Probability Theory and Related Fields
https://doi.org/10.1007/s00440-018-0895-0

Optimal Hölder continuity and dimension properties for SLE with Minkowski content parametrization

Dapeng Zhan[1]

Received: 1 May 2018 / Revised: 13 November 2018

Abstract
We make use of the fact that a two-sided whole-plane Schramm–Loewner evolution (SLE_κ) curve γ for $\kappa \in (0, 8)$ from ∞ to ∞ through 0 may be parametrized by its d-dimensional Minkowski content, where $d = 1 + \frac{\kappa}{8}$, and become a self-similar process of index $\frac{1}{d}$ with stationary increments. We prove that such γ is locally α-Hölder continuous for any $\alpha < \frac{1}{d}$. In the case $\kappa \in (0, 4]$, we show that γ is not locally $\frac{1}{d}$-Hölder continuous. We also prove that, for any deterministic closed set $A \subset \mathbb{R}$, the Hausdorff dimension of $\gamma(A)$ almost surely equals d times the Hausdorff dimension of A.

Mathematics Subject Classification 60G · 30C

1 Introduction

1.1 Overview

The Schramm–Loewner evolution (SLE_κ), introduced by Oded Schramm in 1999 [27], is a one-parameter ($\kappa \in (0, \infty)$) family of probability measures on non-self-crossing curves, which has received a lot of attention since then. It has been shown that, modulo time parametrization, the interface of several discrete lattice models have SLE_κ with different parameters κ as their scaling limits. The reader may refer to [8,26] for basic properties of SLE.

The regularity property of SLE curves have been studied by a number of authors. Rohde and Schramm proved in [26] that, for $\kappa \neq 8$, an SLE_κ curve exists and is Hölder continuous in its original capacity parametrization. Lind improved the estimates by Rohde and Schramm and derived a better Hölder exponent for SLE in the capacity parametrization [19], which was later proved [7] to be optimal.

✉ Dapeng Zhan
zhan@math.msu.edu

1 Michigan State University, East Lansing, USA

People tried to find better Hölder exponents of SLE curves with other parametrizations. It is know that [1,26] the Hausdorff dimension of SLE_κ is $\min\{1+\frac{\kappa}{8},2\}=:d$. Thus an SLE_κ curve can not be reparametrized to be Hölder continuous of any order greater than $1/d$. On the other hand, using a variation estimate, Werness proved [28] that, for $\kappa\le 4$, for any $\alpha<1/d$, an SLE_κ curve may be reparametrized to be Hölder continuous of order α. The result was later extended to all $\kappa\ne 8$ by Friz and Tran [2].

The natural parametrization of SLE_κ is defined for $\kappa\in(0,8)$ [13,17], and is expected to be the scaling limit of the natural length of various lattice models, while the convergence has been established for loop-erased random walk [14,15]. Lawler and Rezaei proved [9] that the natural parametrization of an SLE_κ curve agrees with the d-dimensional Minkowski content of the curve. So the natural parametrization is determined by the SLE_κ curve.

It was proven in [9] that the transition function Θ between the capacity parametrization and the natural parametrization is Hölder continuous, where Θ_t is defined to be the Minkowski content of the SLE_κ curve under capacity parametrization during the time interval $[0,t]$, i.e., if γ is an SLE curve with capacity parametrization, then $\gamma\circ\Theta^{-1}$ is the same SLE curve with natural parametrization. To get any Hölder regularity of SLE in the natural parametrization based on the Hölder regularity of SLE in the capacity parametrization, one needs the Hölder continuity of Θ^{-1}, which was not derived in [9].

1.2 Main results

We fix $\kappa\in(0,8)$ and let $d=1+\frac{\kappa}{8}$. The main purpose of this paper is to solve a conjecture proposed by Greg Lawler: an SLE_κ curve under Minkowski content parametrization is locally Hölder continuous of any order less than $1/d$. We will discard the boundary effect, and focus on the Hölder continuity of any compact subcurve that is bounded away from the boundary.

There are several different versions of SLE. Thanks to the local equivalence between them (cf. [11]), we have the freedom to work on any version of SLE_κ in any domain. The main object studied in this paper is a two-sided whole-plane SLE_κ curve γ in the Riemann sphere $\widehat{\mathbb{C}}$ from ∞ to ∞ passing through 0. The nice property of such SLE curve is that the Minkowski content parametrization makes it a self-similar process with stationary increment.

Proposition 1.1 [29, Corollary 4.7] *A two-sided whole-plane SLE_κ curve γ from ∞ to ∞ passing through* 0 *may be parametrized by its d-dimensional Minkowski content, and become a self-similar process of index $\frac{1}{d}$ with stationary increments.*

We will refer the γ in Proposition 1.1 as an sssi SLE_κ curve. Here we say that a curve γ is parametrized by d-dimensional Minkowski content, if for any $a<b$ in the definition domain of γ, the d-dimensional Minkowski content of $\gamma([a,b])$ is $b-a$. A random curve γ is called a self-similar process of index H with stationary increments if it is defined on $\mathbb{R}$ with $\gamma(0)=0$ such that (i) for any $a>0$, $(\gamma(at))$ has the same law as $(a^H\gamma(t))$; and (ii) for any $a\in\mathbb{R}$, $(\gamma(a+t)-\gamma(a))$ has the same law as $(\gamma(t))$.

Theorem 1.2 *An sssi* SLE_κ *curve is a.s. locally Hölder continuous of any order less than* $1/d$.

The following theorem resembles Mckean's dimension theorem for Brownian motion [20]. We use $\dim_H$ to denote the Hausdorff dimension. It is closely related to the work in [5,6], which proves analogous results when the Liouville quantum gravity variant of the Minkowski content is used.

Theorem 1.3 *For an sssi* SLE_κ *curve* γ *and any deterministic closed set* $A \subset \mathbb{R}$, *almost surely*

$$\dim_H(\gamma(A)) = d \cdot \dim_H(A). \tag{1.1}$$

Theorem 1.4 *If* $\kappa \in (0, 4]$, *an sssi* SLE_κ *curve* γ *is a.s. not* $1/d$*-Hölder continuous on any open interval.*

Corollary 1.5 *Let* γ *be a chordal* SLE_κ, *radial* SLE_κ, *or* $SLE_\kappa(\rho)$ *curve,* $\kappa \in (0, 8)$, *in any simply connected domain* D. *Let* β *be a compact subcurve of* γ, *which has positive distance from* ∂D *and the set of marked points of* γ *(including the force points and the target). If* β *is parametrized by its* d*-dimensional Minkowski content measure, then it is a.s. Hölder continuous of any order less than* $1/d$, *and a.s. not* $1/d$*-Hölder continuous when* $\kappa \leq 4$.

Proof This follows from Theorems 1.2 and 1.4, the conformal covariance of Minkowski content (Proposition 2.2) and the local equivalence between different versions of SLE. □

Remark 1.6 Theorems 1.2, 1.3 and 1.4 also hold for $\kappa = 8$. See Remark 4.4. We expect that Theorem 1.4 also holds for $\kappa \in (4, 8)$. It is not known to the author whether an SLE_κ curve possesses some parametrization that makes it exactly locally $1/d$-Hölder continuous.

1.3 Some proofs

The proofs of the main theorems follow some standard arguments. We now give the proofs of Theorems 1.2 and 1.3 assuming that the following finite moment lemma holds.

Lemma 1.7 *If* γ *is an sssi* SLE_κ *curve, then for any* $c \in (-d, \infty)$, $\mathbb{E}[|\gamma(1)|^c] < \infty$.

Proof of Theorem 1.2. Let $\alpha < 1/d$. Let $I \subset \mathbb{R}$ be a bounded interval. We use the Garsia–Rodemich–Rumsey inequality [4] in the following form: For any $p > 1/\alpha$, there exists a constant $C_{\alpha,p} \in (0, \infty)$ such that for any $s, t \in I$,

$$|\gamma(t) - \gamma(s)|^p \leq C_{\alpha,p}|t - s|^{\alpha p - 1} \int_I \int_I \frac{|\gamma(s) - \gamma(t)|^p}{|s - t|^{\alpha p + 1}} dsdt. \tag{1.2}$$

Using Proposition 1.1, Lemma 1.7 and Fubini Theorem, we see that

$$\mathbb{E}\left[\int_I\int_I \frac{|\gamma(s)-\gamma(t)|^p}{|s-t|^{\alpha p+1}}\,dsdt\right] = \int_I\int_I \frac{\mathbb{E}[|\gamma(s)-\gamma(t)|^p]}{|s-t|^{\alpha p+1}}\,dsdt$$
$$= \int_I\int_I \frac{|s-t|^{p/d}\mathbb{E}[|\gamma(1)|^p]}{|s-t|^{\alpha p+1}}\,dsdt$$
$$= \mathbb{E}[|\gamma(1)|^p]\int_I\int_I |s-t|^{(1/d-\alpha)p-1}dsdt < \infty.$$

Thus, a.s. $\int_I\int_I \frac{|\gamma(s)-\gamma(t)|^p}{|s-t|^{\alpha p+1}}dsdt < \infty$. From (1.2) we see that γ is a.s. Hölder continuous of order $\alpha - 1/p$ on I. We can now finish the proof by choosing α close to $1/d$ and p arbitrarily big. □

Remark 1.8 We actually proved that, for any $\alpha < 1/d$, the α-Hölder norm of γ restricted to a bounded interval has finite p-moment for any $p > 0$.

Proof of Theorem 1.3. Since γ is a.s. locally α-Hölder continuous for any $\alpha < 1/d$, we get the upper bound of (1.1): $\dim_H(\gamma(A)) \le d \cdot \dim_H(A)$ (cf. [22, Remark 4.12]).

For the lower bound, we use a corollary of Frostman's Lemma (cf. [3]). Let $\mathcal{P}(E)$ denote the space of probability measures on a metric space (E, ρ). For $\mu \in \mathcal{P}(E)$ and $\alpha \ge 0$, the α-energy of μ is

$$I_\alpha(\mu) := \int_E\int_E \frac{\mu(dx)\mu(dy)}{\rho(x,y)^\alpha}.$$

Frostman's lemma implies that (cf. [22, Theorem 4.36]) for any closed set $A \subset \mathbb{R}^n$,

$$\dim_H(A) = \sup\{\alpha : \exists \mu \in \mathcal{P}(A) \text{ with } I_\alpha(\mu) < \infty\}. \tag{1.3}$$

We now prove the lower bound of (1.1). Let $A \subset \mathbb{R}$ be closed. By replacing A with $A \cap [-n, n]$, we may assume that A is compact. If $\dim_H(A) = 0$, the statement of lower bound is trivial. Suppose $\dim_H(A) > 0$. Let $\alpha \in (0, \dim_H(A))$. By (1.3), there is a $\mu \in \mathcal{P}(A)$ such that $I_\alpha(\mu) < \infty$. Now $\gamma(A)$ is a compact subset of $\mathbb{C}$, and $\gamma_*(\mu) \in \mathcal{P}(\gamma(A))$. We compute that

$$\mathbb{E}[I_{d\alpha}(\gamma_*(\mu))] = \int_A\int_A \mathbb{E}[|\gamma(s)-\gamma(t)|^{-d\alpha}]\mu(ds)\mu(dt)$$
$$= I_\alpha(\mu)\mathbb{E}[|\gamma(1)|^{-d\alpha}] < \infty,$$

using Proposition 1.1, Lemma 1.7 and the fact that $-d\alpha > -d$, which follows from $\alpha < \dim_H(A) \le 1$. Thus, we have a.s. $I_{d\alpha}(\gamma_*(\mu)) < \infty$. From (1.3), we then get a.s. $\dim_H(\gamma(A)) \ge d\alpha$. By choosing α arbitrarily close to $\dim_H(A)$, we finish the proof of the lower bound. □

The rest of the paper is devoted to proving Lemma 1.7 and Theorem 1.4. We review some preliminary facts on two-sided whole-plane SLE and Minkowski content in

Sect. 2, and then prove Lemma 1.7 and Theorem 1.4 in Sect. 3 assuming two lemmas about crossing estimates for radial $\mathrm{SLE}_\kappa(2)$ curves. In Sect. 4, we prove these crossing estimate lemmas. At the end, we discuss sssi SLE_κ curves for $\kappa \ge 8$ and the Hausdorff measure of SLE_κ curves for $\kappa < 8$.

2 Preliminary

2.1 Two-sided whole-plane SLE

A two-sided whole-plane SLE_κ curve ($\kappa \in (0,8)$) is composed of two arms: the first arm is a whole-plane $\mathrm{SLE}_\kappa(2)$ curve from a marked point in $\widehat{\mathbb{C}}$, say a, to another marked point in $\widehat{\mathbb{C}}$, say b; and given the first arm, the second arm is a chordal SLE_κ curve from b to a in one complement domain of the first arm.

A whole-plane $\mathrm{SLE}_\kappa(2)$ curve γ grows in $\widehat{\mathbb{C}}$ from one (interior) point, say a, to another point, say b, with the force point located at the initial point a. It is related with radial $\mathrm{SLE}_\kappa(2)$ processes by the domain Markov property: if τ is a stopping time for γ, which happens after γ leaves a and before γ reaches b, then conditionally on the part of γ before τ, the rest of the γ is a radial $\mathrm{SLE}_\kappa(2)$ curve in a complement domain of the explored portion of γ at time τ from $\gamma(\tau)$ to b with the force point located at a. Here a lies on the boundary of that domain, and determines a prime end. The above also holds with 2 replaced by some $\rho \ge \frac{\kappa}{2} - 2$.

We now recall the definition of a radial $\mathrm{SLE}_\kappa(2)$ curve. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and $\mathbb{T} = \partial \mathbb{D}^* = \{|z| = 1\}$. Let a, b be distinct points on $\mathbb{T}$. Let $x, y \in \mathbb{R}$ be such that $a = e^{ix}$, $b = e^{iy}$ and $x - y \in (0, 2\pi)$. Let (B_t) be a standard Brownian motion. Let (λ_t) and (q_t) be the solution of the SDE:

$$\begin{cases} d\lambda_t = \sqrt{\kappa} dB_t + \cot((\lambda_t - q_t)/2)dt, & \lambda_0 = x \\ dq_t = \cot((q_t - \lambda_t)/2)dt, & q_0 = y. \end{cases} \tag{2.1}$$

Let g_t, $t \ge 0$, be the solution of the following radial Loewner equation:

$$\partial_t g_t(z) = g_t(z) \cdot \frac{e^{i\lambda_t} + g_t(z)}{e^{i\lambda_t} - g_t(z)}, \quad g_0(z) = z.$$

For each $t \ge 0$, let K_t denote the set of $z \in \mathbb{D}^*$ such that the solution $s \mapsto g_s(z)$ blows up before or at time t. Then g_t maps $\mathbb{D}^* \setminus K_t$ conformally onto $\mathbb{D}^*$, fixes ∞, and satisfies $g_t(z)/z \to e^{-t}$ as $z \to \infty$. It turns out that, for each $t \ge 0$, g_t^{-1} extends continuously to $\overline{\mathbb{D}^*}$, and $\gamma(t) := g_t^{-1}(e^{i\lambda_t})$, $t \ge 0$, is a continuous curve in $\overline{\mathbb{D}^*}$ that starts from $e^{i\lambda_0} = e^{ix} = a \in \mathbb{T}$, and tends to ∞ as $t \to \infty$. Moreover, for each $t \ge 0$, $\mathbb{D}^* \setminus K_t$ is the unbounded component of $\mathbb{D}^* \setminus \gamma([0,t])$.

Such γ is called a radial $\mathrm{SLE}_\kappa(2)$ curve in $\mathbb{D}^*$ from a to ∞ with a (value 2) force point at b; λ_t and q_t are called the driving process and force point process for γ, respectively; and g_t are called the radial Loewner maps for γ.

If D is a simply connected domain with two distinct marked prime ends a, b and one marked interior point z_0, there exist $a' \ne b' \in \mathbb{T}$ and a conformal map f from

$(\mathbb{D}^*; a', b'; \infty)$ onto $(D; a, b; z_0)$. Then the f-image of a radial $\mathrm{SLE}_\kappa(2)$ curve in $\mathbb{D}^*$ from a' to ∞ with a force point at b' is called a radial $\mathrm{SLE}_\kappa(2)$ curve in D from a to z_0 with a force point at b.

The radial $\mathrm{SLE}_\kappa(2)$ curve γ in D satisfies the following domain Markov property: if T is a stopping time for γ that happens before z_0 is reached, then conditional on the part of γ before T, the part of γ after T is a radial $\mathrm{SLE}_\kappa(2)$ curve from $\gamma(T)$ to z_0 with a force point at b in the complement domain of $\gamma([0, T])$ in D that contains z_0.

The above radial $\mathrm{SLE}_\kappa(2)$ curve γ also acts the role of the first arm of a two-sided radial SLE_κ curve in D from a to b through z_0, which is composed of two arms. Conditional on the first arm, the second arm is a chordal SLE_κ curve from z_0 to b in the complement domain of the first arm, whose boundary contains b.

Throughout, we use $\mu^\#_{D;a\to b}$ to denote the law of a chordal SLE_κ curve in D from a to b; use $\nu^\#_{D;a\to z_0;b}$ to denote the law of a radial $\mathrm{SLE}_\kappa(2)$ curve in D from a to z_0 with a force point at b; and use $\nu^\#_{D;a\to z_0\to b}$ to denote the law of a two-sided radial SLE_κ curve in D from a to b passing through z_0, all modulo time parametrization.

The Green's function for the chordal SLE_κ curve in D from a to b is the function $G_{D;a,b}(z)$ defined by

$$G_{D;a,b}(z_0) = \lim_{r\downarrow 0} r^{d-2}\mu^\#_{D;a\to b}\{\gamma : \mathrm{dist}(\gamma, z_0) < r\}, \quad z_0 \in D.$$

The limit is known to exist, and satisfies conformal covariance, i.e., if f maps $(D_1; a_1, b_1)$ conformally onto $(D_2; a_2, b_2)$, then

$$G_{D_1;a_1,b_1}(z) = |f'(z)|^{2-d} G_{D_2;a_2,b_2}(f(z)), \quad z \in D_1. \tag{2.2}$$

2.2 Minkowski content measure

In this subsection, we review the notion of Minkowski content, Minkowski content measure, and cite some propositions from [29, Sections 2.3 and 2.4].

Throughout, the Minkowski content will always be d-dimensional. Since $d = 1+\frac{\kappa}{8}$ is fixed, we will omit the word "d-dimensional". Let m and m^2 denote the 1 and 2-dimensional Lebesgue measures, respectively. Let $S \subset \mathbb{C}$ be a closed set. The Minkowski content of S is defined to be $\mathrm{Cont}(S) = \lim_{r\downarrow 0} r^{d-2}\mathrm{m}^2(\bigcup_{z\in S} B(z; r))$, where $B(z; r)$ is the open disc with radius r centered at z, provided that the limit exists.

Definition 2.1 Let $S, U \subset \mathbb{C}$. Suppose $\mathcal{M}$ is a measure supported by $S \cap U$ such that for every compact set $K \subset U$, $\mathrm{Cont}(K \cap S) = \mathcal{M}(K) < \infty$. Then we say that $\mathcal{M}$ is the Minkowski content measure on S in U, or S possesses Minkowski content measure in U, and denote the measure by $\mathcal{M}_{S;U}$. If $U = \mathbb{C}$, we may omit the phrase "in U", and write the measure as $\mathcal{M}_S$.

Proposition 2.2 [29, Lemma 2.6] *Suppose that S possesses Minkowski content measure $\mathcal{M}_{S;U}$ in an open set $U \subset \mathbb{C}$. Suppose f is a conformal map defined on U such that $f(U) \subset \mathbb{C}$. Then the Minkowski content measure of $f(S \cap U)$ in $f(U)$ exists, which is absolutely continuous w.r.t. $f_*(\mathcal{M}_{S;U})$, and the Radon–Nikodym derivative is $|f'(f^{-1}(\cdot))|^d$.*

It was proved in [9] that the natural parametrization function Θ_t of a chordal SLE$_\kappa$ curve γ (under capacity parametrization) in $\mathbb{H}$ from 0 to ∞ at any time t agrees with the Minkowski content of $\gamma([0,t])$. Using their result, one can easily show that such curve γ possesses Minkowski content measure in $\mathbb{H}$, which is the pushforward measure of $d\Theta$ under the function γ. The existence of Minkowski content measure extends to any simply connected domain [29, Remark 2.7], although the natural parametrization (in the usual sense) does not always exist because the Minkowski content of $\gamma([0,t])$ may be infinite due to the roughness of the domain boundary. Moreover, the following proposition holds.

Proposition 2.3 [29, Proposition 2.8] *Let D be a simply connected domain with two distinct prime ends a and b. Then*

$$\mu^{\#}_{D;a\to b}(d\gamma)\otimes \mathcal{M}_{\gamma;D}(dz)=\nu^{\#}_{D;a\to z\to b}(d\gamma)\overleftarrow{\otimes}(G_{D;a,b}\cdot \mathrm{m}^2)(dz).$$

This proposition first appeared in [30, Corollary 4.3] (in a slightly different form). It means that, there are two ways to sample a random curve-point pair (γ, z) according to the same measure. Method 1: first sample a chordal SLE$_\kappa$ curve γ in D from a to b, and then sample a point z on γ according to the Minkowski content measure of γ in D. Method 2: first sample a point z according to the density $G_{D;a,b}$, and then sample a two-sided radial SLE$_\kappa$ curve γ in D from a to b passing through z. The reader is referred to [29, Section 2.1] for the meaning of the symbols $\otimes$ and $\overleftarrow{\otimes}$ for the products between a measure and a kernel.

3 Finite Moments

We prove Lemma 1.7 and Theorem 1.4 in this section. We divide the proof of Lemma 1.7 into two subsections according to the sign of the exponent c.

Let γ be as in Proposition 1.1. Let $\gamma_+=\gamma|_{[0,\infty)}$ and $\gamma_-=\gamma|_{(-\infty,0]}$ be the two arms of γ, both connecting 0 with ∞. Let $\widehat{\mathbb{C}}_{\gamma_-}$ be the unbounded connected component of $\mathbb{C}\backslash\gamma_-$. Recall that γ_- is a whole-plane SLE$_\kappa(2)$ curve from ∞ to 0; and given γ_-, γ_+ is a chordal SLE$_\kappa$ curve from 0 to ∞ in $\widehat{\mathbb{C}}_{\gamma_-}$. Let $\gamma_-^R(t)=\gamma(-t)$, $0\le t<\infty$, be the time-reversal of γ_-. From the reversibility of whole-plane SLE$_\kappa(2)$ [21, Theorem 1.20], we know that γ_-^R is a whole-plane SLE$_\kappa(2)$ curve from 0 to ∞.

For $0\le t<\infty$, let D_t denote the complement domain of $\gamma_-^R([0,t])$ in $\widehat{\mathbb{C}}$ that contains ∞. Then $K_t:=\widehat{\mathbb{C}}\backslash D_t$ is an interior hull in $\mathbb{C}$. An interior hull in $\mathbb{C}$ is a nonempty connected compact set $K\subset\mathbb{C}$ such that $\mathbb{C}\backslash K$ is connected. It is called nondegenerate if it contains more than one point. For a nondegenerate interior hull K, there exists a unique conformal map g_K from $\widehat{\mathbb{C}}\backslash K$ onto $\mathbb{D}^*$ such that $g_K(\infty)=\infty$ and $g_K'(\infty):=\lim_{z\to\infty} z/g_K(z)>0$. The capacity of K is defined to be $\mathrm{cap}(K)=\log(g_K'(\infty))$. If K is degenerate, i.e., a singlet, we define $\mathrm{cap}(K)=-\infty$. From Schwarz Lemma and Koebe 1/4 Theorem, we know that $e^{\mathrm{cap}(K)}\le \mathrm{diam}(K)\le 4e^{\mathrm{cap}(K)}$ for any interior hull K in $\mathbb{C}$.

Let $(\mathcal{F}_t)_{t\ge0}$ denote the filtration generated by $(\gamma_-^R(t))_{t\ge0}$. If $T\in(0,\infty)$ is an $(\mathcal{F}_t)$-stopping time, then conditional on $\mathcal{F}_T$, the part of γ_-^R after T is an SLE$_\kappa(2)$ curve in D_T from $\gamma_-^R(T)$ to ∞ with a force point at 0. Let $u(t) = \mathrm{cap}(K_t)$. Then u is an increasing homeomorphism from $[0,\infty)$ onto $[-\infty,\infty)$ and for every $r \in \mathbb{R}$, $u^{-1}(r)$ is an $(\mathcal{F}_t)$-stopping time.

For $R, r > 0$, let τ_R denote the hitting time (i.e., the first visiting time) of the circle $\{|z| = R\}$; and let τ_r^R denote the last visiting time of the circle $\{|z| = r\}$ before τ_R. Here we set τ_R to be ∞ when the former time does not exist, and set τ_r^R to be τ_R when the latter time does not exist. Note that τ_R is a stopping time, but τ_r^R is not in general. For any $R > 0$, since $K_{\tau_R} \subset \{|z| \le R\}$ and $\mathrm{diam}(K_{\tau_R}) \ge R$, we have $\log(R) \ge \mathrm{cap}(K_{\tau_R}) \ge \log(R/4)$, and so $\tau_R \le u^{-1}(\log R) \le \tau_{4R}$.

3.1 Negative power

Proof of Lemma 1.7 for $c \le 0$. Let $f_c(z) = \mathbf{1}_{\{|z|\le 1\}}|z|^c$. We will show that

$$\mathbb{E}\left[\int_0^\infty f_c(\gamma(t))dt\right] < \infty. \tag{3.1}$$

From the scaling property of γ, we get

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty f_c(\gamma(t))dt\right] &\ge \int_0^1 \mathbb{E}\left[f_c(\gamma(t))\right]dt = \int_0^1 \mathbb{E}\left[f_c(t^{1/d}\gamma(1))\right]dt\\ &= \int_0^1 \mathbb{E}\left[\mathbf{1}_{\{|\gamma(1)|\le t^{-1/d}\}}t^{c/d}|\gamma(1)|^c\right]dt \ge \mathbb{E}\left[f_c(\gamma(1))\right]\int_0^1 t^{c/d}dt.\end{aligned}$$

Thus, (3.1) implies that $\mathbb{E}[f_c(\gamma(1))] < \infty$, which then implies that $\mathbb{E}[|\gamma(1)|^c] < \infty$.

Since the whole-plane $\mathrm{SLE}_\kappa(2)$ curve $\gamma(t)$, $-\infty < t < 0$, visits its target $\gamma(0) = 0$ only at the end, we have a.s. $\gamma(0) \notin \gamma_- = \gamma((-\infty, 0))$. Since γ has stationary increment, we see that for any $t > 0$, a.s. $\gamma(t) \notin \gamma_-$. By Fubini Theorem, the Lebesgue measure of the set of $t > 0$ such that $\gamma(t) \in \gamma_-$ is a.s. zero. Since γ is parametrized by Minkowski content measure $\mathcal{M}_\gamma$, i.e., $\mathcal{M}_\gamma$ is the pushforward measure of the Lebesgue measure under the curve function γ, we then conclude that, a.s. the set $\gamma_+ \cap \gamma_-$ has Minkowski content measure zero. We know that γ_+ is contained in the closure of $\widehat{\mathbb{C}}_{\gamma_-}$, and when γ_+ touches the boundary of $\widehat{\mathbb{C}}_{\gamma_-}$, it intersects γ_-. Thus, the set of points on γ_+ that is not strictly contained in $\widehat{\mathbb{C}}_{\gamma_-}$ has zero Minkowski content measure almost surely. So we have a.s. $\mathcal{M}_{\gamma_+} = \mathcal{M}_{\gamma_+;\widehat{\mathbb{C}}_{\gamma_-}}$.

Since γ_+ is a chordal SLE_κ curve from 0 to ∞ in $\widehat{\mathbb{C}}_{\gamma_-}$, from Proposition 2.3 and that a.s. $\mathcal{M}_{\gamma_+} = \mathcal{M}_{\gamma_+;\widehat{\mathbb{C}}_{\gamma_-}}$, we get

$$\mu^{\#}_{\widehat{\mathbb{C}}_{\gamma_-};0\to\infty}(d\gamma_+) \otimes \mathcal{M}_{\gamma_+}(dz) = \nu^{\#}_{\widehat{\mathbb{C}}_{\gamma_-};0\to z\to\infty}(d\gamma_+) \overleftarrow{\otimes} \left(G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty} \cdot \mathrm{m}^2\right)(dz). \tag{3.2}$$

Since γ is parametrized by Minkowski content, we have $\int_0^\infty f_c(\gamma(t))dt = \int_{\mathbb{C}} f_c(z)\mathcal{M}_{\gamma_+}(dz)$. Since the law of γ_+ given γ_- is $\mu^\#_{\widehat{\mathbb{C}}_{\gamma_-};0\to\infty}$, using (3.2) we get

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{C}} f_c(z)\mathcal{M}_{\gamma_+}(dz)\Big|\gamma_-\right] &= \int f_c(z)\left[\mu^\#_{\widehat{\mathbb{C}}_{\gamma_-};0\to\infty}(d\gamma_+)\otimes\mathcal{M}_{\gamma_+}\right](dz)\\ &= \int f_c(z)\left[\nu^\#_{\widehat{\mathbb{C}}_{\gamma_-};0\to z\to\infty}\overleftarrow{\otimes}\left(G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}\cdot \mathrm{m}^2\right)\right](dz)\\ &= \int f_c(z)G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}(z)\,\mathrm{m}^2(dz).\end{aligned}$$

Thus, we have

$$\mathbb{E}\left[\int_0^\infty f_c(\gamma(t))dt\right] = \int_{\mathbb{C}} f_c(z)\mathbb{E}\left[G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}(z)\right]\mathrm{m}^2(dz). \tag{3.3}$$

We will prove that

$$\mathbb{E}\left[G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}(z_1)\right] < \infty, \quad \forall z_1\in\mathbb{C}\backslash\{0\}. \tag{3.4}$$

If this holds, by scaling and rotation invariance of γ_- and conformal covariance of chordal SLE$_\kappa$ Green's function, we see that $\mathbb{E}[G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}(z)] = C|z|^{d-2}$, $z\in\mathbb{C}\backslash\{0\}$, for some constant $C = C_\kappa\in(0,\infty)$. Combining it with (3.3), we get $\mathbb{E}[\int_0^\infty f_c(\gamma(t))dt] = C\int_{\{|z|\le 1\}}|z|^{c+d-2}\,\mathrm{m}^2(dz)$. Using $c>-d$, we get (3.1) and finish the proof.

Now it remains to prove (3.4). We claim that there are constants $C, R\in(1,\infty)$ such that the following holds. Let $D\subsetneq\mathbb{C}$ be a simply connected domain with distinct prime ends a, b. Let z_0, z_1 be distinct points in D such that $\mathrm{dist}(z_0,\partial D)\ge R|z_1-z_0|$. Let β have the law $\nu^\#_{D;a\to z_0;b}$, and D_β be the complement domain of β in D that has b as a prime end. Then

$$\mathbb{E}\left[G_{D_\beta;z_0,b}(z_1)\right]\le C|z_1-z_0|^{d-2} \tag{3.5}$$

If the claim holds true, we can finish the proof as follows. Fix $z_1\in\mathbb{C}\backslash\{0\}$. Let $T=\tau_{R|z_1|}$. Let D^T be the complement domain of $\gamma_-((-\infty,T])$ that contains 0. Conditional on the part of γ_- before T, the remaining part of γ_-, denoted by β_0, is a radial SLE$_\kappa(2)$ curve in D^T from $\gamma(T)$ to 0 with a force point at ∞. Then we have $(D^T)_{\beta_0} = \widehat{\mathbb{C}}_{\gamma_-}$, and $\mathrm{dist}(0,\partial D^T) = R|0-z_1|$. Applying the claim to $D=D^T$, $a=\gamma(T)$, $b=\infty$, $z_0=0$, and $\beta=\beta_0$, we get (3.4) because

$$\begin{aligned}\mathbb{E}\left[G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}(z_1)\right] &= \mathbb{E}\left[\mathbb{E}\left[G_{\widehat{\mathbb{C}}_{\gamma_-};0,\infty}(z_1)|\gamma_{(-\infty,T]}\right]\right]\\ &= \mathbb{E}\left[G_{(D^T)_{\beta_0};0,\infty}(z_1)\right]\le C|z_1|^{d-2}<\infty.\end{aligned}$$

To prove the claim, by Koebe distortion Theorem and the conformal covariance of chordal SLE$_\kappa$ Green's function, it suffices to work on the case that $D=\mathbb{H} := \{z\in$

$\mathbb{C} : \operatorname{Im} z > 0\}$, $a = 0$ and $b = \infty$. To estimate $\mathbb{E}[G_{\mathbb{H}_\beta;z_0,b}(z_1)]$, we use the formula

$$\overrightarrow{G}_{\mathbb{H};0,\infty}(z_0, z_1) = G_{\mathbb{H};0,\infty}(z_0)\mathbb{E}\left[G_{\mathbb{H}_\beta;z_0,\infty}(z_1)\right], \tag{3.6}$$

where $\overrightarrow{G}_{\mathbb{H};0,\infty}(z_0, z_1)$ is the *ordered* two-point Green's function for the chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞, i.e.,

$$\overrightarrow{G}_{\mathbb{H};0,\infty}(z_0, z_1) := \lim_{r_0,r_1\downarrow 0} r_0^{d-2} r_1^{d-2} \mu^\#_{\mathbb{H};0\to\infty}\left[\tau^{z_0}_{r_0} < \tau^{z_1}_{r_1} < \infty\right],$$

where $\tau^{z_j}_{r_j}$ is the hitting time of $\{|z - z_j| = r_j\}$.

Formula (3.6) was first derived in [16, Theorem 1], where the Euclidean distance is replaced with conformal radius. The Euclidean distance version of (3.6) was later derived in [9, Proposition 4.5]. A sharp bound for the *unordered* two-point Green's function:

$$G_{\mathbb{H};0,\infty}(z_0, z_1) := \lim_{r_0,r_1\downarrow 0} r_0^{d-2} r_1^{d-2} \mu^\#_{\mathbb{H};0\to\infty}\left[\tau^{z_0}_{r_0}, \tau^{z_1}_{r_1} < \infty\right]$$

is obtained in [10]. Combining it with (3.6) and the exact formula for $G_{\mathbb{H};0,\infty}(z_0)$, we then get an upper bound for $\mathbb{E}[G_{\mathbb{H}_\beta;z_0,\infty}(z_1)]$.

Now we show the computation. For convenience, we use an upper bound of $G_{\mathbb{H};0,\infty}(z_0, z_1)$ in the form derived in [25, Remark 2 after Theorem 1.2]:

$$G_{\mathbb{H};0,\infty}(z_0, z_1) \le C_2 \frac{y_0^{\alpha-(2-d)}}{P_{y_0}(l_0)} \cdot \frac{y_1^{\alpha-(2-d)}}{P_{y_1}(l_1)}, \tag{3.7}$$

where $C_2 \in (0, \infty)$ is a constant, $\alpha = \frac{8}{\kappa} - 1$, $y_j = \operatorname{Im} z_j$, $j = 0, 1$, $l_0 = |z_0|$, $l_1 = |z_1| \wedge |z_1 - z_0|$, and $P_y(x) = x^\alpha$ if $x \ge y$; $P_y(x) = y^{\alpha-(2-d)} x^{2-d}$ if $0 \le x \le y$. From [9, Theorem 4.2] we know that there is a constant $C_1 \in (0, \infty)$ such that

$$G_{\mathbb{H};0,\infty}(z_0) = C_1 y_0^{\alpha-(2-d)} |z|^{-\alpha} = C_1 \frac{y_0^{\alpha-(2-d)}}{P_{y_0}(l_0)}. \tag{3.8}$$

Since $\overrightarrow{G}_{\mathbb{H};0,\infty}(z_0, z_1) \le G_{\mathbb{H};0,\infty}(z_0, z_1)$, combining (3.6,3.7,3.8) we get

$$\mathbb{E}\left[G_{\mathbb{H}_\beta;z_0,\infty}(z_1)\right] \le \frac{C_2}{C_1} \cdot \frac{y_1^{\alpha-(2-d)}}{P_{y_1}(l_1)}.$$

Suppose $y_0 = \operatorname{dist}(z_0, \mathbb{R}) \ge R|z_1 - z_0|$ with $R = 2$. Then $|z_1 - z_0| \le y_0 - |z_1 - z_0| \le y_1 \le |z_1|$. So $l_1 = |z_1 - z_0| \le y_1$, and the above displayed formula implies (3.5) because $P_{y_1}(l_1) = y_1^{\alpha-(2-d)} |z_1 - z_0|^{2-d}$. So the proof is complete. □

3.2 Positive power

Let $\mathcal{D}$ denote the set of simply connected domains $D \subsetneq \widehat{\mathbb{C}}$ such that $\widehat{\mathbb{C}} \setminus D$ is a nondegenerate interior hull in $\mathbb{C}$ containing 0. Note that $\infty \in D$ for $D \in \mathcal{D}$.

Lemma 3.1 *There exist $R > r > 4$ such that for any $p > 0$ there exists $l > 0$ such that for any $D \in \mathcal{D}$ with distinct prime ends a, b,*

$$\nu^{\#}_{D;a\to\infty;b}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau^{Re^{\mathrm{cap}(D^c)}}_{re^{\mathrm{cap}(D^c)}}, \tau_{Re^{\mathrm{cap}(D^c)}}\right]\right)\right) < l * e^{d*\mathrm{cap}(D^c)}\right\}\right) < p.$$

We now use Lemma 3.1 to prove Lemma 1.7 for positive powers, and postpone the proof of Lemma 3.1 to the next section.

Proof of Lemma 1.7 for $c > 0$. We work on the whole-plane $\mathrm{SLE}_\kappa(2)$ curve γ^R_- from 0 to ∞. By the stationary increments of γ, $\mathrm{diam}(\gamma([0,1]))$ has the same law as $\mathrm{diam}(\gamma^R_-([0,1]))$. So it suffices to show that $\mathrm{diam}(\gamma^R_-([0,1]))$ has finite moments of any positive order.

Let $R > r \in (4, \infty)$ be given by Lemma 3.1. Fix $p > 0$. Let $l = l(p, r, R) > 0$ be given by Lemma 3.1. Let $s_n = \log(R^n/l^{1/d})$ for $n \in \mathbb{N} \cup \{0\}$. Applying Lemma 3.1 to $D = D_{u^{-1}(s_k)}$, $a = \gamma^R_-(u^{-1}(s_k))$ and $b = 0$, we get

$$\mathbb{P}\left[u^{-1}(s_{k+1}) - u^{-1}(s_k) < l * e^{d*s_k} | \mathcal{F}_{u^{-1}(s_k)}\right] < p, \quad k = 0, 1, 2, \ldots.$$

Here we use the facts that given $\mathcal{F}_{u^{-1}(s_k)}$, $\gamma^R_-(u^{-1}(s_k) + t)$, $t \geq 0$, has the law of $\nu^{\#}_{D;a\to z_0;b}$; $u^{-1}(s_{k+1}) - u^{-1}(s_k) = \mathrm{Cont}(\gamma^R_-([u^{-1}(s_k), u^{-1}(s_{k+1})]))$; $u^{-1}(s_{k+1}) \geq \tau_{e^{s_k+1}} = \tau_{Re^{s_k}}$, $u^{-1}(s_k) \leq \tau_{4e^{s_k}} < \tau_{re^{s_k}} \leq \tau^{Re^{s_k}}_{re^{s_k}}$; and $s_k = \mathrm{cap}(D^c)$. Since $u^{-1}(s_k) - u^{-1}(s_{k-1})$ is $\mathcal{F}_{u^{-1}(s_k)}$-measurable and $l * e^{d*s_k} = R^{d*k} \geq 1$, we get

$$\begin{aligned}\mathbb{P}[s_n < u(1)] &= \mathbb{P}\left[u^{-1}(s_n) < 1\right] \\ &\leq \mathbb{P}\left[\bigcap_{k=1}^{n}\{u^{-1}(s_k) - u^{-1}(s_{k-1}) < l * e^{d*s_k}\}\right] < p^n, \quad n \in \mathbb{N}.\end{aligned}$$

Since $\mathrm{diam}(K_1) \geq \mathrm{diam}(\gamma^R_-([0,1]))$ and $u(1) = \mathrm{cap}(K_1) \geq \log(\mathrm{diam}(K_1)/4)$, we see that $\mathrm{diam}(\gamma^R_-([0,1])) > 4R^n/l^{1/d}$ implies that $u(1) > s_n$. Thus, $\mathbb{P}[\mathrm{diam}(\gamma^R_-([0,1])) > 4R^n/l^{1/d}] < p^n$. This shows that $\mathbb{E}[\mathrm{diam}(\gamma^R_-([0,1]))^c] < \infty$ if $c < \log(1/p)/\log(R)$. Since R is fixed, taking p arbitrarily small completes the proof. □

3.3 Critical exponent

Lemma 3.2 *Suppose $\kappa \in (0, 4]$. There exist $R > r > 4$ such that for any $l > 0$ there exists $p > 0$ such that for any $D \in \mathcal{D}$ with distinct prime ends a, b,*

$$\nu^{\#}_{D;a\to\infty;b}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau^{Re^{\mathrm{cap}(D^c)}}_{re^{\mathrm{cap}(D^c)}}, \tau_{Re^{\mathrm{cap}(D^c)}}\right]\right)\right) < l * e^{d*\mathrm{cap}(D^c)}\right\}\right) > p.$$

We now use Lemma 3.2 to prove Theorem 1.4, and postpone the proof of Lemma 3.2 to the next section.

Proof of Theorem 1.4. By self-similarity and stationarity of increments, it suffices to prove that γ^R_- is a.s. not Hölder continuous on $[0, \tau_1]$. Let $R > r > 4$ be given by Lemma 3.2. Fix $l > 0$. Let $p = p(l, r, R)$ be given by Lemma 3.2.

Let $T_n = \tau_{R^{-n}}$, $n \in \mathbb{N} \cup \{0\}$. Let E_n denote the event that $\tau_{Re^{u(T_n)}} - \tau^{Re^{u(T_n)}}_{re^{u(T_n)}} \le l * e^{d*u(T_n)}$. Applying Lemma 3.2 to $D = D_{T_n}$, $a = \gamma^R_-(T_n)$ and $b = 0$, and using the fact that the law of $\gamma^R_-(T_n + t)$, $t \ge 0$, given $\mathcal{F}_{T_n}$, is $\nu^{\#}_{D;a\to\infty;b}$, we get $\mathbb{P}[E^c_n|\mathcal{F}_{T_n}] < 1 - p$. From $u(T_n) \le \log(R^{-n})$, we get $Re^{u(T_n)} \le R^{1-n}$, and so E_n is measurable w.r.t. $\mathcal{F}_{T_{n-1}}$. Thus, $\mathbb{P}[\bigcup_{n=1}^{\infty} E_n] = 1 - \mathbb{P}[\bigcap_{n=1}^{\infty} E^c_n] \ge 1 - (1 - p)^{\infty} = 1$. If E_n happens for some $n \in \mathbb{N}$, then $\gamma^R_-([0, \tau_1])$ contains a subcurve crossing the annulus $\{re^{u(T_n)} < |z| < Re^{u(T_n)}\}$ with time duration no more than $l * e^{d*u(T_n)}$, which implies that

$$\sup_{0\le s<t\le\tau_1} \frac{|\gamma^R_-(t) - \gamma^R_-(s)|}{(t - s)^{1/d}} \ge \frac{Re^{u(T_n)} - re^{u(T_n)}}{(l * e^{d*u(T_n)})^{1/d}} = \frac{R - r}{l^{1/d}}.$$

Thus, a.s. $\sup_{0\le s<t\le\tau_1} \frac{|\gamma^R_-(t)-\gamma^R_-(s)|}{(t-s)^{1/d}} \ge \frac{R-r}{l^{1/d}}$. Since $R > r > 0$ are fixed, taking $l > 0$ arbitrarily small completes the proof. □

4 Crossing estimates

The purpose of the last section is to prove Lemmas 3.1 and 3.2. First, we use Koebe distortion theorem to derive the following proposition. For $a > 0$, let $\mathcal{U}_a$ denote the set of univalent analytic functions f defined on $\mathbb{D}$ such that $f(0) = 0$ and $|f'(0)| = a$.

Proposition 4.1 (i) *For every $a > 0$ and $R_2 > R_1 \in (0, 1)$ such that $R_2 < 1/6$, there exist $r_2 > r_1 > 0$ such that $r_2 < \frac{3}{2}aR_2$, and*

$$\{r_1 \le |z| \le r_2\} \supset f(\{R_1 \le |z| \le R_2\}), \quad \forall f \in \mathcal{U}_a. \tag{4.1}$$

(ii) *For every $a > 0$ and $R_2 > R_1 \in (0, 1)$ such that $R_2/R_1 \ge 6$, there exist $r_2 > r_1 > 0$ satisfying $r_2 < a/4$ and $r_2/r_1 > (R_2/R_1)/6$ such that*

$$\{r_1 \le |z| \le r_2\} \subset f(\{R_1 \le |z| \le R_2\}), \quad \forall f \in \mathcal{U}_a. \tag{4.2}$$

(iii) *For every $a > 0$ and $r_2 > r_1 > 0$ such that $r_2 < a/4$ and $r_2/r_1 \ge 12$, there exist $R_2 > R_1 \in (0, 1)$ such that $R_2/R_1 > (r_2/r_1)/12$, and*

$$\{R_1 \le |z| \le R_2\} \subset f^{-1}(\{r_1 \le |z| \le r_2\}), \quad \forall f \in \mathcal{U}_a. \tag{4.3}$$

Proof By scaling we may assume that $a = 1$. By Koebe distortion theorem, for any $f \in \mathcal{U}_1$,

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D}. \tag{4.4}$$

(i) Suppose $1/6 \ge R_2 > R_1 > 0$. Let $r_2 = \frac{R_2}{(1-R_2)^2}$ and $r_1 = \frac{R_1}{(1+R_1)^2}$. Then $r_2 > R_2 > R_1 > r_1$, $r_2 \le \frac{R_2}{(1-1/6)^2} < \frac{3}{2}R_2$, and we get (4.1) using (4.4).

(ii) Suppose $R_1, R_2 \in (0,1)$ and $R_2/R_1 \ge 6$. Let $r_2 = \frac{R_2}{(1+R_2)^2}$ and $r_1 = \frac{R_1}{(1-R_1)^2}$. Then from $R_1 \le 1/6$ and $R_2 \le 1$, we get $\frac{r_2}{r_1} \ge \frac{(1-1/6)^2}{(1+1)^2} \cdot \frac{R_2}{R_1} > \frac{1}{6} \cdot \frac{R_2}{R_1}$. Since $R_2/R_1 \ge 6$, we get $r_2 > r_1$. Formula (4.2) now follows from (4.4).

(iii) Suppose $r_1, r_2 > 0$, $r_2 < 1/4$, and $r_2/r_1 \ge 12$. There exist $R_1, R_2 \in (0,1)$ such that $r_2 = \frac{R_2}{(1-R_2)^2}$ and $r_1 = \frac{R_1}{(1+R_1)^2}$. If $(1-R_2)^2 \le 1/3$, then $R_2 = r_2(1-R_2)^2 < 1/3$, which then implies $(1-R_2)^2 > (1-1/3)^2 > 1/3$, which is a contradiction. So $(1-R_2)^2 > 1/3$, and from $R_1 < 1$, we have $\frac{R_2}{R_1} = \frac{(1-R_2)^2}{(1+R_1)^2} \cdot \frac{r_2}{r_1} > \frac{1/3}{4}\frac{r_2}{r_1} = \frac{1}{12} \cdot \frac{r_2}{r_1}$. Since $r_2/r_1 > 12$, we get $R_2 > R_1$. Formula (4.3) now follows from (4.4). □

Let λ_t and q_t be the solutions of (2.1). Let $X_t = \lambda_t - q_t$. Then X_t is a random process staying in $(0, 2\pi)$ that satisfies the SDE

$$dX_t = \sqrt{\kappa} dB_t + 2\cot(X_t/2)dt. \tag{4.5}$$

Proposition 4.2 *The process (X_t) in (4.5) is Markovian with a transition density $p_t(x,y)$. Moreover, there exist constants $t_0 \ge 1$ and $C_1, C_2 > 0$ depending only on κ such that for $t \ge t_0$,*

$$C_1 \sin(y/2)^{8/\kappa} \le p_t(x,y) \le C_2 \sin(y/2)^{8/\kappa}, \quad \forall x, y \in (0, 2\pi).$$

Proof Let $Y_t = \cos(\frac{1}{2}X_{\frac{4}{\kappa}\cdot t})$. Then for some standard Brownian motion $\widetilde{B}_t$, (Y_t) solves the SDE

$$dY_t = -\sqrt{1-Y_t^2}d\widetilde{B}_t - \Big(\frac{4}{\kappa} + \frac{1}{2}\Big)Y_t dt. \tag{4.6}$$

This agrees with [31, Formula (8.2)] with $\delta = 1 + \frac{8}{\kappa}$. From [31, Proposition 8.1], (Y_t) has a transition density $p_t^{(Y)}(x,y)$ given by

$$p_t^{(Y)}(x,y) = \sum_{n=0}^{\infty} \frac{(1-y^2)^{\frac{4}{\kappa}-\frac{1}{2}} C_n^{(\frac{4}{\kappa})}(x) C_n^{(\frac{4}{\kappa})}(y)}{\Big\langle C_n^{(\frac{4}{\kappa})}, C_n^{(\frac{4}{\kappa})} \Big\rangle_{\frac{4}{\kappa}-\frac{1}{2}}} e^{-\frac{n}{2}(n+\frac{8}{\kappa})t},$$

where $C_n^{(\frac{4}{\kappa})}$, $n = 0, 1, 2, \ldots$, are the Gegenbauer polynomials [23] with index $\frac{4}{\kappa}$, and $\langle \cdot \rangle_{\frac{4}{\kappa}-\frac{1}{2}}$ is the inner product defined by $\langle f, g \rangle_{\frac{4}{\kappa}-\frac{1}{2}} = \int_{-1}^{1} (1-x^2)^{\frac{4}{\kappa}-\frac{1}{2}} f(x)g(x)dx$.

Since $Y_t = \cos(X_{\frac{4}{\kappa}\cdot t}/2)$, (X_t) also has a transition density, which is

$$p_t(x,y) = \sum_{n=0}^{\infty} \frac{\sin(y/2)^{\frac{8}{\kappa}} C_n^{(\frac{4}{\kappa})}(\cos(x/2)) C_n^{(\frac{4}{\kappa})}(\cos(y/2))}{2\left\langle C_n^{\left(\frac{4}{\kappa}\right)}, C_n^{(\frac{4}{\kappa})}\right\rangle_{\frac{4}{\kappa}-\frac{1}{2}}} e^{-n(1+\frac{\kappa}{8}n)t}.$$

From [23] we know that $C_0^{(\frac{4}{\kappa})} \equiv 1$, and for all $n \geq 0$,

$$\left\langle C_n^{(\frac{4}{\kappa})}, C_n^{(\frac{4}{\kappa})}\right\rangle_{\frac{4}{\kappa}-\frac{1}{2}} = \frac{\pi\Gamma\left(n+\frac{8}{\kappa}\right)}{2^{\frac{8}{\kappa}-1}\left(n+\frac{4}{\kappa}\right)n!\Gamma\left(\frac{4}{\kappa}\right)^2}, \quad \left\|C_n^{(\frac{4}{\kappa})}\right\|_{\infty} = \frac{\Gamma\left(n+\frac{8}{\kappa}\right)}{n!\Gamma\left(\frac{8}{\kappa}\right)}.$$

Thus, we have

$$\left|\frac{\kappa\pi\Gamma\left(\frac{8}{\kappa}\right)}{2^{\frac{8}{\kappa}}\Gamma\left(\frac{4}{\kappa}\right)^2}\cdot\frac{p_t(x,y)}{\sin(y/2)^{\frac{8}{\kappa}}} - 1\right| \leq \sum_{n=1}^{\infty}\frac{\left(\frac{\kappa}{4}n+1\right)\Gamma\left(n+\frac{8}{\kappa}\right)}{n!\Gamma\left(\frac{8}{\kappa}\right)} e^{-n(1+\frac{\kappa}{8}n)t}.$$

Using Stirling's formula, we can conclude that there exists $t_0 \geq 1$ such that for $t \geq t_0$, the RHS of the above formula is less than $1/2$, which implies the conclusion. □

Proof of Lemma 3.1. Using Proposition 2.2, Koebe distortion theorem and Proposition 4.1 (i) (applied to a conformal map between $\mathbb{D}^*$ and $D \in \mathcal{D}$ that fixes ∞, precomposed and postcomposed by the reciprocal function $z \mapsto 1/z$), we find that Lemma 3.1 follows from the following statement:

$$\begin{aligned}&\exists R > r > 6,\ \forall p > 0,\ \exists l > 0,\ \forall z \in \mathbb{T}\backslash\{1\},\\ &\nu^{\#}_{\mathbb{D}^*;z\to\infty;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau_r^R, \tau_R\right]\right)\right) < l\right\}\right) < p.\end{aligned} \tag{4.7}$$

To see this, first by scaling we may assume that $\mathrm{cap}(D^c) = 0$. Let g be a conformal map from D onto $\mathbb{D}^*$ with $g(\infty) = \infty$. Let $J(z) = 1/z$ and $f = J \circ g^{-1} \circ J$. Then $f \in \mathcal{U}_1$. Suppose (4.7) holds with some $R > r > 6$. Then by Proposition 4.1 (i), there are $0 < r_1 < r_2 < \frac{3/2}{r}$ such that $\{r_1 < |z| < r_2\} \supset f(\{1/R < |z| < 1/r\})$. Let $\widetilde{R} = 1/r_1$ and $\widetilde{r} = 1/r_2$. Then $\widetilde{R} > \widetilde{r} > \frac{2}{3}r > 4$ and $\{\widetilde{r} < |z| < \widetilde{R}\} \supset g^{-1}(\{r < |z| < R\})$. We may then check that Lemma 3.1 holds for such $\widetilde{R}$ and $\widetilde{r}$. Let $p > 0$. Let $l > 0$ be given by (4.7) for such p (and the R and r). Let $D \in \mathcal{D}$ have $\mathrm{cap}(D^c) = 0$, and let a, b be distinct prime ends of D. We may assume that the above g satisfies $g(b) = 1$. Then $g(a) \in \mathbb{T}\backslash\{1\}$. Suppose β has the law $\nu^{\#}_{\mathbb{D}^*;g(a)\to\infty;1}$. Then $\widetilde{\beta} := g^{-1}(\beta)$ has the law $\nu^{\#}_{D;a\to\infty;b}$. From (4.7) we know that $\mathbb{P}[\mathrm{Cont}(\beta([\tau_r^R(\beta), \tau_R(\beta)]) < l] < p$. Since $|\log|(g^{-1})'||$ is uniformly bounded on $\{r < |z| < R\}$, by Proposition 2.2, there is $\widetilde{l} \in (0, l)$ such that $\mathrm{Cont}(\widetilde{\beta}([\tau_r^R(\beta), \tau_R(\beta)]) < \widetilde{l}$ implies that $\mathrm{Cont}(\beta([\tau_r^R(\beta), \tau_R(\beta)]) < l$. From $\{\widetilde{r} < |z| < \widetilde{R}\} \supset g^{-1}(\{r < |z| < R\})$ we get $\tau_{\widetilde{R}}(\widetilde{\beta}) \geq \tau_R(\beta)$ and $\tau_{\widetilde{r}}^{\widetilde{R}}(\widetilde{\beta}) \leq \tau_r^R(\beta)$, which implies that $\mathbb{P}[\mathrm{Cont}(\widetilde{\beta}([\tau_{\widetilde{r}}^{\widetilde{R}}(\widetilde{\beta}),\tau_{\widetilde{R}}(\widetilde{\beta})]) < \widetilde{l}] \le \mathbb{P}[\widetilde{\beta}([\tau_r^R(\beta),\tau_R(\beta)]) < \widetilde{l}] < p$, as desired.

Let $z \in \mathbb{T}\backslash\{1\}$. Let β^z be a radial SLE$_\kappa(2)$ curve in $\mathbb{D}^*$ from z to ∞ with a force point at 1 (parametrized by capacity). Let λ_t and q_t be the driving process and force point process for β^z, respectively; let D_t be the complement of $\mathbb{D}^*\backslash\beta^z([0,t])$ that contains ∞; and let g_t be the radial Loewner maps for β^z. Then λ_t and q_t solve (2.1) with $x \in (0,2\pi)$ such that $e^{ix} = z$ and $y = 0$. Let $X_t = \lambda_t - q_t$, which is a diffusion process that solves (4.5) with initial value $X_0 = x$. Let $t_0 \ge 1$ be as in Proposition 4.2. Then the law of X_{t_0} has a density w.r.t. $\mathrm{m}\,|_{(0,2\pi)}$, which is comparable to $\sin(y/2)^{\frac{8}{\kappa}}$.

Let $(\mathcal{F}_t)$ be the filtration generated by β^z. Since $e^{t_0} \ge e > 2$, the statement (4.7) follows from the following:

$$\begin{aligned}&\exists R > r > 4e^{t_0},\ \forall p > 0,\ \exists l > 0,\ \forall z \in \mathbb{T}\backslash\{1\},\\ &\mathbb{P}\left[\mathrm{Cont}\left(\beta^z\left(\left[\tau_r^R,\tau_R\right]\right)\right) < l|\mathcal{F}_{t_0}\right] < p.\end{aligned} \tag{4.8}$$

Let $\widehat{g}_{t_0}(z) = g_{t_0}(z)/e^{iq_{t_0}}$. Then $\widehat{g}_{t_0}$ maps D_{t_0} conformally onto $\mathbb{D}^*$, fixes ∞ and 1, and maps $\beta^z(t_0)$ to $e^{iX_{t_0}}$. From the domain Markov property for β^z, the conditional law given $\mathcal{F}_{t_0}$ of the $\widehat{g}_{t_0}$-image of the part of β^z after t_0, denoted by $\widehat{\beta}^z_{t_0}$, is $\nu^\#_{\mathbb{D}^*;X_{t_0}\to\infty;1}$. Using Proposition 2.2, Koebe distortion theorem and Proposition 4.1 (i) applied to $a = e^{-t_0}$ and $f = 1/\widehat{g}_{t_0}^{-1}(1/z)$, and using the fact that $\tau_r^R > t_0$ since $\mathrm{diam}(\beta^z([0,t_0])) \le 4e^{t_0} < r < R$, we find that the statement (4.8) follows from the following:

$$\begin{aligned}&\exists R > r > 6,\ \forall p > 0,\ \exists l > 0,\ \forall z \in \mathbb{T}\backslash\{1\},\\ &\mathbb{P}\left[\mathrm{Cont}\left(\widehat{\beta}^z_{t_0}\left(\left[\tau_r^R,\tau_R\right]\right)\right) < l|\mathcal{F}_{t_0}\right] < p.\end{aligned} \tag{4.9}$$

Here we used an argument that is similar to that used right after the statement (4.7). Note that if (4.9) holds for some $R > r > 6$, then we may find $\widetilde{R} > \widetilde{r} > 4e^{t_0}$ such that $\{\widetilde{r} < |z| < \widetilde{R}\} \supset \widehat{g}_{t_0}^{-1}(\{r < |z| < R\})$. This has to do with that $\widehat{g}_{t_0}(z)/z \to e^{-t_0}$ as $z \to \infty$.

Let $\widetilde{\beta}$ be a radial SLE$_\kappa(2)$ curve in $\mathbb{D}^*$ from a random point $e^{i\theta}$ to ∞ with a force point at 1, where θ is distributed on $(0,2\pi)$ with a density function proportional to $\sin(x/2)^{\frac{8}{\kappa}}$ w.r.t. the Lebesgue measure. Since the Minkowski content of any subcurve of $\widetilde{\beta}$ crossing $\{7 \le |z| \le 8\}$ is a.s. strictly positive, we find that

$$\forall p > 0,\quad \exists l > 0,\quad \mathbb{P}\left[\mathrm{Cont}\left(\widetilde{\beta}\left(\left[\tau_7^8,\tau_8\right]\right)\right) < l\right] < p. \tag{4.10}$$

Since the conditional law of X_{t_0} given $\mathcal{F}_{t_0}$ is absolutely continuous w.r.t. the law of θ, and the Radon–Nikodym derivative is uniformly bounded away from ∞ and 0, the same is true for the relation between the conditional law of $\widehat{\beta}^z_{t_0}$ given $\mathcal{F}_{t_0}$ and the law of $\widetilde{\beta}$. So from (4.10) we find that (4.9) holds with $R = 8$ and $r = 7$. The proof is now complete. □

Proof of Lemma 3.2. Using Proposition 2.2, Koebe distortion theorem and Proposition 4.1 (ii) (applied to a conformal map between $\mathbb{D}^*$ and $D \in \mathcal{D}$ that fixes ∞, precomposed and postcomposed by the reciprocal function), we find that the statement of the lemma follows from the following statement:

$$\exists R > 6r > 6,\ \forall l > 0,\ \exists p > 0,\ \forall z \in \mathbb{T}\backslash\{1\},$$
$$\nu^{\#}_{\mathbb{D}^*;z\to\infty;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau_r^R, \tau_R\right]\right)\right) < l\right\}\right) > p. \tag{4.11}$$

Here we used an argument that is similar to the one used after (4.7).

Let $z \in \mathbb{T}\backslash\{1\}$. Let $t_0 \geq 1$ be as in Proposition 4.2. Let β^z, (X_t), $(\mathcal{F}_t)$, $\widehat{g}_{t_0}$ and $\widehat{\beta}^z_{t_0}$ be as in the proof of Lemma 3.1. Using Proposition 2.2, Koebe distortion theorem and Proposition 4.1 (ii) applied to $a = e^{-t_0}$ and $f = 1/\widehat{g}_{t_0}^{-1}(1/z)$, we find that the statement with (4.11) follows from the following: there exist $R > r > 1$ with $R/r > 6^2$ such that for any $l > 0$ there exists $p > 0$ such that for any $z \in \mathbb{T}\backslash\{1\}$,

$$\mathbb{P}\left[\mathrm{Cont}\left(\widehat{\beta}^z_{t_0}\left(\left[\tau_r^R, \tau_R\right]\right)\right) < l|\mathcal{F}_{t_0}\right] > p. \tag{4.12}$$

Since the conditional law of $\widehat{\beta}^z_{t_0}$ given $\mathcal{F}_{t_0}$ is $\nu^{\#}_{\mathbb{D}^*;e^{iX_{t_0}}\to\infty;1}$, and the law of X_{t_0} has a density w.r.t. $\mathrm{m}\,|_{(0,2\pi)}$, which is bounded below on $[\frac{2}{3}\pi, \frac{4}{3}\pi]$ by a uniform positive constant, the statement with (4.12) follows from the following: there exist $\varepsilon \in (0, 1)$ and $R > r > 1$ with $R/r \geq 6^2$ such that for any $l > 0$ there exists $p > 0$ such that for any $z \in \mathbb{T}$ with $|z + 1| < \varepsilon$,

$$\nu^{\#}_{\mathbb{D}^*;z\to\infty;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau_r^R, \tau_R\right]\right)\right) < l\right\}\right) > p.$$

Using reciprocal maps, we convert this statement into the following: there exist $\varepsilon < R < r \in (0, 1)$ with $R/r < 1/6^2$ such that for any $l > 0$ there exists $p > 0$ such that for any $z \in \mathbb{T}$ with $|z + 1| < \varepsilon$,

$$\nu^{\#}_{\mathbb{D};z\to 0;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau_r^R, \tau_R\right]\right)\right) < l\right\}\right) > p. \tag{4.13}$$

If $z \in \mathbb{T}$ satisfies $|z + 1| < \varepsilon$ for some $\varepsilon < 1$, then there exists a Möbius transformation that maps $(\mathbb{D}; z, 0, 1)$ onto $(\mathbb{D}; -1, w, 1)$ for some $w \in \mathbb{C}$ with $|w| < \varepsilon$. By applying Koebe distortion theorem to the map and using Proposition 2.2, we see that the statement with (4.13) follows from the following: there exist $0 < \varepsilon < R < r < 1$ with $\varepsilon/R < 1/2$ and $R/r < 1/6^3$ such that for any $l > 0$ there exists $p > 0$ such that for any $w \in \mathbb{C}$ with $|w| < \varepsilon$,

$$\nu^{\#}_{\mathbb{D};-1\to w;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau_r^R, \tau_R\right]\right)\right) < l\right\}\right) > p. \tag{4.14}$$

We claim that for any $0 < \varepsilon < R < 1$ with $\varepsilon/R < 1/2$ and for any $w \in \mathbb{C}$ with $|w| < \varepsilon$, $\nu^{\#}_{\mathbb{D};-1\to w;1}|\mathcal{F}_{\tau_R}$ is absolutely continuous w.r.t. $\nu^{\#}_{\mathbb{D};-1\to 0;1}|\mathcal{F}_{\tau_R}$, and the Radon–Nikodym derivative is uniformly bounded above and below by finite positive constants depending only on κ, ε, R. To see that this is true, we may compare $\nu^{\#}_{\mathbb{D};-1\to w;1}|\mathcal{F}_{\tau_R}$ with the chordal SLE_κ measure $\mu^{\#}_{\mathbb{D};-1\to 1}$ restricted to the σ-algebra

$\mathcal{F}_{\tau_R}$. We use the fact that $\nu^{\#}_{\mathbb{D};-1\to w;1}|\mathcal{F}_{\tau_R}$ is absolutely continuous w.r.t. $\mu^{\#}_{D;-1\to 1}|\mathcal{F}_{\tau_R}$, and the Radon–Nikodym derivative equals $\mathbf{1}_{\{\tau_R<\infty\}}G_{\mathbb{D}_{\tau_R};\beta(\tau_R),1}(w)/G_{\mathbb{D};-1,1}(w)$ (cf. [16, Proposition 2.12]). To prove the claim, we need to show that there is a constant $C\in(1,\infty)$ depending on κ, ε, R such that for any simply connected domain D that contains the disc $\{|z|<R\}$, for any distinct prime ends a, b of D, and for any $w\in\mathbb{C}$ with $|w|\le\varepsilon$, we have $1/C\le G_{D;a,b}(w)/G_{D;a,b}(0)\le C$. For this purpose, we use the explicit expression of the chordal SLE$_\kappa$ Green's function (cf. [9, Formula (8)]):

$$G_{D;a,b}(w)=\widehat{c}\,\Upsilon_D(w)^{1-\frac{\kappa}{8}}\sin\left(\pi * h_D\left(w;\widehat{ab}\right)\right)^{\frac{8}{\kappa}-1}, \tag{4.15}$$

where $\widehat{c}$ is a positive constant depending only on κ, $\Upsilon_D(w)$ is the conformal radius of D viewed from w, and $h_D(w;\widehat{ab})$ is the harmonic measure of the clockwise arc on ∂D from a to b viewed from w. Using Koebe distortion theorem and Harnack inequality, we can prove that, if $|w|\le\varepsilon$, then $\Upsilon_D(w)\asymp\Upsilon_D(0)$ and $\sin(\pi * h_D(w;\widehat{ab}))\asymp\sin(\pi * h_D(0;\widehat{ab}))$, and the implicit constants depend only on κ, ε, R. Then the claim is proved.

Thus, the statement with (4.14) follows from the following: there exist $0<R<r<1$ with $R/r\le 1/6^3$ such that for any $l>0$,

$$\nu^{\#}_{\mathbb{D};-1\to 0;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau_r^R,\tau_R\right]\right)\right)<l\right\}\right)>0.$$

It will be more convenient to work on square domains. Let Q be the open square $\{z\in\mathbb{C} : |\mathrm{Re}\, z|, |\mathrm{Im}\, z|<1\}$. For $R, r>0$, we define $\widehat{\tau}_R$ to be the hitting time (i.e., the first visiting time) of $\partial(RQ)$, and define $\widehat{\tau}_r^R$ to be the last visiting time of $\partial(rQ)$ before $\widehat{\tau}_R$. Using Proposition 2.2, Koebe distortion theorem and Proposition 4.1 (iii), we find that the previous statement follows from the following: there exist $R<r\in(0,1)$ with $R/r\le 1/(12*\sqrt{2}*6^3)$ such that

$$\forall l>0,\quad \nu^{\#}_{Q;-1\to 0;1}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\widehat{\tau}_r^R,\widehat{\tau}_R\right]\right)\right)<l\right\}\right)>0.$$

Since $\nu^{\#}_{Q;-1\to 0;1}|\mathcal{F}_{\widehat{\tau}_R}$ is absolutely continuous w.r.t. the chordal SLE$_\kappa$ measure $\mu^{\#}_{Q;-1\to 1}|\mathcal{F}_{\widehat{\tau}_R}$, and the Radon–Nikodym derivative is strictly positive on the event $\{\widehat{\tau}_R<\infty\}$, the above statement follows from

$$\forall R, l\in(0,1),\quad \mu^{\#}_{Q;-1\to 1}\left(\{\beta : \widehat{\tau}_R(\beta)<\infty, \mathrm{Cont}\left(\beta\left([0,\widehat{\tau}_R]\right)\right)<l\}\right)>0. \tag{4.16}$$

Define $U_r=\{z\in\mathbb{C} : |\mathrm{Re}\, z|<1, |\mathrm{Im}\, z|<r\}$ for $r\in(0,1]$. From the generalized restriction property for SLE$_\kappa$, $\kappa\in(0,4]$, we know that, for any $\delta\in(0,1)$, $\mu^{\#}_{U_\delta;-1\to 1}$ is absolutely continuous w.r.t. $\mu^{\#}_{Q;-1\to 1}|_{\{\cdot\cap(Q\setminus U_\delta)=\emptyset\}}$, and the Radon–Nikodym derivative is expressed in terms of Brownian loop measures and the central charge for SLE$_\kappa$ [12,18]. So (4.16) follows from

$$\forall R, l \in (0,1), \exists \delta \in (0,R), \quad \mu^{\#}_{U_\delta;-1\to 1}(\{\beta : \mathrm{Cont}(\beta([0,\widehat{\tau}_R])) < l\}) > 0. \tag{4.17}$$

Using Formula (4.15), Koebe 1/4 theorem, and the fact that the expectation of the Minkowski content measure of a chordal SLE$_\kappa$ curve equals the integral of its Green's function (cf. [9, Theorem 1.1]), we get

$$\int \mathrm{Cont}(\beta)\mu^{\#}_{U_\delta;-1\to 1}(d\beta) = \int_{U_\delta} G_{U_\delta;-1,1}(z)\,\mathrm{m}^2(dz) \lesssim \int_{U_\delta} \mathrm{dist}(z, U_\delta^c)^{\frac{\kappa}{8}-1}\,\mathrm{m}^2(dz).$$

Let Q_1 be the first quadrant $\{\mathrm{Re}\, z, \mathrm{Im}\, z > 0\}$. For $z \in U_\delta \cap Q_1$, we have $\mathrm{dist}(z, U_\delta^c) = \min\{1 - \mathrm{Re}\, z, \delta - \mathrm{Im}\, z\}$. Since $\frac{\kappa}{8} - 1 < 0$, we have

$$\int_{U_\delta \cap Q_1} \mathrm{dist}(z, U_\delta^c)^{\frac{\kappa}{8}-1}\,\mathrm{m}^2(dz) \le \int_0^1 \int_0^\delta (1-x)^{\frac{\kappa}{8}-1} + (\delta - y)^{\frac{\kappa}{8}-1} dx dy$$
$$= \frac{8}{\kappa}(\delta + \delta^{\frac{\kappa}{8}}).$$

By symmetry, this inequality also holds with Q_1 replaced by other quadrants. So we have $\int \mathrm{Cont}(\beta)\mu^{\#}_{U_\delta;-1\to 1}(d\beta) \lesssim \delta^{\frac{\kappa}{8}}$. We can make $\int \mathrm{Cont}(\beta)\mu^{\#}_{U_\delta;-1\to 1}(d\beta) < l$ by choosing $\delta \in (0, R)$ small enough. So we get (4.17) and finish the proof. □

Remark 4.3 One step of the above argument does not work for $\kappa \in (4, 8)$, i.e., the generalized restriction property does not hold for $\kappa > 4$. One possible way to get around this issue is to work on a chordal SLE$_\kappa$ curve in U_δ from -1 to 1 conditioned to avoid the two horizontal boundary segments of U_δ. This law of such SLE is absolutely continuous w.r.t. $\mu^{\#}_{Q;-1\to 1}|_{\{\cdot \cap (Q\setminus U_\delta) = \emptyset\}}$. If we can prove that the integral of the Green's function for the conditional SLE$_\kappa$ curve in U_δ tends to 0 as $\delta \to 0$, then we can follow the above proof to extend Lemma 3.2 and Theorem 1.4 to $\kappa \in (4, 8)$. This requires some estimate of the Green's functions for this SLE curve, such as $G(z) \lesssim \mathrm{dist}(z, \partial U_\delta)$, which we do not have now.

Remark 4.4 For $\kappa \ge 8$, we have a self-similar SLE$_\kappa$ process of index $1/2$ with stationary increments. This is the SLE$_\kappa$ loop rooted at ∞ (now we have a probability measure, cf. [29, Remark 4.8]). It is space-filling (visits every point in $\mathbb{C}$), and parametrized by the Lebesgue measure. We can prove that Lemma 1.7 still holds for $c \in (-2, 0)$. To see this, note that (3.1) holds because

$$\int_0^\infty f_c(\gamma(t))dt \le \int_{-\infty}^\infty f_c(\gamma(t))dt = \int_{\mathbb{C}} f_c(z)\,\mathrm{m}^2(dz) = \int_{|z|\le 1} |z|^c\,\mathrm{m}^2(dz) < \infty.$$

We may decompose an SLE$_\kappa$ loop γ rooted at ∞ into two arms. The first arm $\gamma_- := \gamma|_{\mathbb{R}_-}$ is a whole-plane SLE$_\kappa(\kappa - 6)$ curve from ∞ to 0; and given γ_-, the second arm $\gamma_+ := \gamma|_{\mathbb{R}_+}$ is a chordal SLE$_\kappa$ curve from 0 to ∞ in $\widehat{\mathbb{C}}\setminus\gamma_-$. For $\kappa = 8$, from the reversibility of whole-plane SLE$_8(2)$ [21], we know that the time-reversal γ_-^R of γ_- is a whole-plane SLE$_8(2)$ curve from 0 to ∞. Then Lemmas 3.1 and 3.2 also hold for $\kappa = 8$. In fact, the proof of Lemma 3.1 still works in the case $\kappa = 8$ without any change. The proof of Lemma 3.2 works from its start all the way to the statement (4.16), but (4.16) should be proved in a different way. We have (4.16) for $\kappa = 8$ because for any $\delta < R < 1$, there is a positive probability that a chordal SLE_8 curve in Q from -1 to 1 will reach the vertical line $\{\mathrm{Re}\, z = -R\}$ before any of the horizontal lines $\{\mathrm{Im}\, z = \delta\}$ and $\{\mathrm{Im}\, z = -\delta\}$, and when this happens, $\mathrm{Cont}(\beta([0, \widehat{\tau}_R])) = \mathrm{m}(\beta([0, \widehat{\tau}_R])) \le 2\delta(1 - R) < l$ if δ is small enough. Thus, Theorems 1.2, 1.3 and 1.4 all hold in the case $\kappa = 8$. The above argument does not work for $\kappa > 8$ due to the lack of time-reversibility of whole-plane $\mathrm{SLE}_\kappa(\kappa - 6)$.

Remark 4.5 Mohammad Rezaei proved in [24] that for $\kappa \in (0, 8)$, the d-dimensional Hausdorff measure of an SLE_κ curve is 0. Using the sssi SLE_κ curve, we may gain another perspective on that statement. We now need a new crossing estimate: there exist $R > r > 4$ such that for any $L > 0$ there exists $p > 0$ such that for any $D \in \mathcal{D}$ with distinct prime ends a, b,

$$\nu^{\#}_{D;a\to\infty;b}\left(\left\{\beta : \mathrm{Cont}\left(\beta\left(\left[\tau^{Re^{\mathrm{cap}(D^c)}}_{re^{\mathrm{cap}(D^c)}}, \tau_{Re^{\mathrm{cap}(D^c)}}\right]\right)\right) > L * e^{d*\mathrm{cap}(D^c)}\right\}\right) > p. \tag{4.18}$$

The estimate is similar to [24, Lemma 3.2], and its proof should also follow the proof of that lemma, which is technically involved. With this estimate, one may prove that $\gamma([0, 1])$ has Hausdorff measure zero using the following approach. For $t \in [0, 1]$ and $n \in \mathbb{N}$, let Q^n_t denote the dyadic square containing $\gamma(t)$ of side length 2^{-n}. Fix $\varepsilon > 0$ and $M > m \in \mathbb{N}$. For each t, define Q_t such that if there is $n \in [m, M]$ such that $\mathrm{diam}(Q^n_t)^d < \varepsilon\, \mathrm{Cont}(\gamma \cap Q^n_t)$, then let Q_t be the largest of such Q^n_t; otherwise let $Q_t = Q^M_t$. We then get a finite covering $\{Q_t : t \in [0, 1]\}$ of $\gamma([0, 1])$. The sum of the d-th powers of the diameters of these squares is naturally decomposed into two parts. The first is over the squares with side length $> 2^{-M}$, whose sum is less than $\varepsilon\, \mathrm{Cont}(\gamma([0, 1]))$. The second is over the squares with side length 2^{-M}. Since $\gamma([0, 1])$ has Minkowski content 1, the typical number of the dyadic squares with side length 2^{-M} that intersect $\gamma([0, 1])$ is comparable to 2^{dM}. Given one of such square, the probability that it belongs to the covering $\{Q_t : t \in [0, 1]\}$ is small when M is big by (4.18). So we can estimate the expectation of the second sum, and show that it tends to zero as $M \to \infty$.

Acknowledgements The author thanks Greg Lawler and Nina Holden for inspiring discussions and valuable comments, acknowledges the support from the National Science Foundation (DMS-1056840) and from the Simons Foundation (#396973), and thanks Columbia University and KIAS, where part of this work was carried out during two conferences held by them. The author also thanks the anonymous referees, whose comments improved the quality of the paper.

References

1. Beffara, V.: The dimension of SLE curves. Ann. Probab. **36**(4), 1421–1452 (2008)
2. Friz, P., Tran, H.: On the regularity of SLE trace (2016). arXiv:1611.01107
3. Frostman, O.: Potential déquilibre et capacité des ensembles avec quelques applications a la théorie des fonctions. Meddel. Lunds Univ. Math. Sem. **3**, 1–118 (1935)

4. Garsia, A.M., Rodemich, E., Rumsey Jr., H.: A real variable lemma and the continuity of paths of some Gaussian processes. Indiana Univ. Math. J. **20**, 565–578 (1970)
5. Gwynne, E., Holden, N., Miller, J.: Dimension transformation formula for conformal maps into the complement of an SLE curve (2016). arXiv:1603.0516
6. Gwynne, E., Holden, N., Miller, J.: An almost sure KPZ relation for SLE and Brownian motion (2015). arXiv:1512.01223
7. Johansson, F., Lawler, G.: Optimal Hölder exponent for the SLE path. Duke Math. J. **159**(3), 351–383 (2011)
8. Lawler, G.: Conformally Invariant Processes in the Plane. American Mathematical Society, Providence (2005)
9. Lawler, G., Rezaei, M.A.: Minkowski content and natural parametrization for the Schramm–Loewner evolution. Ann. Probab. **43**(3), 1082–1120 (2015)
10. Lawler, G., Rezaei, M.A.: Up-to-constants bounds on the two-point Green's function for SLE curves. Electron. Commun. Probab. **20**(45), 1–13 (2015). https://doi.org/10.1214/ECP.v20-4246
11. Lawler, G., Schramm, O., Werner, W.: Values of Brownian intersection exponents I: half-plane exponents. Acta Math. **187**(2), 237–273 (2001)
12. Lawler, G., Schramm, O., Werner, W.: Conformal restriction: the chordal case. J. Am. Math. Soc. **16**(4), 917–955 (2003)
13. Lawler, G., Sheffield, S.: A natural parametrization for the Schramm–Loewner evolution. Ann. Probab. **39**(5), 1896–1937 (2011)
14. Lawler, G., Viklund, F.: Convergence of loop-erased random walk in the natural parametrization (2016). arXiv:1603.05203
15. Lawler, G., Viklund, F.: Convergence of radial loop-erased random walk in the natural parametrization (2017). arXiv:1703.03729
16. Lawler, G., Werness, B.: Multi-point Green's function for SLE and an estimate of Beffara. Ann. Prob. **41**, 1513–1555 (2013)
17. Lawler, G., Zhou, W.: SLE curves and natural parametrization. Ann. Probab. **41**(3A), 1556–1584 (2013)
18. Lawler, G., Werner, W.: The Brownian loop soup. Probab. Theory Relat. Fields **128**(4), 565–588 (2004)
19. Lind, J.: Hölder regularity of the SLE trace. Trans. Am. Math. Soc. **360**(7), 3557–3579 (2008)
20. McKean Jr., H.P.: Hausdorff–Besicovitch dimension of Brownian motion paths. Duke Math. J. **22**, 229–234 (1955)
21. Miller, J., Sheffield, S.: Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees (2013). arXiv:1302.4738
22. Mörters, P., Peres, Y.: Brownian Motion. Cambridge University Press, Cambridge (2010)
23. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/18, Release 1.0.6 of 2013-05-06
24. Rezaei, M.A.: Hausdorff measure of SLE curves. arXiv:1212.5847 **(to appear in Stoch. Proc. Appl.)**
25. Rezaei, M.A., Zhan, D.: Higher moments of the natural parametrization for SLE curves. Ann. Inst. H. Poincaré Probab. Statist. **53**(1), 182–199 (2017)
26. Rohde, S., Schramm, O.: Basic properties of SLE. Ann. Math. **161**, 879–920 (2005)
27. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Isr. J. Math. **118**, 221–288 (2000)
28. Werness, B.: Regularity of Schramm–Loewner evolutions, annular crossings, and rough path theory. Electron. J. Probab. **17**(81), 1–21 (2012). https://doi.org/10.1214/EJP.v17-2331
29. Zhan, D.: SLE loop measures (2017). arXiv:1702.08026
30. Zhan, D.: Decomposition of Schramm–Loewner evolution along its curve (2016). In preprint, arXiv:1509.05015
31. Zhan, D.: Ergodicity of the tip of an SLE curve. Prob. Theory Relat. Fields **164**(1), 333–360 (2016)