Homework 13 Solutions

30.1 Find the following limit if it exists. (a) $\lim_{x\to 0} \frac{e^{2x}-\cos x}{x}$.

Solution. Differentiating the numerator and denominator, we get

$$\lim_{x\to 0} \frac{2e^{2x} - \sin x}{1} = 2e^0 - \sin 1 = 2.$$

Since

$$\lim_{x\to 0}(e^{2x} - \cos x) = e^0 - \cos 0 = 1 - 1 = 0, \quad \lim_{x\to 0} x = 0,$$

we can apply L'Hospital's rule to conclude that

$$\lim_{x\to 0} \frac{e^{2x} - \cos x}{x} = \lim_{x\to 0} \frac{2e^{2x} - \sin x}{1} = 2.$$

□

30.2 Find the following limit if it exists. (c) $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x})$.

Solution. We transform the limit in the form that L'Hospital's rule can apply:

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x}.$$

Since

$$\lim_{x\to 0}(x - \sin x) = 0 - \sin 0 = 0 = 0 \sin 0 = \lim_{x\to 0} x \sin x,$$

by L'Hospital's rule, $\lim_{x\to 0} \frac{x-\sin x}{x \sin x}$ equals

$$\lim_{x\to 0} \frac{1 - \cos x}{\sin x + x \cos x},$$

if the latter limit exists. Since

$$\lim_{x\to 0}(1 - \cos x) = 1 - \cos 0 = 0 = \sin 0 + 0 \cos 0 = \lim_{x\to 0} \sin x + x \cos x,$$

by L'Hospital's rule, $\lim_{x\to 0} \frac{1-\cos x}{\sin x + x \cos x}$ equals

$$\lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x \sin x},$$

if the latter limit exists. We can not (and do not need to) apply L'Hospital's rule for one more time because

$$\lim_{x\to 0} \sin x = \sin 0 = 0, \quad \lim_{x\to 0} \cos x + \cos x - x \sin x = \cos 0 + \cos 0 - 0 \sin 0 = 2 \neq 0.$$

In fact, the above formulas imply that

$$\lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0.$$

Thus,

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = 2.$$

□

30.3 Find the following limit if it exists. (c) $\lim_{x\to 0^+} \frac{1+\cos x}{e^x - 1}$.

Solution. We can not apply L'Hospital's rule because

$$\lim_{x\to 0^+} (e^x - 1) = e^0 - 1 = 0, \quad \lim_{x\to 0^+} (1 + \cos x) = 1 + \cos 0 = 2 \neq 0.$$

Instead, we can calculate the limit directly. Since for $x > 0$, $e^x > e^0 = 1$, we have $e^x - 1 > 0$. For any sequence (x_n) in $(0, \infty)$ with $x_n \to 0$, we have $e^{x_n} - 1 \to 0$ and $e^{x_n} - 1 > 0$. Thus, $\frac{1}{e^{x_n}-1} \to +\infty$. Since $1 + \cos x_n \to 2 > 0$, we get $\frac{1+\cos x_n}{e^{x_n}-1} \to +\infty$. Thus, $\lim_{x\to 0^+} \frac{1+\cos x}{e^x-1} = +\infty$. □

30.5 Find the following limit if it exists. (a) $\lim_{x\to 0}(1 + 2x)^{1/x}$.

Solution. Since $(1 + 2x)^{1/x} = e^{\frac{1}{x}\log_e(1+2x)}$, we evaluate

$$\lim_{x\to 0} \frac{\log_e(1 + 2x)}{x}.$$

Since

$$\lim_{x\to 0} \log_e(1 + 2x) = \log_e(1) = 0 = \lim_{x\to 0} x,$$

by L'Hospital's rule, $\lim_{x\to 0} \frac{\log_e(1+2x)}{x}$ equals

$$\lim_{x\to 0} \frac{\frac{2}{1+2x}}{1},$$

if the latter limit exists. It is clear that

$$\lim_{x\to 0} \frac{\frac{2}{1+2x}}{1} = \lim_{x\to 0} \frac{2}{1 + 2x} = \frac{2}{1} = 2.$$

So $\lim_{x\to 0} \frac{\log_e(1+2x)}{x} = 2$, which implies that

$$\lim_{x\to 0}(1 + 2x)^{1/x} = \exp\left(\lim_{x\to 0} \frac{\log_e(1 + 2x)}{x}\right) = e^2.$$

□

26.3 (a) Use Exercise 26.2 to derive an explicit formula for $\sum_{n=1}^{\infty} n^2 x^n$.

Solution. From Exercise 26.2 we know that

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}, \quad |x| < 1.$$

Differentiating both sides, we get

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}, \quad |x| < 1.$$

Multiplying both sides by x, we get

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}, \quad |x| < 1.$$

□

26.4 (a) Observe that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for $x \in \mathbb{R}$, since we have $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$.

(b) Express $F(x) = \int_0^x e^{-t^2} dt$ as a power series.

Solution. (a) Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$, replacing x by $-x^2$, we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}, \quad x \in \mathbb{R}.$$

(b) By integrating the series term by term and observing the $F(0) = 0$, we get

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}, \quad x \in \mathbb{R}.$$

□

26.6 Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for $x \in \mathbb{R}$.

(a) Prove $s' = c$ and $c' = -s$.

(b) Prove $(s^2 + c^2)' = 0$.

(c) Prove $s^2 + c^2 = 1$.

Actually $s(x) = \sin x$ and $c(x) = \cos x$, but you do not need these facts.

Proof. (a) For $x \in \mathbb{R}$, we have

$$s'(x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = c(x),$$

$$c'(x) = -\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \cdots = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -s(x).$$

(b) By (a) and product rule, we have $(s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0$. (c) By (b), $s^2 + c^2$ is constant. Since $s(0) = 0$ and $c(0) = 1$, the constant is 1. So $s^2 + c^2 = 1$. □

31.1 Find the Taylor series for $\cos x$ about 0 and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

Solution. We know that $\cos' x = -\sin x$ and $\sin' x = \cos x$. By repeatedly differentiating $\cos x$, we find that for all integer $k \geq 0$,

$$\cos^{(4k)} x = \cos x, \quad \cos^{(4k+1)} x = -\sin x, \quad \cos^{(4k+2)} x = -\cos x, \quad \cos^{(4k+3)} x = -\sin x.$$

Evaluating these derivatives at $x = 0$, we get

$$\cos^{(4k)} 0 = 1, \quad \cos^{(4k+1)} 0 = 0, \quad \cos^{(4k+2)} 0 = -1, \quad \cos^{(4k+3)} 0 = 0.$$

This means $\cos^n x = 0$ when n is odd, and $\cos^n x = (-1)^{n/2}$ when n is even. Thus, the Taylor series for $\cos x$ about 0 is

$$\sum_{2|n, n\geq 0} \frac{(-1)^{n/2}}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

We observe the $\cos^{(n)}$ is one of $\cos x$, $-\cos x$, $\sin x$, $-\sin x$, and so are all bounded in absolute value by 1. Applying Corollary 31.4, we conclude that the Taylor series converges to $\cos x$ for every $x \in \mathbb{R}$. □