Homework 10 (due on 11/8)

- Read Sections 25, 26, and 28 for the next week.

23.1 For each of the following power series, find the radius of convergence and determine the exact interval of convergence. (a) $\sum n^2 x^n$; (c) $\sum(\frac{2^n}{n^2})x^n$; (e) $\sum(\frac{2^n}{n!})x^n$; (g) $\sum(\frac{3^n}{n \cdot 4^n})x^n$.

23.4 For $n = 0, 1, 2, 3, \ldots$, let $a_n = [\frac{4+2(-1)^n}{5}]^n$.

(a) Find $\limsup |a_n|^{1/n}$, $\liminf |a_n|^{1/n}$, $\limsup |\frac{a_{n+1}}{a_n}|$ and $\liminf |\frac{a_{n+1}}{a_n}|$.

(b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Explain briefly.

(c) Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.

23.5 Consider a power series $\sum a_n x^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many of them are nonzero, then $R \leq 1$.

(b) Prove that if $\limsup |a_n| > 0$, then $R \leq 1$. Hint: You may work out (b) first and use it to prove (a).

24.2 For $x \in [0, \infty)$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

24.3 Repeat Exercise 24.2 for $f_n(x) = \frac{1}{1+x^n}$.

24.10 (a) Prove that if $f_n \to f$ uniformly on S and $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.

24.11 Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for all $x \in \mathbb{R}$.

(a) Observe $f_n \to f$ uniformly on R [obvious!] and $g_n \to g$ uniformly on R [almost obvious].

(b) Observe the sequence $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. Compare Exercise 24.2.

24.12 Prove the assertion in Remark 24.4: A sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S if and only if

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0.$$