Homework 5 (due on 10/4)

10.6 (a) Let (s_n) be a sequence such that $|s_{n+1} - s_n| \leq 2^{-n}$ for all $n \in \mathbb{N}$. Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

Proof. We first find an upper bound of $|s_n - s_m|$ for $n, m \in \mathbb{N}$. First, suppose $n > m$. We write $s_n - s_m = \sum_{k=m}^{n-1}(s_{k+1} - s_k)$. By triangle inequality, we get

$$|s_n - s_m| \leq \sum_{k=m}^{n-1} |s_{k+1} - s_k| \leq \sum_{k=m}^{n-1} 2^{-k} = \sum_{s=0}^{n-m-1} 2^{-(m+s)} = 2^{-m} \sum_{s=0}^{n-m-1} (\frac{1}{2})^s.$$

Here we used a change of index $s = k - m$. Using Exercise 9.18, we get

$$\sum_{s=0}^{n-m-1} (\frac{1}{2})^s = \frac{1 - (1/2)^{n-m}}{1 - 1/2} < \frac{1}{1 - 1/2} = 2.$$

Thus, $|s_n - s_m| < 2 * 2^{-m}$. If $n < m$, then $|s_n - s_m| = |s_m - s_n| < 2 * 2^{-n}$. If $n = m$, then $|s_n - s_m| = 0$. So in any case, we have $|s_n - s_m| < 2 * 2^{-\min\{n,m\}}$.

From Theorem 9.7, we have $2 * 2^{-n} \to 0$. Let $\varepsilon > 0$. Since $2 * 2^{-n} \to 0$, there is $N \in \mathbb{N}$ such that for $n > N$, $2 * 2^{-n} < \varepsilon$. Suppose now $n, m > N$. Then $\min\{n, m\} > N$. From the last paragraph, we get $|s_n - s_m| < 2 * 2^{-\min\{n,m\}} < \varepsilon$. So (s_n) is a Cauchy sequence. By Theorem 10.11, it is a convergent sequence. □

- Exercise 11.2, 11.3, 11.4 for sequences (b_n), (c_n), (s_n), (w_n).

Solution. (b_n): (a) (b_n) itself is a monotone subsequence of (b_n). (b) Since $b_n \to 0$, its set of subsequential limits is $\{0\}$. (c) Since $\lim b_n = 0$, $\limsup b_n = \liminf b_n = 0$. (d) (b_n) converges. (e) (b_n) is bounded.

(c_n): (a) (c_n) itself is a monotone subsequence of (c_n). (b) Since $c_n \to +\infty$, its set of subsequential limits is $\{+\infty\}$. (c) Since $\lim c_n = +\infty$, $\limsup c_n = \liminf c_n = +\infty$. (d) (c_n) diverges to $+\infty$. (e) (c_n) is not bounded.

(s_n): We observe that (s_n) has period 6, i.e., $s_{n+6} = s_n$ for all n, and $(s_1, s_2, s_3, s_4, s_5, s_6) = (\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1)$. (a) Let $n_k = 6k$, then we get a constant subsequence $(s_{6k}) = (1, 1, \dots)$ of (s_n), which is monotonic. (b) The set of subsequential limits is $\{\frac{1}{2}, -\frac{1}{2}, -1, 1\}$. (c) $\limsup s_n$ is the biggest subsequential limit: 1; $\liminf s_n$ is the smallest subsequential limit: -1. (d) (s_n) neither converges nor diverges to $+\infty$ or $-\infty$. (e) (s_n) is bounded.

(w_n): (a) Let $n_k = 2k$. Then we get an increasing subsequence $(w_{2k}) = ((-2)^{2k}) = (4^k)$ of (w_n). (b) The set of subsequential limits is $\{+\infty, -\infty\}$. (c) $\limsup w_n = +\infty$ and $\liminf w_n = -\infty$. (d) (w_n) neither converges nor diverges to $+\infty$ or $-\infty$. (e) (w_n) is unbounded. □

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$. This means that every element in $\mathbb{Q} \cap (0, 1]$ appears in the sequence exactly once.

(a) Give the set of subsequential limits for (q_n).

(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

Proof. (a) The set of subsequential limits for (q_n) is $[0, 1]$. Since (q_n) is bounded above by 1, any subsequence is also bounded above by 1. By Exercise 8.9, if a subsequence has a limit, then the limit is ≤ 1. Since (q_n) is bounded below by 0, any subsequence is also bounded below by 0. By Exercise 8.9, if a subsequence has a limit, then the limit is ≥ 0. Thus, the set of subsequential limits of (q_n) is a subset of $[0, 1]$.

Next, we show that any $x \in [0, 1]$ is a subsequential limit of (q_n). By Theorem 11.2, it suffices to show that for any $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains infinitely many elements in (q_n). Since (q_n) is an enumeration of all the rationals in the interval $(0, 1]$, this is equivalent to that $(x - \varepsilon, x + \varepsilon) \cap (0, 1]$ contains infinitely many rational numbers. We may choose $0 \leq a < b \leq 1$ such that $(a, b) \subset (x - \varepsilon, x + \varepsilon) \cap (0, 1]$. Then it suffices to show that the interval (a, b) contains infinitely many rationals. To see this, we define $a_n = b - \frac{b-a}{2^n}$ for $n = 0, 1, 2, \ldots$. We then have $b - a_n = \frac{b-a}{2^n}$. Since $b - a > 0$, we have $b - a = b - a_0 > b - a_1 > b - a_2 > \cdots > 0$. Thus, $a = a_0 < a_1 < a_2 < \cdots < b$. So $(a_0, a_1), (a_1, a_2), (a_2, a_3), \ldots$ are mutually disjoint open subintervals of (a, b). By denseness of $\mathbb{Q}$, each of them contains at least one rational. Since these intervals are mutually disjoint, those rationals are different from each other. So (a, b) contains infinitely many rationals. This concludes the proof of (a)

(b) Since the set of subsequential limits for (q_n) is $[0, 1]$, by Theorem 11.8, $\limsup q_n = 1$ and $\liminf q_n = 0$. □

11.8 Use Definition 10.6 and Exercise 5.4 to prove $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n).

Proof. By definition,

$$\liminf s_n = \lim_{N \to \infty} \inf\{s_n : n > N\};$$

$$\limsup(-s_n) = \lim_{N \to \infty} \sup\{-s_n : n > N\}.$$

By Exercise 5.4, for every N,

$$\inf\{s_n : n > N\} = -\sup\{-s_n : n > N\}.$$

By Theorem 9.2, we get the conclusion. □

11.9 (a) Show the closed interval $[a, b]$ is a closed set.

(b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

Proof. (a) We need to show that if (s_n) is a convergent sequence of points in $[a, b]$, then $\lim s_n \in [a, b]$. Since $s_n \le b$ for all n, by Exercise 8.0, $\lim s_n \le b$. Since $s_n \ge a$ for all n, by Exercise 8.9 again, $\lim s_n \ge a$. Thus, $\lim s_n \in [a, b]$.

(b) There does not exist a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits. Suppose such sequence (s_n) exists. Then by Theorem 11.9, $(0, 1)$ is a closed set. This is a contradiction because $(0, 1)$ is not a closed set. To see this, we construct a sequence (t_n) from $(0, 1)$ by $t_n = \frac{1}{n+1}$, and note that $t_n \to 0 \notin (0, 1)$. □

12.4 Show $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n).

Proof. By definition,

$$\limsup(s_n + t_n) = \lim_{N\to\infty} \sup\{s_n + t_n : n > N\};$$

$$\limsup s_n = \sup\{s_n : n > N\};$$

$$\limsup t_n = \sup\{t_n : n > N\}.$$

We claim that

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

For any fixed $n > N$, we have $s_n \le \sup\{s_n : n > N\}$ and $t_n \le \sup\{t_n : n > N\}$, and so

$$s_n + t_n \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Since this holds for any $n > N$, the claim is proved (because the RHS is an upper bound of $\{s_n + t_n : n > N\}$).

Since (s_n) and (t_n) are bounded, $(\sup\{s_n : n > N\})_N$ converges to $\limsup s_n$, and $(\sup\{t_n : n > N\})_N$ converges to $\limsup t_n$. So $(\sup\{s_n : n > N\} + \sup\{t_n : n > N\})_N$ converges to $\limsup s_n + \limsup t_n$. Since $(\sup\{s_n + t_n : n > N\})$ converges to $\limsup(s_n + t_n)$, by the above claim and Exercise 9.9 (c), we get the desired inequality. □

12.6 Let (s_n) be a bounded sequence, and let k be a nonnegative real number.

(a) Prove $\limsup(ks_n) = k \limsup s_n$.

(b) Do the same for lim inf. Hint: Use Exercise 11.8.

(c) What happens in (a) and (b) if $k < 0$?

Proof. (a) Since $k \geq 0$, for every $N \in \mathbb{N}$,

$$\sup\{ks_n : n > N\} = k \sup\{s_n : n > N\}.$$

So by Theorem 9.2,

$$\limsup ks_n = \lim_{N\to\infty} \sup\{ks_n : n > N\} = \lim_{N\to\infty} k \sup\{s_n : n > N\} = k \limsup s_n.$$

(b) Since $k \geq 0$, for every $N \in \mathbb{N}$,

$$\inf\{ks_n : n > N\} = k \inf\{s_n : n > N\}.$$

So by Theorem 9.2,

$$\liminf ks_n = \lim_{N\to\infty} \inf\{ks_n : n > N\} = \lim_{N\to\infty} k \inf\{s_n : n > N\} = k \liminf s_n.$$

(c) If $k < 0$, then $\limsup(ks_n) = k \liminf s_n$ and $\liminf(ks_n) = k \limsup s_n$. This is because for every $N \in \mathbb{N}$,

$$\sup\{ks_n : n > N\} = k \inf\{s_n : n > N\}, \quad \inf\{ks_n : n > N\} = k \sup\{s_n : n > N\}.$$

□

12.10 Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

Proof. First suppose (s_n) is bounded. Then there is $M \in (0, \infty)$ such that $|s_n| \leq M$ for all n. So any subsequence of $(|s_n|)$ is bounded above by M. By Exercise 8.9, any subsequential limit of $(|s_n|)$ is bounded above by M. By Theorem 11.7, $\limsup |s_n|$ is a subsequential limit of $(|s_n|)$. So $\limsup |s_n| \leq M < +\infty$. On the other hand, suppose $\limsup |s_n| < +\infty$. Let $a = \limsup |s_n|$. By definition, $a = \lim_{N\to\infty} \sup\{|s_n| : n > N\}$. So there is some $N \in \mathbb{N}$ such that $\sup\{|s_n| : n > N\} < a + 1$. Thus, $\{|s_n| : n \in \mathbb{N}\}$ is bounded above by $\max\{a + 1, |s_1|, |s_2|, \ldots, |s_N|\}$, which is finite. So (s_n) is bounded. □

E1 Let (s_n) be a sequence of real numbers and $x \in \mathbb{R}$. Prove that (a) if $\limsup s_n > x$, then there are infinitely many n such that $s_n > x$; (b) if there are infinitely many n such that $s_n \geq x$, then $\limsup s_n \geq x$.

Proof. By definition, $\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$. Since the sequence $(\sup\{s_n : n > N\})$ is decreasing, $\limsup s_n$ can also be expressed by $\inf\{\sup\{s_n : n > N\} : N \in \mathbb{N}\}$.

(a) If $\limsup s_n > x$, then for every $N \in \mathbb{N}$, $\sup\{s_n : n > N\} \geq \inf\{\sup\{s_n : n > N\} : N \in \mathbb{N}\} = \limsup s_n > x$. This then implies that x is not an upper bound of $\{s_n : n > N\}$. So there is some $n > N$ such that $s_n > x$. Thus, for every $N \in \mathbb{N}$, there is $n > N$ such that $s_n > x$. So we conclude that there are infinitely many n such that $s_n > x$.

(b) If there are infinitely many n such that $s_n \geq x$, then for any $N \in \mathbb{N}$, there is some $n > N$ such that $s_n \geq x$, and so we have $\sup\{s_n : n > N\} \geq x$. By Exercise 9.9, we get $\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} \geq x$. □