le tL(x,s,\xi) + (1-t)L(x,s,\eta)$$

for all x, s, ξ, η and $0 \le t \le 1$.

Lemma 3.5. *Let L be C^1 in ξ. Then the convexity of L in ξ is equivalent to the following condition:*

$$L(x,s,\eta) \ge L(x,s,\xi) + L_{\xi_i^k}(x,s,\xi)\,(\eta_i^k - \xi_i^k) \tag{3.11}$$

for all $x \in \Omega$, $s \in \mathbb{R}^N$ and $\xi, \eta \in \mathbb{M}^{N\times n}$.

Proof. Exercise! □

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Theorem 3.6. (Tonelli's Theorem) *Let $L(x,s,\xi) \geq 0$ be* **smooth** *and* **convex** *in ξ. Assume L, L_ξ are both* continuous *in (x,s,ξ). Then the functional $I(u)$ defined above is* weakly (weakly* if $p=\infty$) lower semicontinuous *on $W^{1,p}(\Omega;\mathbb{R}^N)$ for all $1 \leq p \leq \infty$.*

Proof. We only prove $I(u)$ is *w.l.s.c.* on $W^{1,1}(\Omega;\mathbb{R}^N)$; the proof for other p follows easily. To this end, assume $\{u_\nu\}$ is a sequence weakly convergent to u in $W^{1,1}(\Omega;\mathbb{R}^N)$. We need to show

$$I(u) \leq \liminf_{\nu\to\infty} I(u_\nu).$$

By the Sobolev embedding theorem it follows that (via a subsequence) $u_\nu \to u$ in $L^1(\Omega;\mathbb{R}^N)$. We also assume $u_\nu(x) \to u(x)$ for almost every $x \in \Omega$. Now, given $\delta > 0$, we choose a compact set $K \subset \Omega$ such that

(i) $u_\nu \to u$ uniformly on K and $|\Omega \setminus K| < \delta$ (by **Egorov's theorem**);

(ii) u, Du are continuous on K (by **Lusin's theorem**).

Since $L(x,s,\xi)$ is *smooth* and *convex* in ξ, it follows that

$$L(x,s,\eta) \geq L(x,s,\xi) + L_{\xi_i^k}(x,s,\xi)\,(\eta_i^k - \xi_i^k) \quad \forall \xi,\ \eta \in \mathbb{M}^{N\times n}.$$

Therefore, since $L \geq 0$,

$$\begin{aligned}
I(u_\nu) &\geq \int_K L(x,u_\nu,Du_\nu)\,dx \\
&\geq \int_K \Big[L(x,u_\nu,Du) + L_{\xi_i^k}(x,u_\nu,Du)\,(D_i u_\nu^k - D_i u^k)\Big] \\
&= \int_K L(x,u_\nu,Du) + \int_K L_{\xi_i^k}(x,u,Du)\,(D_i u_\nu^k - D_i u^k) \\
&\quad + \int_K [L_{\xi_i^k}(x,u_\nu,Du) - L_{\xi_i^k}(x,u,Du)]\,(D_i u_\nu^k - D_i u^k).
\end{aligned}$$

Since $L(x,s,\xi)$ is uniformly continuous on bounded sets and $u_\nu(x) \to u(x)$ uniformly on K we have

$$\lim_{\nu\to\infty} \int_K L(x,u_\nu,Du)\,dx = \int_K L(x,u,Du)\,dx,$$

$$\lim_{\nu\to\infty} \|L_{\xi_i^k}(x,u_\nu,Du) - L_{\xi_i^k}(x,u,Du)\|_{L^\infty(K)} = 0.$$

Now since $L_{\xi_i^k}(x,u,Du)$ is bounded on K and $D_i u_\nu^k$ converges to $D_i u^k$ weakly in $L^1(\Omega)$ as $\nu \to \infty$, we thus have

$$\lim_{\nu\to\infty} \int_K L_{\xi_i^k}(x,u,Du)\,(D_i u_\nu^k - D_i u^k)\,dx = 0.$$

From these estimates, we have

$$\liminf_{\nu\to\infty} I(u_\nu) \geq \int_K L(x,u,Du). \tag{3.12}$$

If $L(x,u,Du) \in L^1(\Omega)$, i.e., $I(u) < \infty$, then for any given $\epsilon > 0$, we use the **Lebesgue absolute continuity theorem** to determine $\delta > 0$ so that

$$\int_E L(x,u,Du) \geq \int_\Omega L(x,u,Du) - \epsilon, \quad \forall\, E \subset \Omega,\ |\Omega \setminus E| < \delta.$$

On the other hand, if $I(u) = \infty$ then for any number $M > 0$ we choose $\delta > 0$ so that

$$\int_E L(x,u,Du)\,dx > M, \quad \forall\, E \subset \Omega,\ |\Omega \setminus E| < \delta.$$

In either of these two cases, using (3.12) and letting $\epsilon \to 0$ or $M \to \infty$, we obtain

$$\liminf_{\nu\to\infty} I(u_\nu) \ge I(u).$$

The theorem is proved. □

3.2.4. Existence in the Convex Case. Using the theorem, we obtain the following existence result.

Theorem 3.7. *In addition to the hypotheses of the previous theorem, assume there exists $1 < p < \infty$ such that*

$$L(x,s,\xi) \ge c\,|\xi|^p - C(x),$$

where $c > 0$, $C \in L^1(\Omega)$ are given. If for some $\varphi \in W^{1,p}(\Omega;\mathbb{R}^N)$, $I(\varphi) < \infty$, then minimization problem $\inf_{u\in\mathcal{D}_\varphi} I(u)$ has a minimizer in the Dirichlet class $\mathcal{D}_\varphi$.

Proof. This follows from the abstract existence Theorem 3.4 above. □

3.2.5. Weak Solutions of the Euler-Lagrange Equation. Distributional or weak solutions to the Euler-Lagrange equation (3.4) can be defined as long as $A(x,u,Du)$ and $b(x,u,Du)$ are in $L^1_{loc}(\Omega;\mathbb{R}^N)$. We give some **structural conditions** on the Lagrangian $L(x,s,\xi)$ so that the weak solutions to the BVP

$$\begin{cases} -\operatorname{div} A(x,u,Du) + b(x,u,Du) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

can be defined and studied in $W^{1,p}(\Omega;\mathbb{R}^N)$ for some $1 \le p < \infty$.

Standard Structural Conditions. We assume $L(x,s,\xi)$ is C^1 in (s,ξ) and

$$|L(x,s,\xi)| \le c_1(|\xi|^p + |s|^p) + c_2(x), \qquad c_2 \in L^1(\Omega); \tag{3.14}$$

$$|D_s L(x,s,\xi)| \le c_3(|\xi|^{p-1} + |s|^{p-1}) + c_4(x), \quad c_4 \in L^{\frac{p}{p-1}}(\Omega); \tag{3.15}$$

$$|D_\xi L(x,s,\xi)| \le c_5(|\xi|^{p-1} + |s|^{p-1}) + c_6(x), \quad c_6 \in L^{\frac{p}{p-1}}(\Omega), \tag{3.16}$$

where c_1, c_3, c_5 are constants.

Theorem 3.8. *Under the standard structural conditions above, any minimizer u of I on $\mathcal{D}_\varphi$ is a* **weak solution** *of the BVP (3.13) in the sense that $u \in \mathcal{D}_\varphi$ and*

$$\int_\Omega \Big(L_{\xi_i^k}(x,u,Du)\,D_i v^k + L_{s^k}(x,u,Du)\,v^k \Big)\,dx = 0 \quad \forall\, v \in W_0^{1,p}(\Omega;\mathbb{R}^N), \tag{3.17}$$

(as usual, summation notation is used here).

Proof. Let $X = W^{1,p}(\Omega;\mathbb{R}^N)$, $u, v \in X$, and $h(t) = I(u+tv)$. By (3.14), h is finite valued, and we show h is differentiable at $t = 0$ and

$$h'(0) = \langle I'(u), v\rangle = \int_\Omega \big(L_{\xi_i^k}(x,u,Du)\,D_i v^k(x) + L_{s^k}(x,u,Du)\,v^k(x)\big)\,dx. \tag{3.18}$$

We have $\frac{h(t)-h(0)}{t} = \int_\Omega L^t(x)\,dx$, where for almost every $x \in \Omega$,

$$L^t(x) = \frac{1}{t}[L(x,u+tv,Du+tDv) - L(x,u,Du)] = \frac{1}{t}\int_0^t \frac{d}{ds} L(x,u+sv,Du+sDv)\,ds$$

$$= \frac{1}{t}\int_0^t \left[L_{\xi_i^k}(x,u+sv,Du+sDv)\, D_i v^k + L_{s^k}(x,u+sv,Du+sDv)\, v^k \right] ds.$$

Hence

$$\lim_{t\to 0} L^t(x) = L_{\xi_i^k}(x,u,Du)\, D_i v^k + L_{s^k}(x,u,Du)\, v^k \quad a.e.\, x \in \Omega.$$

Using conditions (3.15), (3.16), and **Young's inequality**, we obtain that, for all $0<|t|\le 1$,

$$|L^t(x)| \le C_1(|Du|^p + |Dv|^p + |u|^p + |v|^p) + C_2(x), \quad C_2 \in L^1(\Omega).$$

Thus, by the **Lebesgue dominated convergence theorem**,

$$h'(0) = \lim_{t\to 0}\int_\Omega L^t(x)\,dx = \int_\Omega \Big(L_{\xi_i^k}(x,u,Du)\, D_i v^k + L_{s^k}(x,u,Du)\, v^k \Big) dx,$$

which proves (3.18). If u is a minimizer of I on $\mathcal{D}_\varphi$, then for each $v \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ the function $h(t) = I(u+tv)$ attains the minimum at $t=0$; hence $h'(0) = \langle I'(u), v\rangle = 0$, which proves (3.17). $\square$

3.2.6. Nemytskii Operators and Fréchet Differentiability of I. In fact, we can prove a much stronger result.

Theorem 3.9. *Let $1<p<\infty$. Then, under the same standard structural conditions as above, the functional I is C^1 on $X = W^{1,p}(\Omega;\mathbb{R}^N)$.*

Proof. Given $u \in X$, the formula (3.18) defines an element $I'(u) \in X^*$ and hence $I' \colon X \to X^*$ is well-defined. To show that $I'(u)$ is the **Fréchet derivative** of I at u, let $v \in X$ and $f(t) = I(u+tv) - I(u) - \langle I'(u), v\rangle t$. Then

$$I(u+v) - I(u) - \langle I'(u), v\rangle = \int_0^1 f'(s)ds = \int_0^1 (\langle I'(u+sv), v\rangle - \langle I'(u), v\rangle)\, ds$$

$$= \int_\Omega \int_0^1 \big(L_{\xi_i^k}(x,u+sv,Du+sDv) - L_{\xi_i^k}(x,u,Du) \big) D_i v^k \, ds dx$$

$$+ \int_\Omega \int_0^1 \big(L_{s^k}(x,u+sv,Du+sDv) - L_{s^k}(x,u,Du) \big) v^k \, ds dx.$$

Let

$$A(x,s,\xi) = \max_{0\le\tau\le 1} |L_\xi(x,u(x)+\tau s, Du(x)+\tau\xi) - L_\xi(x,u(x),Du(x))|,$$
$$B(x,s,\xi) = \max_{0\le\tau\le 1} |L_s(x,u(x)+\tau s, Du(x)+\tau\xi) - L_s(x,u(x),Du(x))|.$$

Then A and B are **Carathéodory** with respect to x and (s,ξ) and satisfy (3.20) with $q = \frac{p}{p-1}$ (see below). Note that

$$|I(u+v) - I(u) - \langle I'(u), v\rangle| \le C\, (\|A(x,v,Dv)\|_{L^q} + \|B(x,v,Dv)\|_{L^q})\|v\|_{W^{1,p}}.$$

Thus, the **Fréchet differentiability** of I at u will follow from

$$\lim_{v\to 0}(\|A(x,v,Dv)\|_{L^q} + \|B(x,v,Dv)\|_{L^q}) = 0, \tag{3.19}$$

which is the continuity of general **Nemytskii operators** proved in Lemma 3.10 below.

Finally, from (3.19) it also follows that $I' \colon X \to X^*$ is continuous and I is C^1 on X. $\square$

Nemytskii operators. Let $f\colon \Omega\times\mathbb{R}^d\to\mathbb{R}$ be **Carathéodory**; that is,

(i) for every $\xi\in\mathbb{R}^d$, $f(x,\xi)$ is a measurable function of x on Ω;
(ii) for a.e. $x\in\Omega$, $f(x,\xi)$ is a continuous function of ξ on $\mathbb{R}^d$.

Then, for each measurable function $u\colon\Omega\to\mathbb{R}^d$, the **Nemytskii function** $Nu(x)=f(x,u(x))$ is also measurable on Ω.

Lemma 3.10. *Assume $f(x,\xi)$ is* **Carathéodory** *and*

$$|f(x,\xi)|\le a(x)+b|\xi|^{p/q}\quad\forall\, a.e.\ x\in\Omega,\ \ \forall\,\xi\in\mathbb{R}^d, \tag{3.20}$$

where $1\le p,q<\infty$, $b\ge 0$ are constants, $a(x)\in L^q(\Omega)$ is nonnegative.
Then, the **Nemytskii operator** $N\colon L^p(\Omega;\mathbb{R}^d)\to L^q(\Omega)$ *is continuous.*

Proof. Let $u\in L^p(\Omega;\mathbb{R}^d)$. By (3.20), we have

$$|Nu(x)|^q=|f(x,u(x))|^q\le\text{const}\,(|a(x)|^q+|u(x)|^p).$$

Hence $Nu\in L^q(\Omega)$ and thus $N\colon L^p(\Omega;\mathbb{R}^d)\to L^q(\Omega)$ is well-defined. To show that N is continuous at u, let $u_n\to u$ in $L^p(\Omega;\mathbb{R}^d)$. Then there is a subsequence $\{u_{n'}\}$ and a function $v\in L^p(\Omega)$ such that $u_{n'}(x)\to u(x)$ a.e. and $|u_{n'}(x)|\le v(x)$ a.e. for all n'. Hence

$$\begin{aligned}\|Nu_{n'}-Nu\|^q_{L^q(\Omega)}&=\int_\Omega|f(x,u_{n'}(x))-f(x,u(x))|^q dx\\&\le\text{const}\int_\Omega(|f(x,u_{n'}(x))|^q+|f(x,u(x))|^q)dx\\&\le\text{const}\int_\Omega(|a(x)|^q+|v(x)|^p+|u(x)|^p)dx.\end{aligned}$$

By (ii), $f(x,u_{n'}(x))-f(x,u(x))\to 0$ as $n\to\infty$ for almost all $x\in\Omega$. The dominating convergence theorem implies that $\|Nu_{n'}-Nu\|_{L^q(\Omega)}\to 0$. By repeating this procedure for every subsequence of u_n, it follows that $\|Nu_n-Nu\|_{L^q(\Omega)}\to 0$ which proves that N is continuous at u. □

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Example: The p-Laplace Equations. We consider the BVP for **p-Laplace equations** with $p>1$:

$$\begin{cases}-\sum_{i=1}^n D_i(|Du|^{p-2}D_iu)+f(x,u)=0 & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases} \tag{3.21}$$

where $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the structural condition:

$$|f(x,s)|\le a(x)+b|s|^{p-1}\quad\forall\,x\in\Omega,\ s\in\mathbb{R}, \tag{3.22}$$

where $a\in L^{\frac{p}{p-1}}(\Omega)$ and $b\ge 0$ are given. Let

$$F(x,s)=\int_0^s f(x,t)\,dt.$$

We further assume a structural condition: for some $1\le r<p$

$$F(x,s)\ge -c_1|s|^r-c_2(x)\quad\forall\,x\in\Omega,\ s\in\mathbb{R}, \tag{3.23}$$

where $c_1\ge 0$ and $c_2(x)\in L^1(\Omega)$ are given. Note that (3.23) does not follow from (3.22).

Theorem 3.11. *Under the assumptions* (3.22) *and* (3.23), *the functional*

$$I(u)=\int_\Omega \Big(\frac{1}{p}|Du|^p+F(x,u)\Big)dx$$

has a minimizer on $X=W_0^{1,p}(\Omega)$ *and hence* (3.21) *has a weak solution.*

Proof. Note that (3.22) implies

$$|F(x,s)|\le c(x)+d|s|^p,\quad \forall\, x\in\Omega,\ s\in\mathbb{R}, \tag{3.24}$$

for some $c\in L^1(\Omega)$ and $d\ge 0$. Hence $I\colon X\to\mathbb{R}$ is well-defined. We verify that I is w.l.s.c and weakly coercive on X. Write

$$I(u)=I_1(u)+I_2(u)=\frac{1}{p}\int_\Omega |Du|^p\,dx+\int_\Omega F(x,u)\,dx.$$

Note that, by **Tonelli's Theorem**, $I_1(u)$ is w.l.s.c. on X. And since the embedding $X\subset L^p(\Omega)$ is always *compact*, by (3.24), I_2 is in fact continuous under the weak convergence. Hence I is w.l.s.c. on X. By (3.23), we have for all $u\in X=W_0^{1,p}(\Omega)$

$$I(u)\ge\frac{1}{p}\|Du\|^p_{L^p(\Omega)}-c_1\|u\|^r_{L^r(\Omega)}-C$$

$$\ge\frac{1}{p}\|Du\|^p_{L^p(\Omega)}-c\|u\|^r_{L^p(\Omega)}-C\ge\delta\|u\|^p_{W^{1,p}(\Omega)}-\gamma$$

for some constants $\delta>0$ and $\gamma\ge 0$, where the last inequality follows from **Poincaré's inequality** and **Young's inequality** with ε. Hence $I(u)$ is coercive on $X=W_0^{1,p}(\Omega)$. Thus the result follows from Theorem 3.4. □

Exercise 3.3. Let $n\ge 3$ and $2\le p<n$. Show the theorem is valid if (3.22) above is replaced by

$$|f(x,s)|\le a(x)+b\,|s|^q,$$

where $b\ge 0$ is a constant, $a\in L^{\frac{q+1}{q}}(\Omega)$ and $1\le q<p^*-1$.

3.2.7. Minimality and Uniqueness of Weak Solutions. We study the weak solutions of Euler-Lagrange equation for convex functionals.

Theorem 3.12. (Minimality of weak solutions) *Assume L satisfies the standard structural conditions above and is convex in (s,ξ). Let $u\in W^{1,p}(\Omega;\mathbb{R}^N)$ be a weak solution of the* Euler-Lagrange equation *of I. Then u is a minimizer of I in the Dirichlet class $\mathcal{D}_u$.*

Proof. By the convexity, it follows that

$$L(x,t,\eta)\ge L(x,s,\xi)+D_sL(x,s,\xi)\cdot(t-s)+D_\xi L(x,s,\xi)\cdot(\eta-\xi). \tag{3.25}$$

Assume $v\in\mathcal{D}_u$. Let $t=v(x),\eta=Dv(x),\ s=u(x),\xi=Du(x)$ and integrate over Ω to find

$$I(v)\ge I(u)+\int_\Omega[D_sL(x,u,Du)\cdot(v-u)+D_\xi L(x,u,Du)\cdot(Dv-Du)]dx.$$

Since u is a weak solution of the Euler-Lagrange equation of I and $v-u\in W_0^{1,p}(\Omega;\mathbb{R}^N)$, by (3.17), it follows that

$$\int_\Omega[D_sL(x,u,Du)\cdot(v-u)+D_\xi L(x,u,Du)\cdot(Dv-Du)]dx=0.$$

Hence $I(v)\ge I(u)$ for all $v\in\mathcal{D}_u$. This shows that u is a minimizer of I in the Dirichlet class $\mathcal{D}_u$. □

Under a stronger convexity condition, we can show that the weak solution in a Dirichlet class is **unique**.

Theorem 3.13 (Uniqueness of weak solutions)**.** *Assume, in addition to the standard structural conditions above, L satisfies, for some constant $\theta > 0$,*

$$L(x,t,\eta) \geq L(x,s,\xi) + D_sL(x,s,\xi)\cdot(t-s) + D_\xi L(x,s,\xi)\cdot(\eta-\xi) + \frac{\theta}{2}|\eta-\xi|^2. \tag{3.26}$$

Then a weak solution to Problem (3.13) *is unique.*

Proof. Let $u, v \in \mathcal{D}_\varphi$ be weak solutions to (3.13). Then $I(u) = I(v) = \min_{w\in\mathcal{D}_\varphi} I(w)$. However, as in the proof of previous theorem,

$$I(v) \geq I(u) + \frac{\theta}{2}\int_\Omega |Dv - Du|^2\,dx.$$

From $I(v) = I(u)$, we easily obtain $Du = Dv$ in Ω and hence $v \equiv u$ since $u - v \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. The proof is now completed. □

Lemma 3.14. *If $L = L(x,\xi)$ is independent of s and is C^2 in ξ, then condition* (3.26) *is equivalent to the* **strict convexity or Legendre condition***:*

$$L_{\xi_i^k\xi_j^l}(x,\xi)\,\eta_i^k\,\eta_j^l \geq \theta\,|\eta|^2 \quad \forall\,\xi,\,\eta \in \mathbb{M}^{N\times n}. \tag{3.27}$$

Proof. In this case, Condition (3.26) becomes

$$L(x,\eta) \geq L(x,\xi) + L_{\xi_i^k}(x,\xi)\,(\eta_i^k - \xi_i^k) + \frac{\theta}{2}\,|\eta-\xi|^2. \tag{3.28}$$

Let $\zeta = \eta - \xi$ and $f(t) = L(x,\xi+t\zeta)$. Then, by Taylor's formula,

$$f(1) = f(0) + f'(0) + \int_0^1 (1-t)\,f''(t)\,dt.$$

Note that

$$f'(t) = L_{\xi_i^k}(x,\xi+t\zeta)\,\zeta_i^k, \quad f''(t) = L_{\xi_i^k\xi_j^l}(x,\xi+t\zeta)\,\zeta_i^k\zeta_j^l.$$

From this and the Taylor formula, (3.28) is equivalent to (3.27). □

Lecture 35 – 4/10/19

3.3. Constrained Minimization Problems

3.3.1. Lagrange Multipliers.

Theorem 3.15. (Lagrange Theorem) *Let X be a Banach space. Let $f, g\colon X \to \mathbb{R}$ be C^1 and $g(u_0) = c$. Assume u_0 is a local extremum of f with respect to the constraint $g(u) = c$. Then, either $g'(u_0) = 0$ or there exists $\lambda \in \mathbb{R}$ such that $f'(u_0) = \lambda g'(u_0)$; that is, u_0 is a critical point of $f - \lambda g$.*

Proof. Assume $g'(u_0) \neq 0$; then $g'(u_0)w \neq 0$ for some $w \in X$. Now given any $v \in X$, consider the real-valued functions

$$F(s,t) = f(u_0+sv+tw), \;\; G(s,t) = g(u_0+sv+tw) - c \quad \forall\,(s,t)\in\mathbb{R}^2.$$

Then $F, G \in C^1$ on $\mathbb{R}^2$ and

$$F_s(0,0) = f'(u_0)v, \;\; F_t(0,0) = f'(u_0)w, \quad G_s(0,0) = g'(u_0)v, \;\; G_t(0,0) = g'(u_0)w.$$

Since $G(0,0)=0$ and $G_t(0,0)=g'(u_0)w\neq 0$, the **Implicit function theorem** implies the existence of a C^1 function $t=\phi(s)$ on an open inerval J containing 0 such that

$$\phi(0)=0,\quad G(s,\phi(s))=0\ \ \forall\, s\in J;$$

moreover, $\phi'(0)=-\frac{G_s(0,0)}{G_t(0,0)}=-\frac{g'(u_0)v}{g'(u_0)w}$. Set $z(s)=F(s,\phi(s))=f(u_0+sv+\phi(s)w)$ for $s\in J$. Note that $g(u_0+sv+\phi(s)w)=c$ for all $s\in J$. Since f has a local extremum at u_0, $z(s)$ has a local extremum at $s=0$ and thus

$$0=z'(0)=F_s(0,0)+F_t(0,0)\phi'(0)=f'(u_0)v-\frac{f'(u_0)w}{g'(u_0)w}g'(u_0)v.$$

Hence $f'(u_0)v=\frac{f'(u_0)w}{g'(u_0)w}g'(u_0)v$ for all $v\in X$, and thus the theorem is proved with $\lambda=f'(u_0)w/g'(u_0)w$. □

3.3.2. Nonlinear Eigenvalue Problems.

Theorem 3.16. *Let $1\le\tau<\frac{n+2}{n-2}$ and $k(x),l(x)\in C(\bar\Omega)$ with $l(x)\ge l_0>0$ on $\bar\Omega$. Then, for each $R\in(0,\infty)$, there exists a number $\lambda=\lambda_R$ such that the problem*

$$\begin{cases}\Delta u+k(x)u+\lambda l(x)|u|^{\tau-1}u=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{3.29}$$

has a weak solution $u=u_R\in H_0^1(\Omega)$ satisfying $\frac{1}{\tau+1}\int_\Omega l(x)|u(x)|^{\tau+1}dx=R$.

Proof. 1. Define the functionals

$$f(u)=\frac12\int_\Omega(|Du|^2-k(x)u^2)\,dx,\quad g(u)=\frac{1}{\tau+1}\int_\Omega l(x)|u|^{\tau+1}\,dx.$$

Then both f,g are C^1 on $H_0^1(\Omega)$ with

$$f'(u)v=\int_\Omega(Du\cdot Dv-k(x)uv)dx,\quad g'(u)v=\int_\Omega l(x)|u|^{\tau-1}uv\,dx$$

for all $u,v\in H_0^1(\Omega)$. And, a weak solution to problem (3.29) is exactly a function $u\in H_0^1(\Omega)$ satisfying

$$f'(u)v=\lambda g'(u)v\quad(v\in H_0^1(\Omega)).$$

By the **Lagrange Theorem** above, we minimize $f(u)$ with constraint $g(u)=R$ for any given $R>0$. Let

$$\mathcal{C}_R=\{u\in H_0^1(\Omega):\ g(u)=R\}.$$

Then $\mathcal{C}_R$ is nonempty because, given any $w\neq 0$ in $H_0^1(\Omega)$, we have $tw\in\mathcal{C}_R$ for some $t>0$.

2. We show that there exists a $u_0\in\mathcal{C}_R$ such that $f(u_0)=\min_{u\in\mathcal{C}_R}f(u)$. We prove this by the **direct method**. Take a minimizing sequence $u_j\in\mathcal{C}_R$, so that

$$\lim_{j\to\infty}f(u_j)=\inf_{u\in\mathcal{C}_R}f(u)<\infty.$$

By Hölder's inequality we have

$$\int_\Omega|u|^2dx\le\left(\int_\Omega|u|^{\tau+1}dx\right)^{\frac{2}{\tau+1}}|\Omega|^{\frac{\tau-1}{\tau+1}}\le C(g(u))^{\frac{2}{\tau+1}}.$$

Thus $f(u)\ge\frac12\|Du\|^2_{L^2(\Omega)}-C'(g(u))^{\frac{2}{\tau+1}}$. From this, by Poincaré's inequality and $g(u_j)=R$, we have

$$\|u_j\|^2_{H_0^1(\Omega)}\le C''(f(u_j)+R^{\frac{2}{\tau+1}}).$$

Hence $\{u_j\}$ is bounded in $H_0^1(\Omega)$. By reflexivity and compact embedding, we assume via a subsequence (denoted by u_j again) $u_j \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_j \to u_0$ in $L^2(\Omega)$ and in $L^{\tau+1}(\Omega)$ (since $\tau + 1 < \frac{2n}{n-2} = 2^*$ by the condition $\tau < \frac{n+2}{n-2}$), where $u_0 \in H_0^1(\Omega)$ is a function. From the strong convergence we have $g(u_0) = R$ and hence $u_0 \in \mathcal{C}_R$. Moreover, since

$$\lim_{j\to\infty} \int_\Omega k(x)|u_j|^2\,dx = \int_\Omega k(x)|u_0|^2\,dx.$$

and so, by the weak lower semicontinuity of norm,

$$f(u_0) \le \lim_{j\to\infty} f(u_j) = \inf_{u\in\mathcal{C}_R} f(u).$$

This proves $u_R := u_0 \in \mathcal{C}_R$ is a minimizer of f. Since $g'(u_0)u_0 = (\tau+1)g(u_0) \neq 0$, we have $u_0 \neq 0$, $g'(u_0) \neq 0$. Hence by Theorem 3.15, there exists a number $\lambda = \lambda_R$ such that $f'(u_0) = \lambda g'(u_0)$. Thus $u_R = u_0 \in H_0^1(\Omega)$ is a weak solution to (3.29) with $\lambda = \lambda_R$. Moreover, if $\beta_R = f(u_R) = \min_{u\in\mathcal{C}_R} f(u)$, then $\lambda_R = \frac{2\beta_R}{(\tau+1)R}$. (**Exercise!**) □

Remark 3.4. The problem (3.29) is called a **nonlinear eigenvalue problem**; any nonzero weak solution u of (3.29) is called an **eigenfunction** corresponding to the **eigenvalue** λ.

Corollary 3.17. *For each $1 < \tau < (n+2)/(n-2)$, there exists a nontrivial weak solution of*

$$\Delta u + |u|^{\tau-1}u = 0 \ \text{ in } \ \Omega, \ \ u|_{\partial\Omega} = 0. \tag{3.30}$$

Proof. By Theorem 3.16, with $k(x) = 0$, $l(x) = 1$ and $R = 1$, there exist a function $u_1 \neq 0$ in $H_0^1(\Omega)$ and a number $\lambda_1 \in \mathbb{R}$ such that

$$\Delta u_1 + \lambda_1 |u_1|^{\tau-1}u_1 = 0 \ \text{ in } \ \Omega, \ \ u_1|_{\partial\Omega} = 0, \ \ \int_\Omega |u_1|^{\tau+1}dx = (\tau+1).$$

Thus, testing with $u_1 \in H_0^1(\Omega)$ gives

$$\int_\Omega |Du_1|^2 dx = \lambda_1 \int_\Omega |u_1|^{\tau+1}dx = \lambda_1(\tau+1),$$

and hence $\lambda_1 > 0$. Set $u_1 = ku$ with $k > 0$ to be determined. Then $k\Delta u + \lambda_1 k^\tau |u|^{\tau-1}u = 0$, and so if we choose k to satisfy $\lambda_1 k^{\tau-1} = 1$, then $u = u_1/k$ is a nontrivial weak solution of (3.30). □

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Remark 3.5. (i) Let $n \ge 3$ and Ω be **star-shaped**. Then problem (3.30) has no nontrivial smooth solutions if $\tau > \frac{n+2}{n-2}$ and has no positive smooth solutions in Ω if $\tau = \frac{n+2}{n-2}$.

(ii) However, for certain non star-shaped domains, such as an annulus, (3.30) always has nontrivial solutions for all $\tau > 1$. For example, let $b > a > 0$, $\Omega = \{x \in \mathbb{R}^n \mid a < |x| < b\}$ and $\tau > 1$. Then a nontrivial weak solution u of (3.30) in the radial form $u(x) = v(|x|)$ can be obtained by minimizing

$$f(v) = \int_a^b (v')^2 r^{n-1} dr$$

on the set $C = \left\{ v \in H_0^1(a,b) \,\middle|\, \int_a^b |v|^{\tau+1} r^{n-1} dr = 1 \right\}$. **(Exercise.)**

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3.3.3. Obstacle Problems. We study minimization with certain pointwise, **unilateral constraints**. Many important applied problems can be formulated as such a problem. To discuss the main ideas, we only consider a simple example. Let

$$I(u) = \int_\Omega \Big(\frac{1}{2}|Du|^2 - f(x)u\Big)\,dx$$

and

$$\mathcal{A} = \{u \in \varphi + H_0^1(\Omega) \,|\, u(x) \geq h(x) \ \ a.e.\ x \in \Omega\},$$

where $f \in L^2(\Omega)$, $\varphi \in H^1(\Omega)$, and $h\colon \Omega \to \mathbb{R}$ is a given function called the **obstacle**.

Theorem 3.18. (Existence of minimizers with obstacle) *Assume the admissible class $\mathcal{A}$ is nonempty. Then there exists a unique function $u \in \mathcal{A}$ satisfying*

$$I(u) = \min_{w\in\mathcal{A}} I(w).$$

Proof. 1. The existence of minimizer follows easily from the direct method, considering the fact that I is coercive and $\mathcal{A}$ is weakly closed in $H^1(\Omega)$.

2. The uniqueness of minimizer follows essentially from the strict convexity of I and the convexity of the set $\mathcal{A}$. For instance, notice that

$$I\Big(\frac{u+v}{2}\Big) = \frac{I(u)+I(v)}{2} - \frac{1}{8}\int_\Omega |Du - Dv|^2\,dx \quad (u, v \in H^1(\Omega)).$$

Hence any two minimizers u, v in $\mathcal{A}$ will satisfy, since $\frac{u+v}{2} \in \mathcal{A}$,

$$\frac{1}{8}\int_\Omega |Du - Dv|^2\,dx = \frac{I(u)+I(v)}{2} - I\Big(\frac{u+v}{2}\Big) \leq 0;$$

so $u = v$ in $\varphi + H_0^1(\Omega)$. □

Theorem 3.19. (Variational inequality for minimizers) *$u \in \mathcal{A}$ is a minimizer of I over $\mathcal{A}$ if and only if the* **variational inequality** *holds:*

$$\int_\Omega Du\cdot D(v-u)\,dx \geq \int_\Omega (v-u)f\,dx \quad \forall\, v \in \mathcal{A}. \tag{3.31}$$

Proof. Given $u, v \in \mathcal{A}$ and $0 \leq \tau \leq 1$, let

$$h(\tau) = I((1-\tau)u + \tau v) = I(u + \tau(v-u)).$$

Then $h\colon [0,1] \to \mathbb{R}$ is convex. Since $(1-\tau)u + \tau v \in \mathcal{A}$, it follows that u is a minimizer of I over $\mathcal{A}$ if and only if $h(0) \leq h(\tau)$ for all $\tau \in [0,1]$; that is, $h(0)$ is the minimum of h on $[0,1]$. Since h is convex on $[0,1]$, it follows that $h(0)$ is the minimum of h on $[0,1]$ if and only if $h'(0^+) \geq 0$; however,

$$h'(0^+) = \lim_{\tau\to 0^+}\frac{h(\tau)-h(0)}{\tau} = \int_\Omega (Du\cdot D(v-u) - (v-u)f)dx.$$

This proves the result. □

The Free Boundary Problem. We now assume f, φ, h and $\partial\Omega$ are all smooth. Then a regularity result (not proved here) asserts that the minimizer $u \in W^{2,\infty}(\Omega)$.

Theorem 3.20. *Let $U = \{x \in \Omega \,|\, u(x) > h(x)\}$. Then the unique minimizer u determined above is in $W^{2,\infty}(\Omega)$ and satisfies $u \in C^\infty(U)$ and solves the following* **free boundary problem***:*

$$\begin{cases} u = h, \quad -\Delta u \geq f & \text{a.e. on } \Omega \setminus U, \\ -\Delta u = f & \text{in } U, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

The set $F = \Omega \cap \partial U$ is called the **free boundary** *of the free boundary problem.*

Proof. We first claim that in fact $u \in C^\infty(U)$ and solves Poisson's equation $-\Delta u = f$ in U. To see this, fix any test function $w \in C_0^\infty(U)$. Then if $|\tau|$ is sufficiently small, $w = u + \tau w \geq h$ in Ω and hence $v \in \mathcal{A}$. Then by (3.31) we have

$$\tau \int_\Omega (Du \cdot Dw - wf)\, dx \geq 0.$$

This is valid for both sufficiently small positive and negative τ, and so we have

$$\int_U (Du \cdot Dw - wf)\, dx = 0 \quad (w \in C_0^\infty(U)).$$

This proves that u is a weak solution to equation $-\Delta u = f$ in U; thus, by regularity, $u \in C^\infty(U)$.

If we assume $w \in C_0^\infty(\Omega)$ satisfies $w \geq 0$ and if $\tau \in (0, 1]$, then with $v = u + \tau w \in \mathcal{A}$ as test function in (3.31) we have $\int_\Omega (Du \cdot Dw - wf)\, dx \geq 0$. But since $u \in W^{2,\infty}(\Omega)$, we deduce that

$$\int_\Omega (-\Delta u - f) w dx \geq 0$$

for all $w \in C_0^\infty(U)$ and $w \geq 0$. This implies $-\Delta u \geq f$ a.e. in Ω. Therefore, u solves the free boundary problem (3.32).

Note that as part of the problem *the free boundary $F = \Omega \cap \partial U$ is unknown.* □

3.3.4. Harmonic Maps. We now consider the Dirichlet energy

$$I(\mathbf{u}) = \frac{1}{2} \int_\Omega |D\mathbf{u}|^2\, dx$$

for vector $\mathbf{u} \in H^1(\Omega; \mathbb{R}^N)$ with point-wise constraint $|\mathbf{u}(x)| = 1$ for almost every $x \in \Omega$.

Let $\mathcal{C} = \{\mathbf{u} \in \mathcal{D}_\varphi \,|\, |\mathbf{u}(x)| = 1 \text{ a.e. in } \Omega\}$, where $\mathcal{D}_\varphi$ is a Dirichlet class, and assume $\mathcal{C}$ is non-empty. Then $\mathcal{C}$ is weakly closed in $H^1(\Omega; \mathbb{R}^N)$.

We have the following result.

Theorem 3.21. *There exists $\mathbf{u} \in \mathcal{C}$ satisfying $I(\mathbf{u}) = \min_{\mathbf{v} \in \mathcal{C}} I(\mathbf{v})$. Moreover, $\mathbf{u}$ is a weak solution to the* **harmonic map equation**

$$-\Delta \mathbf{u} = |D\mathbf{u}|^2 \mathbf{u} \quad \text{in } \Omega$$

in the sense that $|\mathbf{u}(x)| = 1$ a.e. in Ω and

$$\int_\Omega D\mathbf{u} : D\mathbf{v}\, dx = \int_\Omega |D\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v}\, dx \tag{3.33}$$

for each $\mathbf{v} \in H_0^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. *Any weak solution of the harmonic map equation is called a* **harmonic map** *from* Ω *into* $\mathbb{S}^{N-1}$.

Remark 3.6. In this case, the Lagrange multiplier corresponding to the constraint $|\mathbf{u}(x)| = 1$ appears as a function $\lambda = |D\mathbf{u}|^2$.

Proof. 1. The existence of minimizers follows by the direct method as above. Given any $\mathbf{v} \in H_0^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$, let ϵ be such that $|\epsilon|\|\mathbf{v}\|_{L^\infty(\Omega)} \le \frac{1}{2}$. Define

$$\mathbf{w}_\epsilon(x) = \frac{\mathbf{u}(x) + \epsilon\mathbf{v}(x)}{|\mathbf{u}(x) + \epsilon\mathbf{v}(x)|}, \quad h(\epsilon) = I(\mathbf{w}_\epsilon).$$

Note that $\mathbf{w}_\epsilon \in \mathcal{C}$ and $h(0) = I(\mathbf{u}) = \min_{\mathcal{C}} I \le h(\epsilon)$ for sufficiently small ϵ; hence, $h'(0) = 0$.

2. Note that

$$h'(0) = \int_\Omega D\mathbf{u} : D\frac{\partial \mathbf{w}_\epsilon}{\partial \epsilon}|_{\epsilon=0}\, dx.$$

Computing directly we have

$$\frac{\partial \mathbf{w}_\epsilon}{\partial \epsilon} = \frac{\mathbf{v}}{|\mathbf{u}+\epsilon\mathbf{v}|} - \frac{[(\mathbf{u}+\epsilon\mathbf{v})\cdot\mathbf{v}](\mathbf{u}+\epsilon\mathbf{v})}{|\mathbf{u}+\epsilon\mathbf{v}|^3},$$

hence $\frac{\partial \mathbf{w}_\epsilon}{\partial \epsilon}|_{\epsilon=0} = \mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u}$. Inserting this into $h'(0) = 0$, we find

$$\int_\Omega \Big(D\mathbf{u} : D\mathbf{v} - D\mathbf{u} : D((\mathbf{u}\cdot\mathbf{v})\mathbf{u}) \Big)\, dx = 0. \tag{3.34}$$

However, using $|\mathbf{u}|^2 = 1$ we have $(D\mathbf{u})^T\mathbf{u} = 0$; namely, $D_i u^k u^k = 0$ for all i. Hence, using indices, we have the identity

$$D\mathbf{u} : D((\mathbf{u}\cdot\mathbf{v})\mathbf{u}) = |D\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{v}) \quad \text{a.e. in } \Omega.$$

This identity combined with (3.34) proves (3.33). □