MTH 320 Section 004 Final Sample

1. (a) [4pts] State Mean Value Theorem.

   (b) [6pts] Suppose that f is differentiable on $\mathbb{R}$ and $f'(x) > 0$ on $\mathbb{R}$. Prove that f is strictly increasing.

   Solution. (a) If f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   (b) Let $x < y \in \mathbb{R}$. By Mean Value Theorem, there is $z \in (x, y)$ such that

   $$\frac{f(y) - f(x)}{y - x} = f'(z).$$

   By the assumption, $f'(z) > 0$. From $y > x$ we get $f(y) - f(x) = (y - x)f'(z) > 0$. So $f(y) > f(x)$. Thus, f is strictly increasing. $\square$

2. Let $f : \mathbb{R} \to \mathbb{R}$.

   (a) (4 pts) What does it mean for f to be differentiable at a?

   (b) (6 pts) Let $f(x) = x \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. Is f differentiable at $x = 0$? Justify your answer.

   Solution. (a) We say that f is differentiable at a if the limit

   $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists and is finite.

   (b) We have

   $$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \sin(\frac{1}{x}).$$

   We know that $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist. To see this, we may choose a sequence $x_n = \frac{1}{n\pi + \pi/2}$, which tends to 0. Then we have $\sin(\frac{1}{x_n}) = \sin(n\pi + \pi/2) = (-1)^n$. But the sequence $((-1)^n)$ does not converge. So $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist, which implies that f is not differentiable at 0. $\square$

3. (a) [5 pts] Prove that the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ in $[-1, 1)$.

   (b) [5 pts] What function does the power series above represent on $(-1, 1)$? Justify your answer.

Solution. (a) This is a power series with coefficients $a_n = \frac{1}{n}$. Since $|\frac{a_{n+1}}{a_n}| = \frac{n}{n+1} \to 1$, by ratio test, the radius of the series is 1. Thus, the power series converges at every point in $(-1, 1)$, and diverges at every point in $[-1, 1]^c$. At $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges by the p-test. At $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternative series test. So the exact interval of convergence is $[-1, 1)$.

(b) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $x \in (-1, 1)$. Differentiating the power series, we get

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Thus, for some constant $C \in \mathbb{R}$, $f(x) = \log_e(\frac{1}{1-x}) + C = -\log_e(1-x) + C$. Since the series has no constant term, $f(0) = 0$. Taking $x = 0$, we get $0 = f(0) = -\log(1) + C = C$. Thus, $f(x) = -\log_e(1-x)$. □

4. (a) [4pts] Define $\liminf s_n$

   (b) [6pts] Prove that if $\limsup s_n = \liminf s_n = s \in \mathbb{R}$, then (s_n) converges to s.

Solution. (a) Let $u_n = \inf\{s_m : m \geq n\}$, $n \in \mathbb{N}$. Either $s_n = -\infty$ for all $n \in \mathbb{N}$, or $s_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. In the first case, we define $\liminf s_n = -\infty$; in the second case, we define $\liminf s_n = \lim u_n$. (b) Let $v_n = \sup\{s_m : m \geq n\}$, $n \in \mathbb{N}$. The $\limsup s_n$ is defined as $\lim v_n$ (if $v_n = +\infty$ for all n, then $\limsup s_n = +\infty$. From $\limsup s_n = \liminf s_n = s \in \mathbb{R}$ we know that $\lim u_n = \lim v_n = s$. Since for every n, $u_n \leq s_n \leq v_n$, by squeeze lemma we have $\lim s_n = s$. □

5. (a) [4 pts.] State Weierstrass M-test.

   (b) [6 pts.] Prove that the series of functions $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ converges to a continuous function on all of $\mathbb{R}$, being careful to justify all of your steps.

Solution. (a) Suppose (f_n) is a sequence of functions defined on $S \subseteq \mathbb{R}$. Let (a_n) be a sequence of nonnegative real numbers such that $\sum a_n$ converges. If $|f_n(x)| \leq a_n$ for each $n \in \mathbb{N}$ and $x \in S$. Then $\sum f_n$ converges uniformly on S.

(b) Applying Weierstrass M-test to $f_n(x) = \frac{\sin(nx)}{n^3}$ and $a_n = \frac{1}{n^3}$, and noting that $|\frac{\sin(nx)}{n^3}| = \frac{|\sin(nx)|}{n^3} \leq \frac{1}{n^3}$ and $\sum \frac{1}{n^3}$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ converges uniformly on $\mathbb{R}$. Since each f_n is continuous on $\mathbb{R}$, the uniform limit of the series is also continuous on $\mathbb{R}$. □

6. Suppose (s_n) is an increasing sequence of real numbers. Prove that $\lim_{n\to\infty} s_n = \sup\{s_n : n \in \mathbb{N}\}$. You need to consider two cases: (i) $\sup\{s_n : n \in \mathbb{N}\} \in \mathbb{R}$; and (ii) $\sup\{s_n : n \in \mathbb{N}\} = +\infty$.

Solution. (a) Let f_n be a sequence of functions defined on $S \subseteq \mathbb{R}$. Let (a_n) be a sequence of nonnegative real numbers such that $\sum a_n$ converges. Suppose $|f_n(x)| \leq a_n$ for every $n \in \mathbb{N}$ and $x \in S$. Then $\sum f_n$ converges uniformly on S.

(b) We have $|\frac{\sin(nx)}{n^3}| \leq \frac{1}{n^3}$ for every $x \in \mathbb{R}$. Since $\sum \frac{1}{n^3}$ converges, by Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ converges uniformly on $\mathbb{R}$. Since every $\frac{\sin(nx)}{n^3}$ is continuous on $\mathbb{R}$, the uniform limit of the series is continuous on $\mathbb{R}$. $\square$

7. (a) [4 pts] State L'Hospital's rule. Be sure to include all conditions.

(b) [6 pts] Find the following limits

- $\lim_{y\to\infty}(1+\frac{2}{y})^y$
- $\lim_{x\to 0}\frac{\cos x - 1}{e^x - 1 - x}$

Proof. (a) Let s be one of $a, a^+, a^-, +\infty, -\infty$, where $a \in \mathbb{R}$. Let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Suppose in a neighborhood of s, f and g are differentiable and $g' \neq 0$. If

$$\lim_{x\to s}\frac{f'(x)}{g'(x)} = L$$

and either

$$\lim_{x\to s} f(x) = \lim_{x\to s} g(x) = 0,$$

or

$$\lim_{x\to s}|g(x)| = +\infty,$$

then

$$\lim_{x\to s}\frac{f(x)}{g(x)} = L.$$

(b) For the first limit, we write $(1+\frac{2}{y})^y = e^{y\log_e(1+\frac{2}{y})}$. Since $1/y \to 0$ as $y \to \infty$, we get

$$\lim_{y\to\infty} y\log_e(1+\frac{2}{y}) = \lim_{y\to\infty}\frac{\log_e(1+\frac{2}{y})}{\frac{1}{y}} = \lim_{x\to 0}\frac{\log_e(1+2x)}{x},$$

if the latter limit exists. Since

$$\lim_{x\to 0}\log_e(1+2x) = \log_e(1) = 0 = \lim_{x\to 0} x,$$

by L'Hospital's rule, we have

$$\lim_{x\to 0}\frac{\log_e(1+2x)}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\log_e(1+2x)}{\frac{d}{dx}x},$$

if the latter limit exists. Direct calculation shows

$$\lim_{x\to 0}\frac{\frac{d}{dx}\log_e(1+2x)}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{2}{1+2x} = 2.$$

Thus, $\lim_{y\to\infty} y\log_e(1+\frac{2}{y}) = 2$, which implies that

$$\lim_{y\to\infty}(1+\frac{2}{y})^y = e^{\lim_{y\to\infty} y\log_e(1+\frac{2}{y})} = e^2.$$

For the second limit, we observe that

$$\lim_{x\to 0}(\cos x - 1) = \cos 0 - 1 = 1 - 1 = 0;$$

$$\lim_{x\to 0}(e^x - 1 - x) = e^0 - 1 = 1 - 1 = 0.$$

Thus, by L'Hospital's rule, we have

$$\lim_{x\to 0}\frac{\cos x - 1}{e^x - 1 - x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(e^x - 1 - x)} = \lim_{x\to 0}\frac{-\sin x}{e^x - 1},$$

if the latter limit exists. Since

$$\lim_{x\to 0}(-\sin x) = -\sin 0 = 0 = e^0 - 1 = \lim_{x\to 0}(e^x - 1),$$

by L'Hospital's rule, we have

$$\lim_{x\to 0}\frac{-\sin x}{e^x - 1} = \lim_{x\to 0}\frac{\frac{d}{dx}(-\sin x)}{\frac{d}{dx}(e^x - 1)} = \lim_{x\to 0}\frac{-\cos x}{e^x},$$

if the latter limit exists. Using the continuity of $\cos x$ and e^x, we get $\lim_{x\to 0}\frac{-\cos x}{e^x} = \frac{-\cos 0}{e^0} = -1$. Thus, $\lim_{x\to 0}\frac{\cos x - 1}{e^x - 1 - x} = -1$. □

8. (a) [4 pts] For $a, L \in \mathbb{R}$, define the expressions $\lim_{x\to a^-} f(x) = L$, $\lim_{x\to a^+} f(x) = L$, and $\lim_{x\to a} f(x) = L$.

   (b) [6 pts] Prove that if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

Solution. (a) We first give definitions using sequences. By saying that $\lim_{x\to a^-} f(x) = L$ we mean that there is $r > 0$ such that f is defined on $(a-r, a)$, and for any sequence (s_n) in $(a-r, a)$ with $s_n \to a$, we have $f(s_n) \to L$. By saying that $\lim_{x\to a^+} f(x) = L$ we mean that there is $r > 0$ such that f is defined on $(a, a+r)$, and for any sequence (s_n) in $(a, a+r)$ with $s_n \to a$, we have $f(s_n) \to L$. By saying that $\lim_{x\to a} f(x) = L$ we mean that there is $r > 0$ such that f is defined on $(a-r, a)\cup(a, a+r)$, and for any sequence (s_n) in $(a-r, a)\cup(a, a+r)$ with $s_n \to a$, we have $f(s_n) \to L$.

We then give definitions using the "$\varepsilon - \delta$" language. We say that $\lim_{x\to a^-} f(x) = L$ if for any $\varepsilon > 0$, there is $\delta > 0$ such for any $x \in (a-\delta, a)$, $f(x)$ is defined and $|f(x) - L| < \varepsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $x \in (a, a+\delta)$, $f(x)$ is defined and $|f(x) - L| < \varepsilon$. We say that $\lim_{x\to a} f(x) = L$ if for any $\varepsilon > 0$, there is $\delta > 0$ such for any $x \in (a-\delta, a)\cup(a, a+\delta)$, $f(x)$ is defined and $|f(x) - L| < \varepsilon$.

(b) We give two proofs. The first is easier, and based on "$\varepsilon - \delta$" definitions. Suppose that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$. Let $\varepsilon > 0$. Then there exist $\delta_+, \delta_- > 0$ such that for any $x \in (a - \delta_-, a)$, $f(x)$ is defined and $|f(x) - L| < \varepsilon$, and for any $x \in (a, a + \delta)$, $f(x)$ is defined and $|f(x) - L| < \varepsilon$. Let $\delta = \min\{\delta_+, \delta_-\} > 0$. Since $(a-\delta, a)\cup(a, a+\delta) \subset (a-\delta_-, a)\cup(a, a+\delta_+)$, we find that for any $x \in (a-\delta, a)\cup(a, a+\delta)$, $f(x)$ is defined and $|f(x) - L| < \varepsilon$. Thus, $\lim_{x\to a} f(x) = L$.

The second proof is longer, and based on the sequential definitions. Suppose that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$. Then there are $r_+, r_- > 0$ such that f is defined on $(a - r_-, a)$ and $(a, a + r_+)$, and for any sequence (s_n) in $(a - r_-, a)$ or $(a, a + r_+)$ with $s_n \to a$, we have $f(s_n) \to L$. Let $r = \min\{r_+, r_-\} > 0$. Then f is defined on $(a - r, a) \cup (a, a + r)$. Let (s_n) be a sequence in $(a - r, a) \cup (a, a + r)$ with $s_n \to a$. We need to show that $f(s_n) \to L$. For this purpose, it suffices to show that L is the only subsequential limit of $(f(s_n))$. If this is not true, then (s_n) contains a subsequence (s_{n_k}) such that $f(s_{n_k}) \to L' \neq L$. Here L' could be $+\infty$ or $-\infty$. One of the following two cases must happen: 1) there are infinitely many k such that $s_{n_k} > a$; 2) there are infinitely many k such that $s_{n_k} < a$. In the first case, we get a subsequence $(s_{n_{k_l}})$ of (s_{n_k}), which lies in $(a, a + r) \subset (a, a + r_+)$. In the second case, we get a subsequence $(s_{n_{k_l}})$ of (s_{n_k}), which lies in $(a - r, a) \subset (a - r_-, a)$. In either case, we have $s_{n_{k_l}} \to a$, but $f(s_{n_{k_l}}) \to L' \neq L$, which contradicts the assumption. Thus, $f(s_n) \to L$. $\square$