Solutions of Extra Problems

1. Let $f(x) = \sin(x^2)$, $x \in \mathbb{R}$. Compute $f'(x)$, $f''(x)$, and $f'''(x)$ for $x \in \mathbb{R}$.

   Solution. Since $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}x^2 = 2x$, by the chain rule, f is differentiable, and

$$f'(x) = \cos(x^2) \cdot 2x.$$

   Since $\frac{d}{dx}\cos x = -\sin x$, by the product rule and chain rule, f' is also differentiable and

$$f''(x) = \frac{d}{dx}[\cos(x^2)] \cdot 2x + \cos(x^2) \cdot \frac{d}{dx}(2x) = -\sin(x^2) \cdot (2x)^2 + 2\cos(x^2).$$

   Further differentiation gives

$$f'''(x) = -\cos(x^2) \cdot (2x)^3 - \sin(x^2) \cdot 8x - 2\sin(x^2) \cdot (2x) = -8\cos(x^2) \cdot x^3 - 12\sin(x^2) \cdot x.$$

   □

2. Let $g(x) = e^{-\frac{1}{x}}$, $x \in (0, \infty)$. Compute $g'(x)$, $g''(x)$, and $g'''(x)$ for $x \in (0, \infty)$.

   Solution. Since $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}(-\frac{1}{x}) = \frac{1}{x^2}$, by the chain rule, g is differentiable, and

$$g'(x) = e^{-\frac{1}{x}} \cdot \frac{1}{x^2}.$$

   Since $\frac{d}{dx}(\frac{1}{x^2}) = \frac{-2}{x^3}$, by the product rule and chain rule, g' is also differentiable and

$$g''(x) = e^{-\frac{1}{x}} \cdot \left(\frac{1}{x^2}\right)^2 + e^{-\frac{1}{x}} \cdot \frac{-2}{x^3} = e^{-\frac{1}{x}} \cdot \left(\frac{1}{x^4} - \frac{2}{x^3}\right).$$

   Since $\frac{d}{dx}(\frac{1}{x^4} - \frac{2}{x^3}) = \frac{-4}{x^5} + \frac{6}{x^4}$, further differentiation gives

$$g'''(x) = e^{-\frac{1}{x}} \cdot \frac{1}{x^2} \cdot \left(\frac{1}{x^4} - \frac{2}{x^3}\right) + e^{-\frac{1}{x}} \cdot \left(\frac{-4}{x^5} + \frac{6}{x^4}\right)$$
$$= e^{-\frac{1}{x}} \cdot \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right).$$

   □