Homework 12 (due on 11/25)

29.2 Prove $|\cos x - \cos y| \le |x-y|$ for all $x, y \in \mathbb{R}$.

Proof. We use the fact that $|\cos' x| = |-\sin x| \le 1$ for all $x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$. If $x = y$, then $|\cos x - \cos y| = 0 \le |x-y|$. Now suppose $x \ne y$. Since $|\cos x - \cos y| = |\cos y - \cos x|$ and $|x-y| = |y-x|$, by symmetry we may assume $y < x$. By Mean Value Theorem, there is $z \in (y, x)$ such that $\frac{\cos x - \cos y}{x-y} = \cos' z$, which implies that $|\cos x - \cos y| = |x-y||\cos' z| \le |x-y|$. □

29.3 Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

(a) Show $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$.

(b) Show $f'(x) = \frac{1}{7}$ for some $x \in (0,2)$.

Proof. (a) Applying Mean Value Theorem to $a = 0$ and $b = 2$, we see that there is $x \in (0,2)$ such that $f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{1}{2}$.

(b) Applying Mean Value Theorem to $a = 0$ and $b = 1$, we see that there is $x_1 \in (0,1)$ such that $f'(x) = \frac{f(1)-f(0)}{1-0} = 1$. Applying Mean Value Theorem to $a = 1$ and $b = 2$, we see that there is $x_2 \in (1,2)$ such that $f'(x) = \frac{f(2)-f(1)}{2-1} = 0$. Applying Intermediate Value Theorem to $c = \frac{1}{7}$, which lies between $f'(x_1) = 1$ and $f'(x_2) = 0$, we see that there is $x \in (x_1, x_2) \subset (0,2)$ such that $f'(x) = c = \frac{1}{7}$. □

29.4 Let f and g be differentiable functions on an open interval I. Suppose a, b in I satisfy $a < b$ and $f(a) = f(b) = 0$. Show $f'(x) + f(x)g'(x) = 0$ for some $x \in (a,b)$. Hint: Consider $h(x) = f(x)e^{g(x)}$.

Proof. Let $h(x) = f(x)e^{g(x)}$. By chain rule and product rule, h is differentiable and

$$h'(x) = f'(x)e^{g(x)} + f(x)\frac{d}{dx}e^{g(x)} = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x) = (f'(x) + f'(x)g(x))e^{g(x)}.$$

From $f(a) = f(b) = 0$ we know $h(a) = h(b) = 0$. By Mean Value Theorem, there is $x \in (a,b)$ such that $h'(x) = 0$. From the above formula and that $e^{g(x)} > 0$ we know that $f'(x) + f(x)g'(x) = 0$. □

29.7 (a) Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b.

(b) Suppose f is three times differentiable on an open interval I and $f''' = 0$ on I. What form does f have? Prove your claim.

Proof. (a) Since $(f')'(x) = 0$ for all $x \in I$, by Corollary 29.4, f' is a constant function on I. Let $a \in \mathbb{R}$ denote this constant. Let $g(x) = f(x) - ax$, $x \in I$. Then $g'(x) = f'(x) - a = 0$, $x \in I$. By Corollary 29.4, g is a constant function on I. Let $b \in \mathbb{R}$ denote this constant. From $f(x) = ax + g(x)$ we get $f(x) = ax + b$, $x \in I$.

(b) We claim that f has the form $f(x) = \frac{1}{2}ax^2 + bx + c$ for suitable constants a, b, c. Since $f''' = (f')''$, by (a) f' has the form $ax + b$ for some constants a, b. Let $h(x) = f(x) - \frac{1}{2}ax^2 - bx$, $x \in I$. Then $h'(x) = f'(x) - ax - b = 0$, $x \in I$. By Corollary 29.4, h is a constant function on I. Let $c \in \mathbb{R}$ denote this constant. From $f(x) = \frac{1}{2}ax^2 + bx + h(x)$ we get $f(x) = \frac{1}{2}ax^2 + bx + c$, $x \in I$. □

29.9 Show $ex \leq e^x$ for all $x \in R$.

Proof. Let $f(x) = e^x - ex$. Then f is differentiable with $f'(x) = e^x - e = e^x - e^1$. Since e^x is increasing, we have $f'(x) \geq 0$ for $x \geq 1$ and $f'(x) \leq 0$ for $x \leq 1$. So f is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$. Since $f(1) = e^1 - e = 0$, we see that $f(x) \geq 0$ for $x \geq 1$ or $x \leq 1$, which implies that $e^x \geq ex$ for all $x \in \mathbb{R}$. □

29.13 Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$.

Proof. Let $h = g - f$. Then h is differentiable with $h' = g' - f' \geq 0$ on $\mathbb{R}$. So h is increasing on $\mathbb{R}$. From $h(0) = g(0) - f(0) = 0$, we see that for $x \geq 0$ $h(x) \geq 0$, which implies that $g(x) \geq f(x)$. □

29.16 Use Theorem 29.9 to obtain the derivative of the inverse $g = \mathrm{Tan}^{-1} = \arctan$ of f where $f(x) = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution. Let $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. We have $\cos x > 0$ and $f'(x) = \cos^{-2} x = 1 + \tan^2 x = 1 + f(x)^2 > 0$ on I. Thus, f is strictly increasing on I. So we may apply Theorem 29.9 to $g = f^{-1}$ and conclude that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1 + f(g(x))^2} = \frac{1}{1 + x^2}.$$

□

29.18 Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$.

(a) Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0)$, $s_2 = f(s_1)$, etc. Prove (s_n) is a convergent sequence.
Hint: To show that (s_n) is Cauchy, first show $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$.

(b) Show f has a fixed point, i.e., $f(s) = s$ for some s in $\mathbb{R}$, and such point is unique.

Proof. (a) We first prove that for any $x, y \in \mathbb{R}$, $|f(x) - f(y)| \le a|x - y|$. If $x = y$, the inequality holds trivially. Suppose $x \neq y$. By symmetry we may assume $y \le x$. By Mean Value Theorem there is $z \in (y, x)$ such that $\frac{f(x)-f(y)}{x-y} = f'(z)$, which implies that $|f(x) - f(y)| = |f'(z)||x - y| \le a|x - y|$.

For $n \ge 1$, since $s_{n+1} = f(s_n)$ and $s_n = f(s_{n-1})$, applying the above inequality to $x = s_n$ and $y = s_{n-1}$, we get $|s_{n+1} - s_n| \le a|s_n - s_{n-1}|$. By induction, we get $|s_{n+1} - s_n| \le a^n|s_1 - s_0|$ for all $n \ge 0$. From $|a| = a < 1$, we know that $\sum_n a^n$ converges, and so $\sum_n a^n|s_1 - s_0|$ converges. By Comparison test, $\sum_{n=0}^{\infty}(s_{n+1} - s_n)$ converges. We observe that the partial sum sequence for the series $\sum_{n=0}^{\infty}(s_{n+1} - s_n)$ is $t_m = \sum_{n=0}^{m}(s_{n+1} - s_n) = s_{m+1} - s_0$. The convergence of $\sum_{n=0}^{\infty}(s_{n+1} - s_n)$ implies that (t_m) converges. So (s_m) also converges.

(b) Let $s = \lim_m s_m$. From the continuity of f, we have $f(s) = \lim_m f(s_m) = \lim_m s_{m+1} = s$. So s is a fixed point of f. Suppose s' is another fixed point of f. Then from $|s - s'| = |f(s) - f(s')| \le a|s - s'|$ and $a < 1$ we get $|s - s'| = 0$, i.e., $s' = s$. So the fixed point of f is unique. $\square$