ON p-QUASICONVEX HULLS OF MATRIX SETS

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Abstract. We present some basic properties and equivalent definitions of the p-quasiconvex hull of a given set of matrices. In particular, we completely characterize the p-quasiconvex hull in terms of the $W^{1,p}$-gradient Young measures studied by [9] and establish an important relationship with the weak convergence in Sobolev spaces. We also give some simple characterization of the p-quasiconvex hulls for certain special sets.

1. Introduction

The p-quasiconvex hull problem, motivated by the notions of convex hull and quasiconvex hull in convex analysis and the calculus of variations, is concerned with determining an appropriate structure for gradients of weak limits of sequences in the Sobolev space $W^{1,p}(\Omega;\mathbf{R}^n)$ whose gradients approach a given set K of $n\times m$ matrices in the certain sense. Such a problem has been completely studied when the given set K is a compact set; in this case, the structure of limit gradients is completely determined by the so-called *quasiconvex hull* of K and does not depend on the power p. However, for unbounded sets K, such a structure in general depends on p and is determined by the p-quasiconvex hull of K to be discussed in this paper.

Throughout this paper, we assume K is a closed subset of $M^{n\times m}$, where $M^{n\times m}$ is the set of all real $n\times m$ matrices with standard Euclidean space structure, and assume $\Omega\subset\mathbf{R}^m$ is a bounded open set with $|\partial\Omega|=0$. We say that a sequence $\{u_j\}$ in $W^{1,p}(\Omega;\mathbf{R}^n)$ is an *approximating sequence* for a closed set $K\subset M^{n\times m}$ provided

$$\lim_{j\to\infty}\int_\Omega d_K(Du_j(x))\,dx=0,$$

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where, for any $u \in W^{1,p}(\Omega;\mathbf{R}^n)$, $Du(x)$ denotes the gradient matrix defined by $(Du)_{ij} = \partial u^i/\partial x_j$ $(1 \le i \le n,\ 1 \le j \le m)$, and $d_K(\xi) = \operatorname{dist}(\xi;K)$ denotes the distance from to ξ to K. Suppose an approximating sequence $\{u_j\}$ is itself weakly (weakly * if $p=\infty$) convergent in $W^{1,p}(\Omega;\mathbf{R}^n)$ to a limit $\bar u$. We would like to study the value distribution of $D\bar u(x)$. For $p=\infty$, this problem is closely related to the *quasiconvex hull* of set K, generalizing the convex hull of K, studied in the recent literature on vectorial calculus of variations [6, 11, 14]. With the similar motivation, the notion of the *p-quasiconvex hull* of K has been introduced [13, 16, 17, 18, 19, 20]; a different notion of p-quasiconvex hulls has been introduced by Zhang [22]. The purpose of this paper is to give some properties and equivalent defintions of p-quasiconvex hulls.

2. Morrey's quasiconvexity and the p-quasiconvex hulls

Let us first recall the notion of *quasiconvexity* introduced by C. B. Morrey [10]. We restrict ourselves only to finite real valued functions. Let $f\colon M^{n\times m} \to \mathbf{R}$ be a given function. According to Morrey [10], we say f is *quasiconvex* on $M^{n\times m}$ if for all $\xi \in M^{n\times m}$

$$f(\xi) \le \frac{1}{|\Omega|}\int_\Omega f(\xi + D\varphi(x))\,dx \quad \forall\, \varphi \in W_0^{1,\infty}(\Omega;\mathbf{R}^n), \tag{2.1}$$

where $W_0^{1,p}(\Omega;\mathbf{R}^n)$ denotes the set of all Lipschitz continuous functions from $\bar\Omega$ to $\mathbf{R}^n$ and vanishing on $\partial\Omega$. Given any $f\colon M^{n\times m} \to \mathbf{R}$, the *quasiconvex envelope* of f, denoted by f^{qc}, is defined to be the largest quasiconvex function below f, which is in fact given by

$$f^{qc}(\xi) = \inf_{\varphi \in W_0^{1,\infty}(\Omega;\mathbf{R}^n)} \frac{1}{|\Omega|}\int_\Omega f(\xi + D\varphi(x))\,dx.$$

It has been well-known that the quasiconvexity of f or f^{qc} is independent of domain Ω. Also, the quasiconvexity of f is equivalent to the lower semicontinuity of functional $I(u) = \int_\Omega f(Du(x))dx$ under the Lipschitz convergence of mappings u; see also [1, 2, 6, 10]. In view of Jensen's inequality, it is also easily seen that every convex function is quasiconvex; however, easy examples show that, if $n,\ m \ge 2$, a quasiconvex function may not be a convex function on $M^{n\times m}$. It is also well-known that any (finite valued) quasiconvex function f is locally Lipschitz continuous (see, e.g., [3]).

In the following, we assume $1 \le p < \infty$ and denote by $\mathcal{C}_p$ the linear space of continuous functions f on $M^{n\times m}$ that satisfy the growth condition

$$|f(\xi)| \le C(|\xi|^p + 1), \quad \forall\, \xi \in M^{n\times m}, \tag{2.2}$$

where $C > 0$ is a constant depending on f. Denote by $\mathcal{Q}_p$ the family of quasiconvex functions in $\mathcal{C}_p$.

Definition 2.1. We define the p-quasiconvex hull of K, denoted by $Q_p(K)$, by

$$Q_p(K) = \left\{ \xi \in M^{n\times m} \,\middle|\, f(\xi) \le \sup_K f,\ \forall\, f \in \mathcal{Q}_p \right\}. \tag{2.3}$$

We say K is *p-quasiconvex* if $Q_p(K) = K$.

***Remark* 2.1.** (a) The usual quasiconvex hull of K, denoted by K^{qc}, is defined by

$$K^{qc} = \left\{ \xi \in M^{n\times m} \,\middle|\, f(\xi) \le \sup_K f,\ \forall\, f \text{ quasiconvex} \right\}. \tag{2.4}$$

It is easy to see that $K^{qc} \subseteq Q_p(K) \subseteq Q_r(K)$ for all $r \le p$.

(b) Our definition of $Q_p(K)$ is different from that given in Zhang [22] and, in some case, our hulls are strictly smaller than those defined in [22]. However, the two definitions agree when K is either compact or L^p-coercive; see Theorems 4.3 and 5.1.

We shall prove the following main result that establishes an important relationship between the p-quasiconvex hull and weak convergence.

Theorem 2.1. *Let $1 \le p < \infty$. Suppose $\{u_j\}$ is an approximating sequence of K in $W^{1,p}(\Omega; \mathbf{R}^n)$. If u_j converges weakly to u in $W^{1,p}(\Omega; \mathbf{R}^n)$, then $Du(x) \in Q_p(K)$ for almost every $x \in \Omega$.*

In the following we discuss some equivalent definitions of p-quasiconvex hulls.

Proposition 2.2. *Let $\mathcal{Q}_p^+(K) = \{f \in \mathcal{Q}_p \,|\, 0 \le f(\xi) \le |\xi|^p + 1,\ f|_K = 0\}$. Then*

$$Q_p(K) = \bigcap \{Z(g) \,|\, g \in \mathcal{Q}_p^+(K)\},$$

where $Z(g) = g^{-1}(0) = \{\eta \in M^{n\times m} \,|\, g(\eta) = 0\}$ denotes the zero set of g.

Proof. Let $A = \cap\{Z(g) \,|\, g \in \mathcal{Q}_p^+(K)\}$. We show $A = Q_p(K)$. Note that $\eta \notin A$ if and only if there exists $g \in \mathcal{Q}_p^+(K) \subset \mathcal{Q}_p$ such that $g(\eta) > 0 = \sup_K g$. Hence it is easy that $Q_p(K) \subseteq A$. On the other hand, if $\eta \notin Q_p(K)$ then there exists $f \in \mathcal{Q}_p$ such that $f(\eta) > \sup_K f := \sigma$. Let $h = \max\{0, f - \sigma\}$. Then h is quasiconvex and satisfies $h|_K = 0$ and $0 \le h(\xi) \le C(|\xi|^p + 1) + |\sigma|$. Let $g = (C + |\sigma|)^{-1} h$. Then $g \in \mathcal{Q}_p^+(K)$ and $g(\eta) > 0 = \sup_K g$. Hence $\eta \notin A$, showing $A \subseteq Q_p(K)$. This completes the proof. □

The following result shows that we can define $Q_p(K)$ as the zero set of the largest quasiconvex function in $\mathcal{Q}_p^+(K)$.

Theorem 2.3. *Let $f = \sup\{g |\, g \in \mathcal{Q}_p^+(K)\}$. Then $f \in \mathcal{Q}_p^+(K)$ and $Z(f) = Q_p(K)$.*

We next characterize the p-quasiconvex hull in terms of the $W^{1,p}$-*gradient Young measures* studied by Kinderlehrer and Pedregal [9]. Following [9], we consider a slightly different subspace $\mathcal{E}_p$ of $\mathcal{C}_p$ defined by

$$\mathcal{E}_p = \left\{ f \in \mathcal{C}_p \;\middle|\; \lim_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|^p + 1} = 0 \right\}. \tag{2.5}$$

For each $f \in \mathcal{E}_p$ define

$$\|f\|_p = \sup_{\xi \in M^{n\times m}} \frac{|f(\xi)|}{|\xi|^p + 1}.$$

It can be shown that $\|\cdot\|_p$ is indeed a norm on $\mathcal{E}_p$ and $\mathcal{E}_p$ becomes a Banach space with the norm. We denote the (normed) dual space of $\mathcal{E}_p$ by $\mathcal{E}_p^*$. Since $\mathcal{E}_0 \subset \mathcal{E}_p$, we have, as sets, $\mathcal{E}_p^* \subset \mathcal{E}_0^*$ via restrictions. The dual space $\mathcal{E}_0^*$ is identical with the space of Radon measures on $M^{n\times m}$ and the pairing of $\mu \in \mathcal{E}_0^*, f \in \mathcal{E}_0$ is written as

$$\mu(f) = \langle \mu, f \rangle = \int_{M^{n\times m}} f(\xi) d\mu(\xi),$$

which can be extended to arbitrary $f \in C(M^{n\times m})$; for $f(\xi) = \xi$ we denote $\bar{\mu} = \int_{M^{n\times m}} \xi d\mu(\xi)$. Then, as a set, $\mathcal{E}_p^*$ is identical with the set of Radon measures μ on $M^{n\times m}$ that satisfy $|\langle \mu, 1\rangle| + |\langle \mu, |\xi|^p \rangle| < \infty$.

Let $\mathcal{M}_p^{qc}$ be the set of probability measures μ in $\mathcal{E}_p^*$ (i.e., $\mu \in \mathcal{E}_p^*, \mu \ge 0$ and $\langle \mu, 1 \rangle = 1$) that satisfy

$$f\left(\int_{M^{n\times m}} \xi d\mu(\xi) \right) \le \int_{M^{n\times m}} f(\xi) d\mu(\xi), \quad \forall\, f \in \mathcal{Q}_p. \tag{2.6}$$

The following important result has been proved by Kinderlehrer and Pedragal [9].

Theorem 2.4. *$\mu \in \mathcal{M}_p^{qc}$ if and only if there exists a sequence $\{u_j\}$ in $W^{1,p}(\Omega; \mathbf{R}^n)$ such that $\{|Du_j|^p\}$ is weakly convergent in $L^1(\Omega)$ and, for all $f \in \mathcal{C}_p$, $f(Du_j) \rightharpoonup \bar{f}$ weakly in $L^1(\Omega)$, where $\bar{f}$ is constant defined by $\bar{f} = \int_{M^{n\times m}} f(\xi) d\mu(\xi)$.*

Define

$$\mathcal{M}_p^{qc}(K) = \{\mu \in \mathcal{M}_p^{qc} \mid \operatorname{supp} \mu \subseteq K\}.$$

Another main result of this paper is to establish the following result.

Theorem 2.5. $Q_p(K) = \{\bar{\mu} \mid \mu \in \mathcal{M}_p^{qc}(K)\}$.

3. Proof of main results

3.1. **Proof of Theorem 2.1.** Before we proceed with the proof of this theorem, we recall the well-known Dunford-Pettis compactness theorem (see e.g. [7, page 27-II]).

Lemma 3.1. *Let $D \subset \mathbf{R}^m$ be a measurable set, and let $\{v_j\}$ converge weakly in $L^1(D)$. Then $\{v_j\}$ is bounded in $L^1(D)$ and there exists a continuous function $h \geq 0$ on $\mathbf{R}$, which can also be chosen to be convex, such that $\sup_j \int_D h(|v_j|) < \infty$, $h(t)/t \to \infty$ as $t \to \infty$.*

We also need the so-called Chacon's biting convergence lemma (see e.g. [4]).

Lemma 3.2. *Let G be a bounded domain in $\mathbf{R}^m$ and let $\{v_j\}$ be a sequence bounded in $L^1(G)$. Then there are a subsequence $\{v_{j_k}\}$ and a sequence of measurable sets $E_\alpha \subset \Omega$ with $|E_\alpha| \to 0$ as $\alpha \to \infty$ such that $\{v_{j_k}\}$ converges weakly in $L^1(G \setminus E_\alpha)$ for each $\alpha = 1, 2, \cdots$.*

Proof of Theorem 2.1. Let $1 \leq p < \infty$. In order to show $Du(x) \in Q_p(K)$ for almost every $x \in \Omega$, by the definition of $Q_p(K)$ and since the set $\mathcal{Q}_p^+(K)$ is separable, we have only to show that $g(Du(x)) = 0$ a.e. in Ω for all g in $\mathcal{Q}_p^+(K)$. Since the sequence $\{|Du_j|^p\}$ is bounded in $L^1(\Omega)$, by Lemma 3.2, we have a subsequence $\{u_{j_k}\}$ and a sequence of sets $E_\alpha \subset \Omega$ with $|E_\alpha| \to 0$ such that $\{|Du_{j_k}|^p\}$ converges weakly in $L^1(\Omega \setminus E_\alpha)$ for each $\alpha = 1, 2, \cdots$. By Lemma 3.1, we then have, for each α, a continuous function $h_\alpha(t) \geq 0$ such that

$$\lim_{t\to\infty} h_\alpha(t)/t = \infty, \qquad \sup_{k=1,2,\dots} \int_{\Omega\setminus E_\alpha} h_\alpha(|Du_{j_k}(x)|^p)\, dx < \infty. \tag{3.1}$$

Let $g \in \mathcal{Q}_p^+(K)$ be given. Since $g|_K = 0$, $0 \le g(\xi) \le |\xi|^p + 1$, one can easily show that, for each $\epsilon > 0$, there exists a constant $C_{\epsilon,\alpha} > 0$ depending on g such that

$$0 \le g(\xi) \le \epsilon\,(1 + h_\alpha(|\xi|^p)) + C_{\epsilon,\alpha} d_K(\xi), \quad \forall\, \xi \in M^{n\times m}. \tag{3.2}$$

By a lower semicontinuity theorem (cf. [1, Thm. II.4]), the functional $G_\alpha(u) = \int_{\Omega\setminus E_\alpha} g(Du)$ is (sequentially) weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbf{R}^n)$. Hence

$$\int_{\Omega\setminus E_\alpha} g(Du(x))\,dx \le \liminf_{k\to\infty} \int_{\Omega\setminus E_\alpha} g(Du_{j_k}(x))\,dx$$

and thus, by (3.2) and $\int_\Omega d_K(Du_j) \to 0$, we have

$$\int_{\Omega\setminus E_\alpha} g(Du) \le \liminf_{k\to\infty} \int_{\Omega\setminus E_\alpha} g(Du_{j_k}) \le \epsilon\, \liminf_{k\to\infty} \int_{\Omega\setminus E_\alpha} (1 + h_\alpha(|Du_{j_k}|^p))\,.$$

Letting $\epsilon \to 0^+$, we obtain $g(Du(x)) = 0$ for almost every $x \in \Omega \setminus E_\alpha$. This implies $g(Du(x)) = 0$ for almost every $x \in \Omega$ since $|E_\alpha| \to 0$. The proof of Theorem 2.1 is thus completed. □

We give an application of Theorem 2.1 in connection to the results on lower semicontinuity of L^∞ functionals obtained in [5].

Theorem 3.3. *Let $f\colon M^{n\times m} \to \mathbf{R}$ be such that each level set*

$$K_t \equiv \{\xi \mid f(\xi) \le t\}$$

is p-quasiconvex for any $t \in \mathbf{R}$. Then, for any open set Ω, the functional

$$F(u;\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} f(Du(x)) \tag{3.3}$$

is sequentially weakly, or weakly if $p = \infty$, lower semi-continuous on $W^{1,p}(\Omega; \mathbf{R}^n)$.*

Proof. Let $\{u_j\}$ converge weakly (or weakly* if $p = \infty$) to u in $W^{1,p}(\Omega; \mathbf{R}^n)$. Let $c = \liminf_j F(u_j; \Omega)$. If $c = \infty$, there is nothing to prove. We thus assume $c < \infty$. Then, for any $\epsilon > 0$, there is a subsequence $j_k \to \infty$ such that $f(Du_{j_k}(x)) \le c + \epsilon$ for a.e. $x \in \Omega$. This implies $Du_{j_k}(x) \in K_{c+\epsilon}$ and hence $\{u_{j_k}\}$ is an approximating sequence of set $K_{c+\epsilon}$. Therefore, by Theorem 2.1 above, we have $Du(x) \in K_{c+\epsilon}$ a.e. in Ω. Therefore, $f(Du(x)) \le c + \epsilon$ for a.e. $x \in \Omega$ and hence $F(u;\Omega) \le c + \epsilon$ for all $\epsilon > 0$. This proves $F(u;\Omega) \le \liminf_j F(u_j;\Omega)$ and thus the theorem follows. □

***Remark* 3.1.** For $p = \infty$, it has been recently proved in [5] that the sequential weak* lower semi-continuity of the functional $F(u;\Omega)$ defined by (3.3) on $W^{1,\infty}(\Omega;\mathbf{R}^n)$ for any open set Ω is equivalent to the condition, called the *strong Morrey quasiconvexity* in [5], that for any $\epsilon > 0$, $K > 0$ there exists a $\delta > 0$ such that the inequality

$$f(A) \leq \operatorname*{ess\,sup}_{x\in\Omega} f(A + D\phi(x)) + \epsilon$$

holds for all $\phi \in W^{1,\infty}(\Omega;\mathbf{R}^n)$ satisfying $\|D\phi\|_{L^\infty(\Omega)} \leq K$, $\max_{x\in\partial\Omega} |\phi(x)| \leq \delta$.

3.2. **Proof Theorem 2.3.** Let $f = \sup\{g |\, g \in \mathcal{Q}_p^+(K)\}$. Then $0 \leq f(\xi) \leq |\xi|^p + 1$ and $f|_K = 0$. From definitions, f is finite valued and quasiconvex, and hence continuous. Therefore $f \in \mathcal{Q}_p^+$. It is easy to see $Q_p(K) = Z(f)$.

3.3. **Proof of Theorem 2.5.** Let $A = Q_p(K)$ and $B = \{\bar{\mu} \,|\, \mu \in \mathcal{M}_p^{qc}(K)\}$. We show $A = B$. By Theorems 2.1 and 2.4, it follows easily that $B \subseteq A$. So, to complete the proof of Theorem 2.5, we prove

Proposition 3.4. *$A \subseteq B$.*

Proof. Given $\xi \in A$, assume $\xi \notin B$ and we would like to obtain a contradiction. Define the following subsets of Radon measures $\mathcal{E}_0^*$:

$$\mathcal{A} = \{\mu \in \mathcal{M}_p^{qc} \,|\, \bar{\mu} = \xi\}, \quad \mathcal{B} = \{\nu \in \mathcal{E}_0^* \mid \operatorname{supp} \nu \subset K\}. \tag{3.4}$$

Then both $\mathcal{A}$ and $\mathcal{B}$ are convex subsets of $\mathcal{E}_0^*$. Under the weak * topology of $\mathcal{E}_0^*$, $\mathcal{B}$ is a closed subspace and $\mathcal{A}$ is a compact set. Since $\xi \notin B$, it follows that $\mathcal{A} \cap \mathcal{B} = \emptyset$. Note that the topological dual of $\mathcal{E}_0^*$ equiped with the weak * topology is $\mathcal{E}_0$ itself. Therefore, by the Hahn-Banach theorem (e.g., [15, Proposition 18.2 on P.190]), there exist nonzero function $f \in \mathcal{E}_0$ and positive number $\alpha > 0$ such that

$$(a) \quad \langle \mu, f\rangle \geq \alpha \quad \forall\ \mu \in \mathcal{A}; \quad (b) \quad \langle \nu, f\rangle = 0 \quad \forall\ \nu \in \mathcal{B}. \tag{3.5}$$

Therefore, by (3.5b), $f|_K = 0$, but $f \neq 0$. Let $g = \frac{1}{\|f\|_p} \max\{f, 0\}$ and $h = g^{qc}$, the quasiconvex envelope of g. Hence $h \in \mathcal{Q}_p^+(K)$. It is also well-known that

$$h(\eta) = \inf_{\varphi\in W_0^{1,p}(\Omega;\mathbf{R}^n)} \frac{1}{|\Omega|} \int_\Omega g(\eta + D\varphi(x))\, dx.$$

Note that for any $\varphi \in W_0^{1,p}(\Omega;\mathbf{R}^n)$ the measure μ defined by

$$\langle \mu, q\rangle = \frac{1}{|\Omega|}\int_\Omega q(\xi + D\varphi(x))\,dx \quad \forall q \in \mathcal{C}_p$$

belongs to the set $\mathcal{A}$ defined above. Hence by (3.5a) we have $h(\xi) \geq \frac{1}{\|f\|_p}\alpha > 0$, which shows that $\xi \notin Z(h)$ and contradicts with $\xi \in A = Q_p(K)$. The proof is now completed. ∎

4. p-quasiconvex hulls of compact sets

We now consider the special case when K is compact. We need following two lemmas.

Lemma 4.1. *Let $g\colon M^{n\times m} \to \mathbf{R}$ be any continuous function. Then for any $\xi \in M^{n\times m}$ there exists a sequence $\{\phi_j\}$ in $W_0^{1,\infty}(\Omega;\mathbf{R}^n)$ such that*

$$g^{qc}(\xi)\,|\Omega| = \lim_{j\to\infty}\int_\Omega g(\xi + D\phi_j(x))\,dx, \quad \lim_{j\to\infty}\|\phi_j\|_{L^\infty(\Omega)} = 0. \tag{4.1}$$

We also need an important result of Zhang [21]; for a recent important generalization of this result, see [12].

Lemma 4.2. *Suppose $u_j \rightharpoonup 0$ weakly in $W_0^{1,1}(\Omega;\mathbf{R}^n)$ and, for some $L > 0$,*

$$\lim_{j\to\infty}\int_{\Omega\cap\{|Du_j|>L\}} |Du_j(x)|\,dx = 0.$$

Then there exists a sequence $\{\psi_j\}$ such that $\psi_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(\Omega;\mathbf{R}^n)$ and $u_j - \psi_j \to 0$ strongly in $W^{1,1}(\Omega;\mathbf{R}^n)$ as $j \to \infty$.

We can now characterize the p-quasiconvex hull of a *compact* set (see also [11]).

Theorem 4.3. *Let K be a compact set. Then $Q_p(K) = Z(d_K^{qc}) = K^{qc}$ for all $p \geq 1$.*

Proof. Since $d_K^{qc} \in \mathcal{Q}_1^+(K)$, we have $Q_1(K) \subseteq Z(d_K^{qc})$. From Remark 2.1, $K^{qc} \subseteq Q_p(K) \subseteq Q_1(K)$. Hence, in order to prove the result in the theorem, it suffices to prove

$$Z(d_K^{qc}) \subseteq K^{qc}. \tag{4.2}$$

Let η be given such that $d_K^{qc}(\eta) = 0$. By Lemma 4.1, there exists a sequence $\{\phi_j\}$ in $W_0^{1,\infty}(\Omega; R^n)$ such that

$$\lim_{j\to\infty} \int_\Omega d_K(\eta + D\phi_j(x))\, dx = 0, \quad \lim_{j\to\infty} \|\phi_j\|_{L^\infty(\Omega)} = 0. \tag{4.3}$$

Let $M = \max_{\xi\in K} |\xi - \eta|$ and $\Omega_j(L) = \{x \in \Omega \,|\, |D\phi_j(x)| \geq L\}$. Then $d_K(\eta + \xi) \geq |\xi| - M$ for all matrices ξ and hence it follows that for $L > 2M$

$$\int_{\Omega_j(L)} d_K(\eta + D\phi_j) \geq \int_{\Omega_j(L)} \big(|D\phi_j| - M\big) \geq \frac{L}{2}\,|\Omega_j(L)|. \tag{4.4}$$

We claim that the sequence $\{|D\phi_j|\}$ is equi-integrable in $L^1(\Omega)$; that is, for any given $\epsilon > 0$ one can find a $\delta > 0$ such that

$$\sup_{j=1,2,\cdots} \int_E |D\phi_j| < \epsilon \quad \forall\ E \subset \Omega,\ |E| < \delta.$$

To prove this, observe that from (4.4) for any $L > 2M$

$$|\Omega_j(L)| \leq \frac{2}{L} \int_\Omega d_K(\eta + D\phi_j), \tag{4.5}$$

and hence, using (4.4) again, for any $L > 2M$

$$\int_{\Omega_j(L)} |D\phi_j| \leq 2 \int_\Omega d_K(\eta + D\phi_j) \to 0 \tag{4.6}$$

as $j \to \infty$. We thus select an integer j_0 such that

$$\sup_{L>2M} \int_{\Omega_j(L)} |D\phi_j| < \epsilon/2 \quad \forall\ j \geq j_0.$$

Since each of the functions $\{|D\phi_1|, \cdots, |D\phi_{j_0}|\}$ is absolutely integrable, we can find $\delta_1 > 0$ such that

$$\sup_{j=1,2,\ldots,j_0} \int_F |D\phi_j| < \epsilon/2 \quad \forall\ F \subset \Omega,\ |F| < \delta_1.$$

From (4.5) we can select a constant $\bar{L} > 2M$ such that $|\Omega_j(\bar{L})| < \delta_1$ for all $j = 1, 2, ..., j_0$ and hence

$$\sup_{j=1,2,\ldots,j_0} \int_{\Omega_j(\bar{L})} |D\phi_j| < \epsilon/2.$$

Finally, let $\delta = \epsilon/(2\bar{L})$. Then for any measurable set $E \subset \Omega$ with $|E| < \delta$ and for all $j = 1, 2, ...$

$$\int_E |D\phi_j| \leq \bar{L} \cdot |E| + \int_{\Omega_j(\bar{L})} |D\phi_j| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves the claim. From this claim and the Dunford-Pettis theorem (see e.g. [7, page 27-II]), the sequence $\{D\phi_j\}$ is weakly convergent in $L^1(\Omega)$. The second condition in (4.3) implies the weak limit must be zero. Therefore, $\phi_j \rightharpoonup 0$ in $W_0^{1,1}(\Omega;\mathbf{R}^n)$ as $j\to\infty$. Using (4.6) with any fixed constant $L>2M$, from Lemma 4.2, we obtain a sequence $\{\psi_j\}$ in $W^{1,\infty}(\Omega;\mathbf{R}^n)$ such that

$$\psi_j \overset{*}{\rightharpoonup} 0, \quad \lim_{j\to\infty}\int_\Omega |D\phi_j - D\psi_j| = 0. \tag{4.7}$$

Let g be any finite valued nonnegative quasiconvex function satisfying $g|_K = 0$. By (4.3), (4.7), we can choose a subsequence of $\{\psi_j\}$ (denoted the same) such that

$$\lim_{j\to\infty} g(\eta + D\psi_j(x)) = 0 \quad \text{for a.e. } x\in\Omega. \tag{4.8}$$

Using the lower semicontinuity of the functional $I(u)=\int_\Omega g(\eta+Du)$ (cf. [1]), we have, by (4.8) and the Lebesgue bounded dominated convergence,

$$g(\eta)|\Omega| = I(0) \le \liminf_{j\to\infty} I(\psi_j) = \liminf_{j\to\infty}\int_\Omega g(\eta + D\psi_j(x))dx = 0.$$

Hence $g(\eta)=0$ and thus $\eta\in Z(g)$ for all such functions g. Therefore, by definition, $\eta\in K^{qc}$ and hence (4.2) follows. We complete the proof. □

***Remark* 4.1.** For any sequence $\{\phi_j\}$ satisfying (4.3), it remains unclear whether one can find a sequence $\{\psi_j\}$ that, in addition to the condition (4.7), satisfies

$$\lim_{k\to\infty} \|d_K(\eta + D\psi_j)\|_{L^\infty(\Omega)} = 0.$$

When K is compact and convex, a positive answer to this question has been recently confirmed by Müller [12].

5. L^p-coercivity and the p-quasiconvex hulls

We study sets K with certain special properties. Throughout this section, we assume $p>1$. We say set K is *L^p-coercive* if for some constants $c>0$ and C,

$$\int_B d_K^p(D\phi(y))\,dy \ge c\int_B (|D\phi(y)|^p - C)\,dy \tag{5.1}$$

for all $\phi \in C_0^\infty(B;\mathbf{R}^n)$, where B is any ball in $\mathbf{R}^m$. Note that, by a density argument, condition (5.1) also holds for all $\phi \in W_0^{1,p}(B;\mathbf{R}^n)$. Therefore, if $\phi \in W_0^{1,p}(\Omega;\mathbf{R}^n)$ and K is L^p-coercive,

$$\int_\Omega |D\phi(x)|^p dx \le \frac{1}{c}\int_\Omega d_K^p(D\phi(x))\,dx + C_0\,|\Omega|. \tag{5.2}$$

Theorem 5.1. *Let K be L^p-coercive. Then $Q_p(K) = Z[(d_K^p)^{qc}]$.*

Proof. This result and other important properties on L^p-coercivity have been proved in [20]. We only remark here that if K is not L^p-coercive then the inclusion $Q_p(K) \subset Z[(d_K^p)^{qc}]$ may be strict (see [13, 20]). □

We now study a set-valued function $F(x,s)$ from $\mathbf{R}^m \times \mathbf{R}^n$ to $M^{n\times m}$, that is, for $x \in \mathbf{R}^m$, $s \in \mathbf{R}^n$, $F(x,s)$ is a set in $M^{n\times m}$. Let $d(x,s,\xi) = d_{F(x,s)}(\xi)$ be the distance function from ξ to the set $F(x,s)$. Then $d(x,s,\xi)$ is Lipschitz continuous in ξ. We also need certain continuity conditions of F on s. To this end, we assume that

(1) $|d^p(x,s_1,\xi) - d^p(x,s_2,\xi)| \le \rho(x,|s_1-s_2|)\,\gamma(|\xi|)$, where $\rho(x,t)$ is *Carathéodory* in the sense that it is measurable in x for all t and continuous in t for almost every x, $\rho(x,0)=0$, and $\gamma(t)\ge 0$ is increasing in t;

(2) $F(x,s)$ admits a Carathéodory selection $a(x,s) \in F(x,s)$ satisfying $|a(x,s)| \le b(x) + c(x)|s|$, where $b(x)$, $c(x) \ge 0$ are finite measurable functions on Ω.

Condition (2) implies $d(x,s,\xi) \le b(x) + c(x)|s| + |\xi|$. Also, by [1, Thm. III.6], Conditions (1), (2) imply that the quasiconvexification $(d^p)^{qc}(x,s,\xi)$ with respect to ξ is continuous in (s,ξ) for a. e. x and measurable in x for all s, ξ.

Theorem 5.2. *Let F satisfy the assumptions above and let the set $F(x,s)$ be L^p-coercive for all $s \in \mathbf{R}^n$ and almost every $x \in \Omega$. Suppose $\{u_j\}$ weakly converges to u in $W^{1,p}(\Omega;\mathbf{R}^n)$ and satisfies $\int_\Omega d^p(x,u_j,Du_j) \to 0$ as $j\to\infty$. Then $Du(x) \in Q_p[F(x,u(x))]$ for almost every $x \in \Omega$.*

Proof. From the L^p-coercivity condition of set $F(x,s)$, we have, by Theorem 5.1,

$$Q_p[F(x,s)] = Z[(d_{F(x,s)}^p)^{qc}] = Z[(d^p)^{qc}(x,s,\cdot)]. \tag{5.3}$$

On the other hand, $d^p(x,s,\xi) \le B(x) + C(x)(|s|^p + |\xi|^p)$, where $B(x)$, $C(x) \ge 0$ are *finite* measurable functions in Ω. For given integers $\nu > 0$, let $\Omega_\nu = \{x \in \Omega \,|\, B(x) < \nu,\ C(x) < \nu\}$ and $f(x,s,\xi) = \chi_{\Omega_\nu}(x)(d^p)^{qc}(x,s,\xi)$, where χ_S is the characteristic function of any given set S. Since $f(x,s,\xi)$ is quasiconvex in ξ, by a lower semicontinuity theorem (see e.g. [1, Thm. II.4]), we infer that

$$0 \le \int_{\Omega_\nu} (d^p)^{qc}(x,u(x),Du(x))\,dx \le \lim_{j\to\infty} \int_\Omega d^p(x,u_j(x),Du_j(x))\,dx = 0.$$

Therefore, by (5.3), $Du(x) \in Q_p[F(x,u(x))]$ for almost every $x \in \Omega_\nu$. Since $\Omega = \cup_\nu \Omega_\nu$, we see that this relation holds for almost every $x \in \Omega$. The theorem is thus proved. □

To close this paper, we give an application of Theorem 5.2 on mappings of *finite distortion*, which provides another proof of a convergence result in [8, 17].

Proposition 5.3. *Let $K(x)$ be a finite measurable function from a domain Ω in $\mathbf{R}^n$ to $\mathbf{R}$. Let $u_j \in W^{1,n}(\Omega;\mathbf{R}^n)$ satisfy*

$$|Du_j(x)|^n \le K(x)\det Du_j(x), \quad j = 1,2,... \tag{5.4}$$

for almost every $x \in \Omega$. Suppose that $\{u_j\}$ weakly converges to u. Then $|Du(x)|^n \le K(x)\det Du(x)$ for almost every $x \in \Omega$.

Proof. Consider the set-valued function F defined by

$$F(x,s) = F(x) = \{\xi \in M^{n\times n} \mid |\xi|^n \le K(x)\det\xi\}.$$

Since $F(x,s)$ is independent of s, it is easy to see F satisfies the assumptions (1), (2) above, and, for almost every $x \in \Omega$, one can also see that the set $F(x)$ is L^n-coercive (cf. [16, 20]). Note that $F(x) = Z(g)$, where

$$g(\xi) = \max\{0,\ |\xi|^n - K(x)\det\xi\}.$$

Note that it is easy to see $g \in \mathcal{Q}_n^+(F(x))$. Hence, by definition, $Q_n[F(x)] = F(x)$. Therefore, the result follows from the theorem above. □

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