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THE SCALING LIMITS OF PLANAR LERW IN FINITELY CONNECTED DOMAINS

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We define a family of stochastic Loewner evolution-type processes in finitely connected domains, which are called continuous LERW (loop-erased random walk). A continuous LERW describes a random curve in a finitely connected domain that starts from a prime end and ends at a certain target set, which could be an interior point, or a prime end, or a side arc. It is defined using the usual chordal Loewner equation with the driving function being $\sqrt{2}B(t)$ plus a drift term. The distributions of continuous LERW are conformally invariant. A continuous LERW preserves a family of local martingales, which are composed of generalized Poisson kernels, normalized by their behaviors near the target set. These local martingales resemble the discrete martingales preserved by the corresponding LERW on the discrete approximation of the domain. For all kinds of targets, if the domain satisfies certain boundary conditions, we use these martingales to prove that when the mesh of the discrete approximation is small enough, the continuous LERW and the corresponding discrete LERW can be coupled together, such that after suitable reparametrization, with probability close to 1, the two curves are uniformly close to each other.

1. Introduction. LERW (loop-erased random walk) (cf. [4]) is obtained by removing loops, in the order they are created, from a simple random walk on a graph that is stopped at some hitting time. Since the loops are erased, so an LERW is a simple lattice path. In this paper, we will consider the loop-erasures of conditional random walks. They have properties that are very similar to loop-erased random walks, so we still call them LERW.

In [16], Schramm introduced stochastic Loewner evolution (SLE), a family of random growth processes of closed fractal subsets in simply connected plane domains. The evolution is described by the classical Loewner equation

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with the driving term being $\sqrt{\kappa}$ times a standard linear Brownian motion for some $\kappa \geq 0$. SLE behaves differently for different values of κ. Schramm conjectured that SLE_2 is the scaling limit of a kind of LERW on the grid approximation of the domain. And he proved the conjecture in that paper under the assumption that the scaling limits of LERW are conformally invariant.

Schramm's processes turned out to be very useful. On the one hand, they are amenable to computations; on the other hand, they are related with some statistical physics models. In a series of papers [6, 7, 8], Lawler, Schramm and Werner used SLE to determine the Brownian motion intersection exponents in the plane. In [10], the conjecture in [16] is completely solved, where no additional assumption is added. In the same paper, SLE_8 is proved to be the scaling limits of UST (uniform spanning tree) Peano curve. Smirnov proved in [18] that chordal SLE_6 is the scaling limit of critical site percolation on the triangular lattice. And Schramm and Sheffield proved in [17] that the harmonic explorer converges to chordal SLE_4. In [9], $\mathrm{SLE}_{8/3}$ is proved to have the restriction property, and so is conjectured to be the scaling limits of self-avoiding walk. For the properties of SLE, see [5, 15] and [19].

At the beginning, the SLE is only defined in simply connected domains, because the definition uses the Riemann mapping theorem. In [20], a kind of SLE-type process, which is called annulus SLE, is defined in doubly connected domains. The definition uses the rotation symmetry and reflection symmetry of an annulus. It is proved there that annulus SLE_2 is the scaling limit of the LERW in the grid approximation of a doubly connected domain that starts from a vertex that is close to a boundary point and stops when it hits the other boundary component.

The definitions of LERW on grid approximations of simply or doubly connected domains could be easily extended to multiply connected domains. It is interesting to study the scaling limits of the LERW in multiply connected domains. This may help us to extend the SLE to multiply connected domains.

In this paper, we will define a family of SLE-type processes, which are called continuous LERW, in finitely connected domains. They are defined using the usual chordal Loewner equation with the driving function being $\sqrt{2}B(t) + S(t)$, where $B(t)$ is a standard linear Brownian motion, and the drift term $S(t)$ is continuously differentiable in t. The drift term is carefully chosen, so that the continuous LERW satisfy the conformal invariance, and preserve a family of local martingales generated by generalized Poisson kernels. The local martingales resemble the discrete martingales preserved by the corresponding discrete LERW on the discrete approximation of that domain. And this resemblance is used to prove the convergence of discrete LERW to continuous LERW.

This paper is organized as follows. In Section 2, we define some notation that will be used in this paper. In Section 3, three kinds of continuous LERW are defined, which are continuous LERW "aimed" at interior points, prime ends and side arcs. And we prove that they all satisfy the conformal invariance. In Section 4, we present the continuous and discrete martingales preserved by continuous and discrete LERW, respectively, and explain the similarity between these martingales.

In Section 5, we give a rigorous proof of the existence and uniqueness of the solution to the equation that is used to define a continuous LERW. The lemmas that are used for the proof are interesting. We first use the idea of Carathéodory topology to define the convergence of plane domains. Then we define a metric on the space of hulls in the upper half plane, so that the set of hulls that are contained in a fixed hull is compact. This compactness property is frequently used in the remaining part of this paper. In this section, we use it to derive many uniform constants without working on concrete functions.

In Section 6, we first consider one kind of LERW, whose targets are interior points. The method given in [10] is used to get a coupling of the driving process for the discrete LERW and that for the continuous LERW such that the two driving processes are uniformly close to each other in probability. In Section 7, we first use some regular properties of the discrete LERW curve to get a local coupling of the LERW curve and the continuous LERW trace so that the two curves are close to each other, before either of them leaves a hull bounded by a crosscut. Finally, we glue all local couplings to get a global coupling of the curves. In the last section, we study the convergence of the other two kinds of LERW. And we get the similar results of the convergence.

2. Some notation.

2.1. *Loop-erased random walk.* In general, an LERW is defined on a connected locally finite graph $G = (V, E)$. We will usually consider the graphs that are discrete approximations of some plane domains. A loop-erasure of a finite lattice path $v = (v(0), \dots, v(n))$ on G is defined as follows. Let $n_0 = \max\{m : v(m) = v(0)\}$. Define the sequence (n_j) inductively by $n_{j+1} = \max\{m : v(m) = v(n_j + 1)\}$ if n_j is defined and $n_j < n$. Let χ be the first j such that $n_j = n$. Let $w(j) = v(n_j)$ for $0 \le j \le \chi$. Then $w = (w(0), \dots, w(\chi))$ is called the loop-erasure of $(v(0), \dots, v(n))$ (see [4]), and is denoted by $\mathrm{LE}(v)$. It is a simple lattice path with $w(0) = v(0)$ and $w(\chi) = v(n)$.

A subset S of V is called reachable in G if for any $v \in V \setminus S$, a (simple) random walk on G started from v will hit S in finitely many steps almost surely. Suppose A and B are disjoint subsets of V such that $A \cup B$ is reachable in G. Suppose $v_0 \in V \setminus (A \cup B)$ and there is a lattice path on G

connecting v_0 and A without passing through B. Then the probability that a random walk started from v_0 hits A before B is positive. We now consider this random walk stopped on hitting $A\cup B$ and conditioned to hit A. It is a random finite lattice path. The loop-erasure of this path is called the LERW on G started from x conditioned to hit A before B.

For a function f defined on V, and $v\in V$, let $\Delta_G f(v)=\sum_{w\sim v}(f(w)-f(v))$, where $w\sim v$ means that w and v are adjacent. If $\Delta_G f(v)=0$, then we say f is discrete harmonic at v. The proof of the following lemma is easy, and can be found in [20].

Lemma 2.1. *Suppose A and B are disjoint subsets of V and $A\cup B$ is reachable in G. Let $x\in V\setminus(A\cup B)$ be such that there is a lattice path connecting x and A without passing through any vertex on B. Then there is a unique nonnegative bounded function h on V such that $h\equiv 0$ on $A\cup B$; $\Delta_G h\equiv 0$ on $V\setminus(A\cup B\cup\{x\})$; and $\sum_{v\in A}\Delta_G h(v)=1$. Moreover, if either A or B is a finite set, then there is a unique nonnegative bounded function g on V such that $g\equiv 0$ on B; $g\equiv 1$ on A; $\Delta_G g\equiv 0$ on $V\setminus(A\cup B\cup\{x\})$; and $\sum_{v\in A}\Delta_G g(v)=0$.*

Suppose E_{-1} and F are disjoint subsets of V and $E_{-1}\cup F$ is reachable in G. Let $x_0\in N$ be such that there is a lattice path connecting x_0 and F without passing through any vertex on E_{-1}. Let $(q(0),\dots,q(\chi))$ be the LERW on G started from x_0 conditioned to hit F before E_{-1}. So $q(0)=x_0$ and $q(\chi)\in F$. For $0\le j<\chi$, let $E_j=E_{-1}\cup\{q(0),\dots,q(j)\}$. Then E_j and F are disjoint. Since $E_j\cup F$ is bigger than $E_{-1}\cup F$, so it is also reachable. Note that for any $0\le j<\chi$, $(q(j),\dots,q(\chi))$ is a lattice path connecting $q(j)$ with F without passing through E_{j-1}. Let h_j be as in Lemma 2.1 with $A=F$, $B=E_{j-1}$ and $x=q(j)$. If either E_{-1} or F is finite, then either E_j or F is finite. Let g_j be the g in Lemma 2.1 with $A=F$, $B=E_{j-1}$ and $x=q(j)$. Let $\overline{F}$ be the union of F with the set of vertices of V that are adjacent to F. Then we have:

Proposition 2.1. *Fix any $v_0\in V$. Then $(g_k(v_0))$ (if E_{-1} or F is finite) and $(h_k(v_0))$ are discrete martingales up to the first time x_k hits $\overline{F}$, or E_k disconnects v_0 from F in G.*

Proof. The result for (g_k) in a special case is Proposition 3.2 in [20]. The proof of that proposition applies to general cases. The proof for (h_k) is similar. □

2.2. *Finitely connected domains.* In this paper, a domain is a nonempty connected open subset of the Riemann sphere $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Here we allow that the domain contains ∞. For $n\in\mathbb{Z}_{\ge 0}$, an n-connected domain is a domain D such that $\widehat{\mathbb{C}}\setminus D$ is the disjoint union of n connected compact sets, each of which contains more than one point. A finitely connected domain is an n-connected domain for some $n\in\mathbb{Z}_{\ge 0}$. A 0-connected domain is just $\widehat{\mathbb{C}}$. A 1-connected domain is conformally equivalent to the unit disc.

We will use dist (resp. $\mathrm{dist}^\#$) to denote the Euclidean (resp. spherical) distance; use diam (resp. $\mathrm{diam}^\#$) to denote the Euclidean (resp. spherical) diameter; and use $\mathbf{B}(z_0;r)$ [resp. $\mathbf{B}^\#(z_0;r)$] to denote the ball centered at z_0 with radius r, in the Euclidean (resp. spherical) metric. Let $\partial^\# D$ denote the boundary of D in $\widehat{\mathbb{C}}$; and let $\partial D=\partial^\# D\cap\mathbb{C}$.

Suppose D is an n-connected domain. Then $\partial^\# D$ has n connected components, each of which is the boundary of a connected component of $\widehat{\mathbb{C}}\setminus D$. If f maps D conformally into $\widehat{\mathbb{C}}$, then $D':=f(D)$ is also an n-connected domain. And f induces a one-to-one correspondence $\breve{f}$ from the set of components of $\partial^\# D$ to the set of components of $\partial^\# D'$ such that for any component A of $\partial^\# D$ and $z\in D$, $z\to A$ iff $f(z)\to\breve{f}(A)$. There exists some f that maps D conformally onto a plane domain that is bounded by n mutually disjoint analytic Jordan curves. We call such f a boundary smoothing map of D. Suppose f_1 and f_2 are two boundary smoothing maps of D, and $E_j=f_j(D)$, $j=1,2$. Then $f_2\circ f_1^{-1}$ maps E_1 conformally onto E_2, and $f_2\circ f_1^{-1}$ induces a one-to-one correspondence J from the set of Jordan curves that bound E_1 to the set of Jordan curves that bound E_2 such that for any Jordan curve σ that bounds E_1 and $z\in E_1$, $z\to\sigma$ iff $f_2\circ f_1^{-1}(z)\to J(\sigma)$. Since σ and $J(\sigma)$ are both analytic, from the Schwarz reflection principle, $f_2\circ f_1^{-1}$ can be extended conformally across σ, and maps σ onto $J(\sigma)$.

Now consider the set of all pairs (f,z) such that f is a boundary smoothing map of D, and $z\in\overline{f(D)}$. Two pairs (f_1,z_1) and (f_2,z_2) are equivalent if the extension of $f_2\circ f_1^{-1}$ maps z_1 to z_2. Let $\widehat{D}$ be the set of all equivalent classes. There is a unique conformal structure on $\widehat{D}$ such that $z\mapsto[(f,z)]$ maps $\overline{f(D)}$ conformally onto $\widehat{D}$ for any boundary smoothing map f. Then $z\mapsto[(f,f(z))]$ is a conformal map from D into $\widehat{D}$ independent of the choice of f. So we may view D as a subset of $\widehat{D}$, and call $\widehat{D}$ the conformal closure of D. It is clear that a conformal map between two finitely connected domains extends uniquely to a conformal map between their conformal closures.

We call $\widehat{\partial} D:=\widehat{D}\setminus D$ the conformal boundary of D. Then $\widehat{\partial} D$ is a union of n disjoint analytic Jordan curves, each of which is called a side of D. Each side σ corresponds to a component A of $\partial^\# D$ such that for $z\in D$, $z\to\sigma$ in $\widehat{D}$ iff $z\to A$. Each point on σ is called a prime end of D on A. This is equivalent to the prime ends defined in [1] and [13]. In fact, the definition in [1] describes the property of a sequence of points in D that converges to a point on $\widehat{\partial} D$, and the definition in [12] describes a neighborhood basis bounded by crosscuts of a point on $\partial\widehat{D}$. A connected subset of a side that contains more than one point is called a side arc.

If $z_0 \in \widehat{\mathbb{C}}$ and a prime end w_0 of D satisfies that for $z \in D$, $z \to z_0$ iff $z \to w_0$ in $\widehat{D}$, then we say the point z_0 and the prime end w_0 correspond to each other, and we do not distinguish the point z_0 from the prime end w_0. For example, if a boundary component of D is a Jordan curve, then each point on this curve corresponds to a prime end. If $z_0 \in \partial D$ and for some $\varepsilon > 0$, $\mathbf{B}(z_0;\varepsilon) \setminus D$ is a simple curve γ connecting z_0 with $\{|z - z_0| = \varepsilon\}$, then z_0 corresponds to a prime end of D. But every other point on γ corresponds to two prime ends of D.

If $\alpha : (a, b) \to D$ is a curve in D, and for some $z_0 \in \partial^\# D$, $\alpha(t) \to z_0$ as $t \to a$, then there is some prime end w_0 of D such that $\alpha(t) \to w_0$ in $\widehat{D}$ as $t \to a$. Such w_0 is called the prime end determined by α at one end. In general, not every prime end of D can be determined by a curve in D in this way.

2.3. *Positive harmonic functions.* Suppose D is a finitely connected domain, and $z_0 \in D$. The Green function $G(D, z_0; \cdot)$ in D with the pole at z_0 is the continuous function defined on $\widehat{D} \setminus \{z_0\}$ which vanishes on $\widehat{\partial} D$, is positive and harmonic in $D \setminus \{z_0\}$, and $G(D, z_0; z)$ behaves like $-\ln|z - z_0|/(2\pi)$ near z_0 if $z_0 \neq \infty$; behaves like $\ln|z|/(2\pi)$ near ∞ if $z_0 = \infty$.

Suppose w_0 is a prime end of D. There is a continuous function P defined on $\widehat{D} \setminus \{w_0\}$ which vanishes on $\widehat{D} \setminus \{w_0\}$, and is harmonic and positive in D. It is called a generalized Poisson kernel in D with the pole at w_0. Such P is not unique. But any two generalized Poisson kernels in D with the pole at w_0 differ by a positive multiple constant. Suppose $z_0 \in \partial D$, and ∂D is analytic near z_0; then z_0 corresponds to a prime end of D, and the Poisson kernel in D with the pole at z_0 in the usual sense is well defined, and is an example of a generalized Poisson kernel in D with the pole at z_0.

Suppose I is a side arc of D. The harmonic measure function $H(D, I; \cdot)$ is a bounded continuous function defined on $\widehat{D}$ taking away the end points of I, which is harmonic in D, vanishes on $\widehat{\partial} D \setminus \overline{I}$, and takes constant value 1 on I except the end points. For any $z \in D$, $H(D, I; z)$ is equal to the probability that the plane Brownian motion started from z first hits ∂D at I.

2.4. *Hulls and Loewner chains.* Suppose D is an n-connected domain, and σ is a side of D. Let $A(\sigma)$ be the connected component of $\widehat{\mathbb{C}} \setminus D$ that corresponds to σ. A closed subset H is called a hull of D on σ if $D \setminus H$ is also an n-connected domain, and $A(\sigma) \cup H$ is a component of $\widehat{\mathbb{C}} \setminus (D \setminus H)$. Then other components of $\widehat{\mathbb{C}} \setminus (D \setminus H)$ are the components of $\widehat{\mathbb{C}} \setminus D$ other than $A(\sigma)$.

In this paper, we define a crosscut to be an open simple curve α in D, whose two ends approach to two points on ∂D, in the Lebesgue metric, such that $D \setminus \alpha$ has two components, one of which is simply connected. If U is a simply connected component of $D\setminus\alpha$, then $U\cup\alpha$ is a hull in D. If $n>1$, that is, D is not simply connected, then U is determined by α, and let $H(\alpha):=U\cup\alpha$ be the hull bounded by α. If $n=1$, then the two components of $D\setminus\alpha$ are both simply connected, so we need some other restrictions to determine $H(\alpha)$. For example, if we say that $H(\alpha)$ is a neighborhood of some prime end w_0 in D, then there is no ambiguity.

Suppose σ is a side of D. A Loewner chain in D on σ is a function L from $[0,T)$ for some $T\in(0,+\infty]$ into the set of hulls in D on σ such that $L(0)=\varnothing$, $L(t_1)\subsetneqq L(t_2)$ if $0\le t_1<t_2<T$, and for any fixed $b\in[0,T)$ and any compact subset F of $D\setminus L(b)$, the extremal length (see [1]) of the family of curves in $D\setminus L(t+\varepsilon)$ that separates F from $L(t+\varepsilon)\setminus L(t)$ tends to 0 as $\varepsilon\to 0^+$, uniformly w.r.t. $t\in[0,b]$. Suppose $L(t)$, $0\le t<T$, is a Loewner chain in D on σ. For each $t\in[0,T)$, let d_t be any metric on $\widehat{D\setminus L(t)}$. From the definition, the d_t-diameter of $L(t+\varepsilon)\setminus L(t)$ tends to 0 as $\varepsilon\to 0^+$. Thus there is a unique prime end $w(t)$ of $D\setminus L(t)$ that lies on the closure of $L(t+\varepsilon)\setminus L(t)$ in $\widehat{D\setminus L(t)}$ for all $\varepsilon>0$. We call $w(t)$ the prime end determined by L at time t. Especially, $w(0)$ is a prime end on σ. We say L is a Loewner chain started from $w(0)$. It is clear that for any $b\in[0,T)$, $t\mapsto L(b+t)$, $0\le t<T-b$, is a Loewner chain in $D\setminus L(b)$ started from $w(b)$.

Suppose $L(t)$, $0\le t<T$, is a Loewner chain in D. Suppose u is a continuous (strictly) increasing function defined on $[0,T)$ with $u(0)=0$. Let $u(T):=\sup u([0,T))$. Then $L'(t):=L(u^{-1}(t))$, $0\le t<u(T)$, is also a Loewner chain in D. Such L' is called a time-change of L through u. Moreover, the prime end determined by L' at time $u(t)$ is the same as the prime end determined by L at time t.

One example of a Loewner chain is constructed by a simple curve. Suppose $\gamma:[0,T)\to\widehat{D}$ is a simple curve that satisfies $\gamma(0)\in\widehat{\partial}D$ and $\gamma(t)\in D$ for $0\le t<T$. Let $L(t)=\gamma((0,t])$, $0\le t<T$. Then L is a Loewner chain in D started from $\gamma(0)$, and $\gamma(t)$ corresponds to the prime end determined by L at time t. We say that L is the Loewner chain generated by γ.

3. Continuous LERW.

3.1. *Chordal Loewner equation.* Let $\mathbb{H}=\{z\in\mathbb{C}:\operatorname{Im} z>0\}$. Then $\mathbb{H}$ is a 1-connected domain whose side is $\widehat{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$. We say H is a hull in $\mathbb{H}$ w.r.t. ∞ if H is a hull in $\mathbb{H}$ and H is bounded (i.e., bounded away from ∞). A Loewner chain L in $\mathbb{H}$ w.r.t. ∞ is a Loewner chain in $\mathbb{H}$ such that each $L(t)$ is a hull in $\mathbb{H}$ w.r.t. ∞. For each hull H in $\mathbb{H}$ w.r.t. ∞, there is a unique function φ_H that maps $\mathbb{H}\setminus H$ conformally onto $\mathbb{H}$ such that for some $c\ge 0$,

$$\varphi_H(z)=z+\frac{c}{z}+O\left(\frac{1}{z^2}\right),$$

as $z\to\infty$. Such c is called the capacity of H in $\mathbb{H}$ w.r.t. ∞, denoted by $\operatorname{hcap}(H)$. The empty set is a hull in $\mathbb{H}$ w.r.t. ∞, and $\varphi_\varnothing=\mathrm{id}$, so $\operatorname{hcap}(\varnothing)=0$.

PROPOSITION 3.1. *Suppose Ω is an open neighborhood of $x_0\in\mathbb{R}$ in $\mathbb{H}$. Suppose W maps Ω conformally into $\mathbb{H}$ such that for some $r>0$, if $z\to(x_0-r,x_0+r)$ in Ω, then $W(z)\to\mathbb{R}$. So W extends conformally across (x_0-r,x_0+r) by the Schwarz reflection principle. Then for any $\varepsilon>0$, there is some $\delta>0$ such that if a hull H in $\mathbb{H}$ w.r.t. ∞ is contained in $\{z\in\mathbb{H}:|z-x_0|<\delta\}$, then $W(H)$ is also a hull in $\mathbb{H}$ w.r.t. ∞, and*

$$|\operatorname{hcap}(W(H))-W'(x_0)^2\operatorname{hcap}(H)|\le\varepsilon|\operatorname{hcap}(H)|.$$

PROOF. This is Lemma 2.8 in [6]. $\square$

For $T\in(0,+\infty]$, let $C([0,T))$ denote the space of real-valued continuous functions on $[0,T)$. Suppose $\xi\in C([0,T))$. We solve the chordal Loewner equation:

$$\partial_t\varphi_t(z)=\frac{2}{\varphi_t(z)-\xi(t)},\qquad \varphi_0(z)=z,$$

for $0\le t<T$. For each $t\in[0,T)$, let K_t be the set of $z\in\mathbb{H}$ such that the solution $\varphi_s(z)$ blows up before or at time t. We say that φ_t and K_t, $0\le t<T$, are chordal Loewner maps and hulls, respectively, driven by ξ.

For $0\le t<T$, K_t is a bounded closed subset of $\mathbb{H}$, φ_t maps $\mathbb{H}\setminus K_t$ conformally onto $\mathbb{H}$, and satisfies

$$\varphi_t(z)=z+\frac{2t}{z}+O\left(\frac{1}{z^2}\right)$$

as $z\to\infty$. So K_t is a hull in $\mathbb{H}$ w.r.t. ∞, $\operatorname{hcap}(K_t)=2t$ and $\varphi_{K_t}=\varphi_t$.

PROPOSITION 3.2. (i) *Suppose φ_t and K_t, $0\le t<T$, are chordal Loewner maps and hulls, respectively, driven by ξ. Then $t\mapsto K_t$, $0\le t<T$, is a Loewner chain in $\mathbb{H}$ w.r.t. ∞ started from $\xi(0)$. And for each $t\in[0,T)$, $\operatorname{hcap}(K_t)=2t$, $\varphi_t=\varphi_{K_t}$, and*

$$\{\xi(t)\}=\bigcap_{\varepsilon>0}\overline{\varphi_t(K_{t+\varepsilon}\setminus K_t)}.$$

(ii) *Suppose $L(t)$, $0\le t<T$, is a Loewner chain in $\mathbb{H}$ w.r.t. ∞. Let $v(t)=\operatorname{hcap}(L(t))/2$, $0\le t<T$. Then v is a continuous increasing function with $v(0)=0$. And $K_t:=L(v^{-1}(t))$, $0\le t<v(T)$, are chordal Loewner hulls driven by some $\xi\in C([0,v(T)))$.*

PROOF. This is almost the same as Theorem 2.6 in [6]. □

Fix $b \in [0,T)$. Let $\varphi_{b,t} = \varphi_t \circ \varphi_b^{-1}$ and $K_{b,t} = \varphi_b(K_t \setminus K_b)$ for $b \le t < T$. Then it is easy to check that $K_{b,b+t}$ and $\varphi_{b,b+t}$, $0 \le t < T-b$, are chordal Loewner hulls and maps driven by $t \mapsto \xi(b+t)$, $0 \le t < T-b$. Thus for any $s < t \in [0,T)$, $\varphi_s(K_t \setminus K_s)$ is a hull in $\mathbb{H}$ w.r.t. ∞, and its capacity in $\mathbb{H}$ w.r.t. ∞ is $2(t-s)$.

3.2. *Continuous LERW aiming at an interior point.* We define an almost $\mathbb{H}$ domain to be a finitely connected domain in $\mathbb{H}$ that is bounded by $\widehat{\mathbb{R}}$ and mutually disjoint analytic Jordan curves in $\mathbb{H}$. Let Ω be an almost $\mathbb{H}$ domain, and $p \in \Omega$. If K is a hull in $\mathbb{H}$ w.r.t. ∞ such that $K \subset \Omega \setminus \{p\}$, let $\Omega_K = \varphi_K(\Omega \setminus K)$. Then Ω_K is also an almost $\mathbb{H}$ domain, and $\varphi_K(p) \in \Omega_K$.

For $a \ge 0$, let $C([0,a])$ be the space of all real-valued continuous functions defined on $[0,a]$ with norm $\|\xi\|_a := \sup\{|\xi(t)| : 0 \le t \le a\}$. For $\xi \in C([0,a])$, let K_t^ξ and φ_t^ξ, $0 \le t \le a$, be chordal Loewner hulls and maps, respectively, driven by ξ. If $K_t^\xi \subset \Omega \setminus \{p\}$, we write Ω_t^ξ for $\Omega_{K_t^\xi}$. Define

$$J_t^\xi(z) = G(\Omega \setminus K_t^\xi, p; \cdot) \circ (\varphi_t^\xi)^{-1}. \tag{3.1}$$

Since $J_t^\xi = G(\Omega_t^\xi, \varphi_t^\xi(p); \cdot)$ is positive and harmonic in $\Omega_t^\xi \setminus \{\varphi_t^\xi(p)\}$, and vanishes on $\mathbb{R}$, so it extends harmonically across $\mathbb{R}$. Let

$$X_t^\xi = (\partial_x \partial_y / \partial_y) J_t^\xi(\xi(t)) = \partial_x \partial_y J_t^\xi(\xi(t)) / \partial_y J_t^\xi(\xi(t)).$$

We begin with a theorem. The proof is postponed to Section 5 in this paper.

THEOREM 3.1. *For any $A \in C([0,\infty))$ and $\lambda \in \mathbb{R}$, the equation*

$$\xi(t) = A(t) + \lambda \int_0^t X_s^\xi \, ds \tag{3.2}$$

has a unique maximal solution $\xi(t) = \xi_A(t)$, $0 \le t < T_A$, where $T_A \in (0,\infty]$. Here "maximal" means that the solution cannot be extended. Moreover, we have:

(i) *For any $a \in (0,\infty)$, the set $\{A \in C([0,\infty)) : T_A > a\}$ is open w.r.t. the metric $\|\cdot\|_a$, and $A \mapsto \xi_A$ is $(\|\cdot\|_a, \|\cdot\|_a)$ continuous on $\{A \in C([0,\infty)) : T_A > a\}$.*

(ii) *There is no crosscut α in $\mathbb{H}$ such that $\bigcup_{0 \le t < T_A} K_t^\xi \subset H(\alpha) \subset \Omega \setminus \{p\}$.*

Suppose D is a finitely connected domain, w_0 is a prime end of D, and $z_e \in D$. There is f that maps D conformally onto an almost $\mathbb{H}$ domain Ω, such that $f(w_0) = 0$. Let $p = f(z_e)$, $B(t)$ be a Brownian motion, and $\xi(t)$, $0 \le t < T$, be the maximal solution to (3.2) with $A(t) = \sqrt{2}B(t)$ and $\lambda = 2$.

Let $\{\mathcal{F}_t\}$ be the filtration generated by $B(t)$. From Theorem 3.1(i), T is an $\{\mathcal{F}_t\}$-stopping time, and $(\xi(t))$ is $\{\mathcal{F}_t\}$-adapted. For $0 \le t < T$, let

$$u(t) = \int_0^t (\partial_y J_s^\xi(\xi(s)))^2\, ds. \tag{3.3}$$

Let $S = u(T)$, and $F(t) = f^{-1}(K^\xi_{u^{-1}(t)})$, $0 \le t < S$. In the next subsection, we will prove the following theorem.

THEOREM 3.2. *For $j = 1, 2$, suppose f_j maps D conformally onto some almost $\mathbb{H}$ domain Ω_j such that $f_j(w_0) = 0$. For $j = 1, 2$, let $p_j = f_j(z_e)$, $B_j(t)$ be a Brownian motion, and $\xi_j(t)$, $0 \le t < T_j$, be the maximal solution to*

$$\begin{aligned} \xi_j(t) &= \sqrt{2} B_j(t) \\ &\quad + 2\int_0^t (\partial_x\partial_y/\partial_y)(G(\Omega_j \setminus K_s^{\xi_j}, p_j; \cdot) \circ (\varphi_s^{\xi_j})^{-1})(\xi_j(s))\, ds; \end{aligned} \tag{3.4}$$

and let $u_j(t)$, $0 \le t < T_j$, be defined by

$$u_j(t) = \int_0^t \partial_y (G(\Omega_j \setminus K_s^{\xi_j}, p_j; \cdot) \circ (\varphi_s^{\xi_j})^{-1})(\xi_j(s))^2\, ds.$$

Let $S_j = u_j(T)$ and $F_j(t) = f_j^{-1}(K^{\xi_j}_{u_j^{-1}(t)})$, $0 \le t < S_j$, $j = 1, 2$. Then $(F_1(t), 0 \le t < S_1)$ and $(F_2(t), 0 \le t < S_2)$ have the same distribution.

Thus the distribution of $(F(t), 0 \le t < S)$ does not depend on the choice of f, and is conformally invariant. We call $(F(t), 0 \le t < S)$ a continuous LERW in D from w_0 to z_e, and let it be denoted by $\mathrm{LERW}(D; w_0 \to z_e)$. From the property of chordal SLE_2 (cf. [15]) and Girsanov's theorem [11, 14], almost surely there is a simple curve $\gamma(t) : [0, S) \to \widehat{D}$ such that $\gamma(0) = w_0$, $\gamma(t) \in D$ for $0 < t < S$, and $F(t) = \gamma((0, t])$ for $0 \le t < S$, that is, F is the Loewner chain generated by γ. We call such γ an $\mathrm{LERW}(D; w_0 \to z_e)$ trace.

REMARK. If D is a 1-connected domain, w_0 is a prime end of D and $z_e \in D$, then an $\mathrm{LERW}(D; w_0 \to z_e)$ has the same distribution as a radial $\mathrm{SLE}_2(D; w_0 \to z_e)$ up to a linear time-change.

3.3. *Conformal invariance.*

PROOF OF THEOREM 3.2. For $j = 1, 2$, let $v_j = u_j^{-1}$ and $L_j(t) = K^{\xi_j}_{u_j^{-1}(t)}$. Then $F_j(t) = f_j^{-1}(L_j(t))$, $0 \le t < S_j$. Let $W = f_2 \circ f_1^{-1}$. Then W maps Ω_1 conformally onto Ω_2, $W(0) = 0$ and $W(p_1) = p_2$. Let $L_{2'}(t) = W(L_1(t))$, $0 \le t < S_1$. It suffices to show that $(L_{2'}(t), 0 \le t < S_1)$ has the same distribution as $(L_2(t), 0 \le t < S_2)$. Let $\beta_1(t)$ be the random simple curve that generates $L_1(t)$, that is, $\beta_1(0) = 0$, $\beta_1(t) \in \Omega_1$, $0 < t < S_1$, and $L_1(t) = \beta_1((0,t])$, $0 \le t < S_1$. Let $\beta_{2'}(t) = W(\beta_1(t))$, $0 \le t < S_1$. Then $\beta_{2'}$ is a simple curve, $\beta_{2'}(0) = 0$, $\beta_{2'}(t) \in \Omega_2 \subset \mathbb{H}$, $0 < t < S_1$, and $L_{2'}(t) = \beta_{2'}((0,t])$, $0 \le t < S_1$. Thus $L_{2'}$ is a Loewner chain in $\mathbb{H}$ w.r.t. ∞. Let $v_{2'}(t) = \text{hcap}(L_{2'}(t))/2$, $0 \le t < S_1$. Let $T_{2'} = v_{2'}(S_1)$ and $u_{2'} = v_{2'}^{-1}$. Then from Proposition 3.2, $L_{2'}(u_{2'}(t)) = K_t^{\xi_{2'}}$, $0 \le t < T_{2'}$, for some $\xi_{2'} \in C([0, T_{2'}))$.

Let $\{\mathcal{F}_t^1\}$ be the filtration generated by $B_1(t)$. Let

$$R_1(t,x) = \partial_y (G(\Omega_1 \setminus K_t^{\xi_1}, p_1; \cdot) \circ (\varphi_t^{\xi_1})^{-1})(x).$$

From Theorem 3.1(i), $(\xi_1(t))$ and $R_1(t,x)$ are $\mathcal{F}_t^1$-adapted, and T_1 is an $\mathcal{F}_t^1$-stopping time. Thus for $0 \le t < T_1$, we have

$$d\xi_1(t) = \sqrt{2}\, dB_1(t) + 2\, \frac{\partial_x R_1(t, \xi_1(t))}{R_1(t,\xi_1(t))}\, dt$$

and

$$u_1'(t) = R_1(t, \xi_1(t))^2.$$

So there is another Brownian motion $\breve{B}_1(t)$ such that for $0 \le t < S_1$,

$$d\xi_1(v_1(t)) = \frac{\sqrt{2}}{R_1(v_1(t), \xi_1(v_1(t)))}\, d\breve{B}_1(t) + 2\, \frac{\partial_x R_1(v_1(t), \xi_1(v_1(t)))}{R_1(v_1(t), \xi_1(v_1(t)))^3}\, dt. \tag{3.5}$$

Note that W maps $\Omega_1 \setminus L_1(t)$ conformally onto $\Omega_2 \setminus L_{2'}(t)$. Let $\Omega_1(t) = \varphi_{v_1(t)}^{\xi_1}(\Omega_1 \setminus L_1(t))$, $\Omega_{2'}(t) = \varphi_{v_{2'}(t)}^{\xi_{2'}}(\Omega_2 \setminus L_{2'}(t))$ and $W_t = \varphi_{v_{2'}(t)}^{\xi_{2'}} \circ W \circ (\varphi_{v_1(t)}^{\xi_1})^{-1}$. Then both $\Omega_1(t)$ and $\Omega_{2'}(t)$ are almost $\mathbb{H}$ domains, and W_t maps $\Omega_1(t)$ conformally onto $\Omega_{2'}(t)$, and maps $\widehat{\mathbb{R}}$ onto itself. For $t \in [0, S_1)$ and $\varepsilon \in [0, S_1 - t)$, define $L_1(t,\varepsilon) = \varphi_{v_1(t)}^{\xi_1}(K_{v_1(t+\varepsilon)}^{\xi_1} \setminus K_{v_1(t)}^{\xi_1})$ and $L_{2'}(t,\varepsilon) = \varphi_{v_{2'}(t)}^{\xi_{2'}}(K_{v_{2'}(t+\varepsilon)}^{\xi_{2'}} \setminus K_{v_{2'}(t)}^{\xi_{2'}})$. Then $\text{hcap}(L_1(t,\varepsilon)) = 2(v_1(t+\varepsilon) - v_1(t))$, $\text{hcap}(L_{2'}(t,\varepsilon) = 2(v_{2'}(t+\varepsilon) - v_{2'}(t)))$, and $W_t(L_1(t,\varepsilon)) = L_{2'}(t,\varepsilon)$. From Proposition 3.2, we have $\{\xi_1(v_1(t))\} = \bigcap_{\varepsilon>0} \overline{L_1(t,\varepsilon)}$ and $\{\xi_{2'}(v_{2'}(t))\} = \bigcap_{\varepsilon>0} \overline{L_{2'}(t,\varepsilon)}$. Thus $\xi_{2'}(v_{2'}(t)) = W_t(\xi_1(v_1(t)))$. From Proposition 3.1, we have $v_{2'}'(t) = W_t'(\xi_1(v_1(t)))^2 v_1'(t)$.

Differentiate the equality $W_t \circ \varphi_{v_1(t)}^{\xi_1} = \varphi_{v_{2'}(t)}^{\xi_{2'}} \circ W$ w.r.t. t. We get

$$\partial_t W_t(\varphi_{v_1(t)}^{\xi_1}(z)) + \frac{2W_t'(\varphi_{v_1(t)}^{\xi_1}(z)) v_1'(t)}{\varphi_{v_1(t)}^{\xi_1}(z) - \xi_1(v_1(t))} = \frac{2v_{2'}'(t)}{\varphi_{v_{2'}(t)}^{\xi_{2'}} \circ W(z) - \xi_{2'}(v_{2'}(t))}$$

for any $z \in \Omega_1 \setminus L_1(t)$. Since $\varphi_{v_1(t)}^{\xi_1}$ maps $\Omega_1 \setminus L_1(t)$ conformally onto $\Omega_1(t)$, so for any $w \in \Omega_1(t)$, we have

$$\partial_t W_t(w) = \frac{2W_t'(\xi_1(v_1(t)))^2 v_1'(t)}{W_t(w) - W_t(\xi_1(v_1(t)))} - \frac{2W_t'(w) v_1'(t)}{w - \xi_1(v_1(t))}.$$

Let $w \to \xi_1(v_1(t))$ in $\Omega_1(t)$; from Taylor expansion of W_t at $\xi_1(v_1(t))$, we get

$$\begin{aligned}\partial_t W_t(\xi_1(v_1(t))) &= -3W_t''(\xi_1(v_1(t)))v_1'(t)\\ &= -3W_t''(\xi_1(v_1(t)))/R_1(v_1(t),\xi_1(v_1(t)))^2.\end{aligned}$$

Since $\xi_{2'}(v_{2'}(t)) = W_t(\xi_1(v_1(t)))$, so from (3.5) and Itô's formula [11, 14], we have

$$\begin{aligned}d\xi_{2'}(v_{2'}(t)) &= \partial_t W_t(\xi_1(v_1(t)))\,dt + W_t'(\xi_1(v_1(t)))\,d\xi_1(v_1(t))\\ &\quad + W_t''(\xi_1(v_1(t)))\,d\langle \xi_1(v_1(t))\rangle/2\\ &= \frac{\sqrt{2}W_t'(\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))}\,d\breve{B}_1(t)\\ &\quad + 2\,\frac{W_t'(\xi_1(v_1(t)))\,\partial_x R_1(v_1(t),\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))^3}\,dt\\ &\quad + \partial_t W_t(\xi_1(v_1(t)))\,dt + \frac{W_t''(\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))^2}\,dt\\ &= \frac{\sqrt{2}W_t'(\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))}\,d\breve{B}_1(t)\\ &\quad + 2\Big(\frac{W_t'(\xi_1(v_1(t)))\,\partial_x R_1(v_1(t),\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))^3}\\ &\qquad\qquad - \frac{W_t''(\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))^2}\Big)\,dt.\end{aligned} \tag{3.6}$$

Since $\varphi^{\xi_1}_{v_1(t)}$ maps $\Omega_1 \setminus L_1(t)$ conformally onto $\Omega_1(t)$, so

$$\begin{aligned}R_1(v_1(t),x) &= \partial_y (G(\Omega_1 \setminus K^{\xi_1}_{v_1(t)}, p_1; \cdot) \circ (\varphi^{\xi_1}_{v_1(t)})^{-1})(x)\\ &= \partial_y G(\Omega_1(t), \varphi^{\xi_1}_{v_1(t)}(p_1); \cdot)(x).\end{aligned}$$

Since W_t maps $\Omega_1(t)$ conformally onto $\Omega_{2'}(t)$, and $W_t(\varphi^{\xi_1}_{v_1(t)}(p_1)) = \varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2)$, so

$$G(\Omega_1(t), \varphi^{\xi_1}_{v_1(t)}(p_1); \cdot) = G(\Omega_{2'}(t), \varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2); \cdot) \circ W_t.$$

Thus

$$\begin{aligned}R_1(v_1(t),x) &= \partial_y G(\Omega_{2'}(t), \varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2); W_t(x))W_t'(x);\\ \partial_x R_1(v_1(t),x) &= \partial_x\partial_y G(\Omega_{2'}(t), \varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2); W_t(x))(W_t'(x))^2\\ &\quad + \partial_y G(\Omega_{2'}(t), \varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2); W_t(x))W_t''(x).\end{aligned}$$

Plugging these equalities into (3.6) and letting $x = \xi_1(v_1(t))$, we get

$$d\xi_{2'}(v_{2'}(t)) = \frac{\sqrt{2}W_t'(\xi_1(v_1(t)))}{R_1(v_1(t),\xi_1(v_1(t)))}\,d\breve{B}_1(t) + 2\frac{\partial_x\partial_y G(\Omega_{2'}(t),\varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2);\xi_{2'}(v_{2'}(t)))}{\partial_y G(\Omega_{2'}(t),\varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2);\xi_{2'}(v_{2'}(t)))^3}\,dt.$$

Since

$$\begin{aligned} v_{2'}'(t) &= W_t'(\xi_1(v_1(t)))^2 v_1'(t) \\ &= \frac{W_t'(\xi_1(v_1(t)))^2}{R_1(v_1(t),\xi_1(v_1(t)))^2} \\ &= \partial_y G(\Omega_{2'}(t),\varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2);\xi_{2'}(v_{2'}(t)))^{-2}, \end{aligned} \tag{3.7}$$

and $G(\Omega_{2'}(t),\varphi^{\xi_{2'}}_{v_{2'}(t)}(p_2);\cdot) = G(\Omega_2\setminus K^{\xi_{2'}}_{v_{2'}(t)},p_2;\cdot)\circ\varphi^{\xi_{2'}}_{v_{2'}(t)}$, so for $0\le t<T_{2'}$,

$$d\xi_{2'}(t) = \sqrt{2}\,dB_{2'}(t) + 2(\partial_x\partial_y/\partial_y)(G(\Omega_2\setminus K^{\xi_{2'}}_t,p_2;\cdot)\circ\varphi^{\xi_{2'}}_t)(\xi_{2'}(t))\,dt$$

for another Brownian motion $B_{2'}(t)$. Since $\xi_{2'}(0) = W_0(\xi_1(0)) = W(0) = 0$, so for $0\le t<T_{2'}$,

$$\begin{aligned} \xi_{2'}(t) &= \sqrt{2}B_{2'}(t) \\ &\quad + 2\int_0^t (\partial_x\partial_y/\partial_y)(G(\Omega_2\setminus K^{\xi_{2'}}_s,p_2;\cdot)\circ\varphi^{\xi_{2'}}_s)(\xi_{2'}(s))\,ds. \end{aligned} \tag{3.8}$$

We claim that $\xi_{2'}(t)$, $0\le t<T_{2'}$, is the maximal solution to (3.8). Suppose the claim is not true. Then it may happen that the solution $\xi_{2'}$ extends to $[0,T_{2'}]$. Note that $W(\infty)$ is a prime end on $\widehat{\mathbb{R}}$ other than $W(0)=0$. We may find a crosscut α in $\mathbb{H}$ such that $K^{\xi_{2'}}_{T_{2'}}\subset H(\alpha)\subset\Omega_2\setminus\{p_2\}$, and $W(\infty)\notin\overline{H(\alpha)}$. Then $W^{-1}(\alpha)$ is also a crosscut in $\mathbb{H}$, and $H(W^{-1}(\alpha)) = W^{-1}(H(\alpha))\subset\Omega_1\setminus\{p_1\}$. So $W^{-1}(K^{\xi_{2'}}_t)\subset H(W^{-1}(\alpha))$ for $0\le t<T_{2'}$, which implies that $K^{\xi_1}_t\subset H(W^{-1}(\alpha))$ for $0\le t<T_1$. This contradicts Theorem 3.1(ii). So the claim is justified.

Since $\xi_{2'}(t)$, $0\le t<T_{2'}$ [resp. $\xi_2(t)$, $0\le t<T_2$], is the maximal solution to (3.8) [resp. equation (3.2) when $j=2$], and $(B_{2'}(t))$ has the same distribution as $(B_2(t))$, so $(\xi_{2'}(t), 0\le t<T_{2'})$ has the same distribution as $(\xi_2(t), 0\le t< T_2)$. From (3.7), $u_{2'} = v_{2'}^{-1}$, and that $u_{2'}(0)=0$, we see that for $0\le t<T_{2'}$,

$$u_{2'}(t) = \int_0^t \partial_y(G(\Omega_2\setminus K^{\xi_{2'}}_s,p_2;\cdot)\circ\varphi^{\xi_{2'}}_s)(\xi_{2'}(s))^2\,ds.$$

Thus $((\xi_{2'}(t),u_{2'}(t)), 0\le t<T_{2'})$ has the same distribution as $((\xi_2(t),u_2(t)), 0\le t<T_2)$. Since $L_{2'}(t) = K^{\xi_{2'}}_{u_{2'}^{-1}(t)}$ for $0\le t<S_1 = u_{2'}(T_{2'})$, and $L_2(t) = K^{\xi_2}_{u_2^{-1}(t)}$

for $0 \le t < S_2 = u_2(T_2)$, so $(L_{2'}(t), 0 \le t < S_1)$ has the same distribution as $(L_2(t), 0 \le t < S_2)$. $\square$

3.4. *Continuous LERW with other kinds of targets.* Suppose D is a finitely connected domain, w_0 is a prime end of D, and I_e is a side arc of D that is bounded away from w_0. Then there is f that maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(w_0) = 0$. If a hull K in $\mathbb{H}$ w.r.t. ∞ is bounded away from $f(I_e)$, and $K \subset \Omega$, then $f(I_e)$ is a side arc of $\Omega \setminus K$. We have the harmonic measure function $H(\Omega \setminus K, f(I_e); \cdot)$.

Now we change the definition of J_t^ξ by replacing $G(\Omega \setminus K_t^\xi, p; \cdot)$ by $H(\Omega \setminus K_t^\xi, f(I_e); \cdot)$ in (3.1), and still let $X_t^\xi = (\partial_x \partial_y / \partial_y) J_t^\xi(\xi(t))$. Let everything else in Section 3.2 be unchanged. Then Theorem 3.1 still holds if the condition on α is replaced by that α is a crosscut in $\mathbb{H}$ such that $H(\alpha) \subset \Omega$ and $H(\alpha)$ is bounded away from $f(I_e)$. Let $u(t)$ be defined by (3.3). Then $(F(t) = f^{-1}(K^\xi_{u^{-1}(t)}), 0 \le t < S = u(T))$ is called a continuous LERW in D from w_0 to I_e, and is denoted by $\mathrm{LERW}(D; w_0 \to I_e)$. It is almost surely generated by a random simple curve, which is called an $\mathrm{LERW}(D; w_0 \to I_e)$ trace. The variation of Theorem 3.2 for $\mathrm{LERW}(D; w_0 \to I_e)$ still holds. Thus the distribution of $\mathrm{LERW}(D; w_0 \to I_e)$ does not depend on the choice of f, and is conformally invariant.

Suppose D is a finitely connected domain, w_0 and w_e are two different prime ends of D. There is f that maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(w_0) = 0$. Then $p := f(w_e)$ is a prime end of Ω other than 0. If a hull K in $\mathbb{H}$ w.r.t. ∞ is bounded away from p, and $K \subset \Omega$, then p is a prime end of $\Omega \setminus K$.

A normalization function is a function h that maps a neighborhood U of p in $\widehat{\Omega}$ conformally onto a neighborhood V of 0 in $\overline{\mathbb{H}}$ such that $h(p) = 0$ and $h(U \cap \widehat{\partial} D) \subset \mathbb{R}$. There is a unique generalized Poisson kernel $P(z)$ in $\Omega \setminus K$ with the pole at p such that the principal part of $P \circ h^{-1}(z)$ at 0 is $\mathrm{Im} \frac{-1}{z}$. Let $P(\Omega \setminus K, p, h; z)$ denote this function.

Now fix a normalizing function h. Change the definition of J_t^ξ by replacing $G(\Omega \setminus K_t^\xi, p; \cdot)$ by $P(\Omega \setminus K_t^\xi, p, h; \cdot)$ in (3.1), and still let $X_t^\xi = (\partial_x \partial_y / \partial_y) J_t^\xi(\xi(t))$. Let everything else in Section 3.2 be unchanged. Then Theorem 3.1 still holds if the condition on α is replaced by that α is a crosscut in $\mathbb{H}$ such that $H(\alpha) \subset \Omega$, and $H(\alpha)$ is bounded away from $p = f(w_e)$. Let $u(t)$ be defined by (3.3). Then $(F(t) = f^{-1}(K^\xi_{u^{-1}(t)}), 0 \le t < S = u(T))$ is called a continuous LERW in D from w_0 to w_e, normalized by h, and is denoted by $\mathrm{LERW}(D; w_0 \to w_e)$. It is almost surely generated by a random simple curve, which is called an $\mathrm{LERW}(D; w_0 \to w_e)$ trace normalized by h. The variation of Theorem 3.2 for $\mathrm{LERW}(D; w_0 \to w_e)$ holds with simple modification: $(F_1(t), 0 \le t < S_1)$ and $(F_2(t/a^2), 0 \le t < a^2 S_2)$ have the same distribution, where $a = (h_2 \circ h_1^{-1})'(0)$ and h_j, $j = 1, 2$, are normalization functions. Thus the distribution of $\mathrm{LERW}(D; w_0 \to w_e)$ up to a linear time-change does not depend on the choices of f and h, and is conformally invariant.

REMARK. (i) If D is a 1-connected domain, and $w_0 \neq w_e$ are two prime ends of D, then an $\mathrm{LERW}(D; w_0 \to w_e)$ has the same distribution as a chordal $\mathrm{SLE}_2(D; w_0 \to w_e)$ up to a linear time-change.

(ii) If D is a 1-connected domain, w_0 is a prime end of D, and I_e is a side arc of D that is bounded away from w_0, then an $\mathrm{LERW}(D; w_0 \to I_e)$ has the same distribution as a strip or dipolar $\mathrm{SLE}_2(D; w_0 \to I_e)$ (cf. [2, 21]) up to a linear time-change.

(iii) If D is a 2-connected domain, w_0 is a prime end of D, and I_e is a side of D that does not contain w_0, then an $\mathrm{LERW}(D; w_0 \to I_e)$ has the same distribution as an annulus $\mathrm{SLE}_2(D; w_0 \to I_e)$ (cf. [20]) up to a deterministic time-change.

4. Observables generated by martingales.

4.1. *Local martingales for continuous LERW.* Suppose D is a finitely connected domain, $z_e \in D$, and w_0 is a prime end of D. Let $\gamma(t)$, $0 \le t < S$, be an $\mathrm{LERW}(D; w_0 \to z_e)$ trace. So γ is a simple curve in $\widehat{D}$ with $\gamma(0) = w_0$ and $\gamma(t) \in D$ for $0 < t < S$. For $0 \le t < S$, let P_t be the generalized Poisson kernel in $D \setminus \gamma((0, t])$ with the pole at $\gamma(t)$, normalized by $P_t(z_e) = 1$.

THEOREM 4.1. *For any fixed $z \in D$, $(P_t(z))$ is a local martingale.*

Let Ω be an almost $\mathbb{H}$ domain, and $p \in \Omega$. If K is a hull in $\mathbb{H}$ w.r.t. ∞ such that $K \subset \Omega \setminus \{p\}$, let $P(K, x, \cdot)$ be the generalized Poisson kernel in Ω_K with the pole at x, normalized by $P(K, x, \varphi_K(p)) = 1$. Suppose $\xi \in C([0, T))$ satisfies $\bigcup_{0 \le t < T} K_t^\xi \subset \Omega \setminus \{p\}$. We write $P^\xi(t, \cdot, \cdot)$ for $P(K_t^\xi, \cdot, \cdot)$, $t \in [0, T)$. It is standard to check that P^ξ is $C^{1,2,h}$ differentiable, where "h" means harmonic.

LEMMA 4.1. *For any $t \in [0, T)$ and $z \in \Omega \setminus K_t^\xi$, we have $\mathcal{V}_t(z) = 0$, where*

$$\begin{aligned}\mathcal{V}_t(z) &= \partial_1 P^\xi(t, \xi(t), \varphi_t^\xi(z)) + 2\partial_2 P^\xi(t, \xi(t), \varphi_t^\xi(z)) X_t^\xi \\ &\quad + \partial_2^2 P^\xi(t, \xi(t), \varphi_t^\xi(z)) + 2\,\mathrm{Re}\left(\partial_{3,z} P^\xi(t, \xi(t), \varphi_t^\xi(z)) \cdot \frac{2}{\varphi_t^\xi(z) - \xi(t)}\right).\end{aligned}$$

Here ∂_1 and ∂_2 are partial derivatives w.r.t. the first two (real) variables, and $\partial_{3,z} = (\partial_{3,x} - i\partial_{3,y})/2$ is the partial derivative w.r.t. the third (complex) variable.

PROOF. For $t \in [0,T)$ and $z \in \partial\Omega \setminus \mathbb{R}$, since $\varphi_t^\xi(z) \in \partial\Omega_t^\xi \setminus \mathbb{R}$, so $P^\xi(t,x,\varphi_t^\xi(z)) = 0$ for any $x \in \mathbb{R}$, which implies that $\partial_2 P^\xi = \partial_2^2 P^\xi = 0$ at $(t,x,\varphi_t^\xi(z))$, and

$$\partial_1 P^\xi(t,x,\varphi_t^\xi(z)) + 2\,\mathrm{Re}\left(\partial_{3,z} P^\xi(t,x,\varphi_t^\xi(z)) \cdot \frac{2}{\varphi_t^\xi(z) - \xi(t)}\right) = 0.$$

Thus $\mathcal{V}_t$ vanishes on $\partial\Omega \setminus \mathbb{R}$ for $t \in [0,T)$. Let $\mathcal{W}_t = \mathcal{V}_t \circ (\varphi_t^\xi)^{-1}$. Then $\mathcal{W}_t$ vanishes on $\partial\Omega_t^\xi \setminus \mathbb{R}$ for $t \in [0,T)$. Note that for $t \in [0,T)$ and $w \in \Omega_t^\xi$,

$$\begin{aligned}\mathcal{W}_t(w) &= \partial_1 P^\xi(t,\xi(t),w) + 2\partial_2 P^\xi(t,\xi(t),w) X_t^\xi \\ &\quad + \partial_2^2 P^\xi(t,\xi(t),w) + 2\,\mathrm{Re}\left(\partial_{3,z} P^\xi(t,\xi(t),w) \cdot \frac{2}{w - \xi(t)}\right).\end{aligned}$$

Since $P^\xi(t,\xi(t),\cdot)$ vanishes on $\mathbb{R} \setminus \{\xi(t)\}$ and $\frac{2}{w-\xi(t)}$ is real on $\mathbb{R} \setminus \{\xi(t)\}$, so $\mathcal{W}_t$ vanishes on $\mathbb{R} \setminus \{\xi(t)\}$. As $w \to \infty$ in $\mathbb{H}$, ∂_1, ∂_2, ∂_2^2 and $\partial_{3,z}$ of P^ξ at $(t,\xi(t),w)$ all tend to 0, and $\frac{2}{w-\xi(t)}$ tends to 0 as well. Thus $\mathcal{W}_t$ vanishes on $\widehat{\mathbb{R}} \setminus \{\xi(t)\}$.

Suppose for some $c(t,x) \in \mathbb{R}$, $\mathrm{Im}\,\frac{c(t,x)}{w-x}$ is the principal part of $P^\xi(t,x,w)$ at x. So there is some analytic function $F(t,x,\cdot)$ defined in some neighborhood of x such that in that neighborhood, $P^\xi(t,x,w) = \mathrm{Im}(F(t,x,w) + \frac{c(t,x)}{w-x})$. Then we have

$$\begin{aligned}\partial_1 P^\xi(t,\xi(t),w) &= \mathrm{Im}\left(\partial_1 F(t,\xi(t),w) + \frac{\partial_1 c(t,\xi(t))}{w-\xi(t)}\right), \\ \partial_2 P^\xi(t,\xi(t),w) &= \mathrm{Im}\left(\partial_2 F(t,\xi(t),w) + \frac{\partial_2 c(t,\xi(t))}{w-\xi(t)}\right. \\ &\qquad\qquad \left. + \frac{c(t,\xi(t))}{(w-\xi(t))^2}\right), \\ \partial_2^2 P^\xi(t,\xi(t),w) &= \mathrm{Im}\left(\partial_2^2 F(t,\xi(t),w) + \frac{\partial_2^2 c(t,\xi(t))}{w-\xi(t)}\right. \\ &\qquad\qquad \left. + \frac{2\partial_2 c(t,\xi(t))}{(w-\xi(t))^2} + \frac{2c(t,\xi(t))}{(w-\xi(t))^3}\right)\end{aligned}$$

and

$$2\,\mathrm{Re}\left(\partial_{3,z} P^\xi(t,\xi(t),w) \cdot \frac{2}{w-\xi(t)}\right) = \mathrm{Im}\left(\frac{2F'(t,\xi(t),w)}{w-\xi(t)} - \frac{2c(t,\xi(t))}{(w-\xi(t))^3}\right).$$

Thus $\mathcal{W}_t$ equals the imaginary part of

$$\partial_1 F(t,\xi(t),w) + \frac{\partial_1 c(t,\xi(t))}{w-\xi(t)}$$

$$+2\Big(\partial_2 F(t,\xi(t),w)+\frac{\partial_2 c(t,\xi(t))}{w-\xi(t)}+\frac{c(t,\xi(t))}{(w-\xi(t))^2}\Big)X_t^\xi$$
$$+\partial_2^2 F(t,\xi(t),w)+\frac{\partial_2^2 c(t,\xi(t))}{w-\xi(t)}+\frac{2\partial_2 c(t,\xi(t))}{(w-\xi(t))^2}+\frac{2c(t,\xi(t))}{(w-\xi(t))^3}$$
$$+\frac{2F'(t,\xi(t),w)}{w-\xi(t)}-\frac{2c(t,\xi(t))}{(w-\xi(t))^3}$$
$$=\partial_1 F(t,\xi(t),w)+2\partial_2 F(t,\xi(t),w)X_t^\xi$$
$$+\partial_2^2 F(t,\xi(t),w)+\frac{A_1(t)}{w-\xi(t)}+\frac{A_2(t)}{(w-\xi(t))^2}$$

for some functions $A_1(t)$ and $A_2(t)$, where $A_2(t)=2c(t,\xi(t))X_t^\xi+2\partial_2 c(t,\xi(t))$.

Since $J_t^\xi=G(\Omega_t^\xi,\varphi_t^\xi(p);\cdot)$, so for $x\in\mathbb{R}$, $\partial_y J_t^\xi(x)$ equals the value at $\varphi_t^\xi(p)$ of the (usual) Poisson kernel in Ω_t^ξ with the pole at x. Note that $P^\xi(t,x,\cdot)$ equals some constant times the Poisson kernel in Ω_t^ξ with the pole at x, of which the principal part at x is $\operatorname{Im}\frac{-1/\pi}{w-x}$. So we have

$$\partial_y J_t^\xi(x)/(-1/\pi)=P^\xi(t,x,\varphi_t^\xi(p))/c(t,x)=1/c(t,x).$$

Thus $c(t,x)\,\partial_y J_t^\xi(x)=-1/\pi$ for any $x\in\mathbb{R}$, which implies that

$$0=c(t,\xi(t))\,\partial_x\,\partial_y J_t^\xi(\xi(t))+\partial_2 c(t,\xi(t))\partial_y J_t^\xi(\xi(t))=A_2(t)\,\partial_y J_t^\xi(\xi(t))/2.$$

So $A_2(t)=0$, and $\mathcal{W}_t$ equals the imaginary part of some analytic function plus $\frac{A_1(t)}{w-\xi(t)}$ near $\xi(t)$. Since $\mathcal{W}_t$ is harmonic in Ω_t^ξ, and vanishes at every prime end of Ω_t^ξ other than $\xi(t)$, so $\mathcal{W}_t=C(t)P^\xi(t,\xi(t),\cdot)$ for some $C(t)\in\mathbb{R}$. From $P^\xi(t,x,\varphi_t^\xi(p))=1$ for any $t\in[0,T)$ and $x\in\mathbb{R}$, we get $\mathcal{W}_t(\varphi_t^\xi(p))=0$. So for $t\in[0,T)$, we have $C(t)=0$, which implies that $\mathcal{W}_t$ vanishes on Ω_t^ξ, and so $\mathcal{V}_t$ vanishes on $\Omega\setminus K_t^\xi$. $\square$

Suppose f maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(w_0)=0$. Let $p=f(z_e)$. Let $v(t)=\operatorname{hcap}(f(\gamma((0,t])))/2$, $0\le t<S$. Let $T=v(S)$, and u be the reversal of v. Then $f(\gamma((0,u(t)]))=K_t^\xi$, $0\le t<T$, where $\xi\in C([0,T))$ solves equation (3.2) with $\lambda=2$ and $A(t)=\sqrt{2}B(t)$ for some Brownian motion $B(t)$. Since $\varphi_t^\xi\circ f$ maps $D\setminus f(\gamma((0,u(t)]))$ conformally onto Ω_t^ξ, $\varphi_t^\xi\circ f(\gamma(u(t)))=\xi(t)$ and $\varphi_t^\xi\circ f(z_e)=\varphi_t^\xi(p)$, so from the conformal invariance, $P_{u(t)}\circ f^{-1}\circ(\varphi_t^\xi)^{-1}$ is the generalized Poisson kernel in Ω_t^ξ with the pole at $\xi(t)$, whose value at $\varphi_t^\xi(p)$ is 1, that is,

$$P_{u(t)}\circ f^{-1}\circ(\varphi_t^\xi)^{-1}=P^\xi(t,\xi(t),\cdot). \tag{4.1}$$

Proof of Theorem 4.1. Let $Q_t(z) = P^\xi(t,\xi(t),\varphi_t^\xi(z))$ for $z\in\Omega\setminus K_t^\xi$. From Itô's formula, $(Q_t(z))$ is a semimartingale, and the drift term equals $\mathcal{V}_t(z)$, which vanishes on $\Omega\setminus K_t^\xi$ by Lemma 4.1. Thus $(Q_t(z))$ is a local martingale for any fixed $z\in\Omega$. From (4.1), $P_t(z) = Q_{v(t)}(f(z))$ for $z\in D$. Since $f(D)=\Omega$, and a time-change preserves a local martingale, so $(P_t(z))$ is a local martingale for any fixed $z\in D$.

Second, we consider an $\mathrm{LERW}(D;w_0\to I_e)$ trace: $\gamma(t)$, $0\le t<S$, where w_0 is a prime end of D, and I_e is a side arc of D. Let P_t be the generalized Poisson kernel in $D\setminus\gamma((0,t])$ with the pole at $\gamma(t)$, normalized by $\int_{I_e}\partial_{\mathbf{n}}P_t(z)\,ds(z)=1$. Here the equality means that if g maps a neighborhood U of I_e in $\widehat{D}$ conformally into $\mathbb{C}$ such that $g(I_e)$ is an analytic arc, then $\int_{g(I_e)}\partial_{\mathbf{n}}(P_t\circ g^{-1})(z)\,ds(z)=1$, where $\mathbf{n}$ is the unit normal vector pointing inward, and ds is the length of the curve. In fact, the value of the integral does not depend on the choice of g.

Suppose f maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(w_0)=0$. Let $J=f(I_e)$. If $K_t^\xi\subset\Omega$, and is bounded away from J, let $P^\xi(t,x,\cdot)$ be the generalized Poisson kernel in Ω_t^ξ, with the pole at x, normalized by $\int_{\varphi_t^\xi(J)}\partial_{\mathbf{n}}P^\xi(t,x,z)\,ds(z)=1$. Then Lemma 4.1 holds in this setting, and the proof is similar. Formula (4.1) still holds, so we have Theorem 4.1.

Third, we consider an $\mathrm{LERW}(D;w_0\to w_e)$ trace: $\gamma(t)$, $0\le t<S$, where $w_0\ne w_e$ are prime ends of D. Fix g that maps a neighborhood U of w_e in $\widehat{D}$ conformally onto a neighborhood V of 0 in $\overline{\mathbb{H}}$ such that $g(w_e)=0$ and $g(U\cap\widehat{\partial}D)\subset\mathbb{R}$. Let P_t be the generalized Poisson kernel in $D\setminus\gamma((0,t])$ with the pole at $\gamma(t)$, normalized by $\partial_y(P_t\circ g^{-1})(0)=1$.

Suppose f maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(w_0)=0$. Let $p=f(w_e)$. If $K_t^\xi\subset\Omega$, and is bounded away from p, let $P^\xi(t,x,\cdot)$ be the generalized Poisson kernel in Ω_t^ξ, with the pole at x, normalized by $\partial_y(P^\xi(t,x,\cdot)\circ f\circ g^{-1})(0)=1$. Then Lemma 4.1 holds in this setting, and the proof is similar. Formula (4.1) still holds, so we have Theorem 4.1.
□

4.2. *Discrete approximations.* Let D be a finitely connected domain. Suppose $0\in\partial D$, and there is some $\delta_D>0$ such that the half open line segment $[\delta_D,0)$ is contained in D. As $z\to0$ along $[\delta_D,0)$, z tends to a prime end of D. We use 0_+ to denote this prime end.

For $\delta>0$, let $\delta\mathbb{Z}^2=\{(j+ik)\delta: j,k\in\mathbb{Z}\}\subset\mathbb{C}$. We also view $\delta\mathbb{Z}^2$ as a graph whose vertices are $(j+ik)\delta$, $j,k\in\mathbb{Z}$, and two vertices are adjacent iff the distance between them is δ. We define a graph $\breve{D}^\delta$ that approximates D in $\delta\mathbb{Z}^2$ as follows. The vertex set $V(\breve{D}^\delta)$ is the union of interior vertex set $V_I(\breve{D}^\delta)$ and boundary vertex set $V_\partial(\breve{D}^\delta)$, where $V_I(\breve{D}^\delta):=\delta\mathbb{Z}^2\cap D$, and $V_\partial(\breve{D}^\delta)$ is the set of ordered pairs $\langle z_1,z_2\rangle$ such that $z_1\in V_I(\breve{D}^\delta)$, $z_2\in\partial D$, and there is

$z_3 \in \delta\mathbb{Z}^2$ that is adjacent to z_1 in $\delta\mathbb{Z}^2$, such that $[z_1, z_2) \subset [z_1, z_3) \cap D$. Two vertices w_1 and w_2 in $V(\breve{D}^\delta)$ are adjacent iff either $w_1, w_2 \in V_I(\breve{D}^\delta)$, w_1 and w_2 are adjacent in $\delta\mathbb{Z}^2$, and $[w_1, w_2] \subset D$; or for $j = 1$ or 2, $w_j \in V_I(\breve{D}^\delta)$ and $w_{3-j} = \langle w_j, z_3\rangle \in V_\partial(\breve{D}^\delta)$ for some $z_3 \in \partial D$.

Every interior vertex of $\breve{D}^\delta$ has exactly four adjacent vertices, and every boundary vertex $w = \langle z_1, z_2\rangle$ has exactly one adjacent vertex, which is the interior vertex z_1. So $\breve{D}^\delta$ is locally finite. If $\langle z_1, z_2\rangle$ is a boundary vertex, then it determines a boundary point, which is z_2, and a prime end of D, which is the limit in $\widehat{D}$ as $z \to z_2$ along $[z_1, z_2)$. If there is no ambiguity, we do not distinguish a boundary vertex from the boundary point or prime end it determines. Suppose $\delta \in (0, \delta_D]$. Then δ is an interior vertex of $\breve{D}^\delta$, and $\langle \delta, 0\rangle$ is a boundary vertex of $\breve{D}^\delta$. A random walk on $\breve{D}^\delta$ started from an interior vertex w_0 up to the first time it leaves D agrees with a random walk on $\delta\mathbb{Z}^2$ started from w_0 up to the first time it uses an edge that intersects ∂D. Let D^δ be the connected component of $\breve{D}^\delta$ that contains δ. Let $V_I(D^\delta) := V(D^\delta) \cap V_I(\breve{D}^\delta)$ and $V_\partial(D^\delta) := V(D^\delta) \cap V_\partial(\breve{D}^\delta)$ be the set of interior and boundary vertices, respectively, of D^δ.

Fix $z_e \in D \setminus \{\infty\}$. Let w_e^δ be the vertex in $\delta\mathbb{Z}^2$ that is closest to z_e. If such vertex is not unique, we choose the one that maximizes $\operatorname{Re} z + \pi \operatorname{Im} z$ to break the tie. Suppose $\delta \in (0, \delta_D]$ is small enough. Then there is a lattice path on $\breve{D}^\delta$ that connects δ with w_e^δ, which does not pass through any boundary vertex. So w_e^δ is an interior vertex of D^δ. Let $F = \{w_e^\delta\}$ and $E_{-1} = V_\partial(D^\delta)$. From the recurrence of the random walks on $\mathbb{Z}^2$, we know that $E \cup F$ is reachable in D^δ. Let $(q_\delta(0), \dots, q_\delta(\chi_\delta))$ be the LERW on D^δ started from δ conditioned to hit F before E_{-1}. So $q_\delta(0) = \delta$ and $q_\delta(\chi_\delta) = w_e^\delta$. Let $q_\delta(-1) = 0$. Extend q_δ to $[-1, \chi_\delta]$ such that q_δ is linear on $[k-1, k]$ for each $k \in \mathbb{Z}_{[0,\chi_\delta]}$. Then q_δ is a simple curve in $D \cup \{0\}$ that connects 0 and w_e^δ.

Since F contains only one point, we may define g_k as in Proposition 2.1. Then for any fixed vertex v_0 on D^δ, $(g_k(v_0))$ is a martingale up to the time $q_\delta(k)$ is next to w_e^δ or $E_k := E_{-1} \cup \{q_\delta(0), \dots, q_\delta(k)\}$ disconnects v_0 from z_e. Note that g_k vanishes on $E_k \setminus \{q_\delta(k)\}$, is discrete harmonic at every interior vertex of D^δ except $q_\delta(0), \dots, q_\delta(k)$, and $g_k(w_e^\delta) = 1$. For $0 \le k \le \chi_\delta - 1$, let $D_k = D \setminus q_\delta([-1, k])$. Then $q_\delta(k)$ corresponds to a prime end of D_k. When δ is small, the function g_k approximates the generalized Poisson kernel P_k in D_k with the pole at $q_\delta(k)$, normalized by $P_k(z_e) = 1$. Note the resemblance of the discrete martingales preserved by (discrete) LERW and the local martingales preserved by continuous LERW. Suppose $\gamma_0(t)$, $0 \le t < S_0$, is an LERW$(D; 0_+ \to z_e)$ trace. In the last several sections, we will prove the following theorem. Note that we do not require that the boundary of D is good.

THEOREM 4.2. (i) *Suppose U is a neighborhood of 0_+ in D. Then for any $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there are a coupling of*

q_δ and γ_0, and a continuous increasing function $\breve{u}$ that maps $(-1,\chi_\delta)$ onto $(0,S_0)$ such that

$$\mathbf{P}[\sup\{|q_\delta(\breve{u}^{-1}(t))-\gamma_0(t)|:T_U(\gamma_0)\le t<S_0\}<\varepsilon]>1-\varepsilon,$$

where $T_U(\gamma_0)$ is the first time that γ_0 leaves U.

(ii) *If the prime end 0_+ is degenerate (see [13]), then* (i) *holds with "$T_U(\gamma_0)\le t$" replaced by "$0<t$."*

Now suppose $w_e\in\partial D\setminus\{0\}$ satisfies $w_e\in\delta_e\mathbb{Z}^2$ for some $\delta_e>0$, and ∂D is flat near w_e, which means that there is $r>0$ such that $D\cap\{z\in\mathbb{C}:|z-w_e|<r\}=(w_e+a\mathbb{H})\cap\{z\in\mathbb{C}:|z-w_e|<r\}$ for some $a\in\{\pm1,\pm i\}$. For $\delta>0$, let $w_e^\delta=w_e+ia\delta$.

Let $\mathcal{M}$ be the set of $\delta>0$ such that $w_e\in\delta\mathbb{Z}^2$. If $\delta\in\mathcal{M}$ is small enough, then $\langle w_e^\delta,w_e\rangle$ is a boundary vertex of $\breve{D}^\delta$, which determines the boundary point and prime end w_e, and there is a lattice path on D^δ that connects δ with w_e without passing through any other boundary vertex. Here we do not distinguish w_e from the boundary vertex $\langle w_e^\delta,w_e\rangle$. Let $F=\{w_e\}$ and $E_{-1}=V_\partial(D^\delta)\setminus F$. Then $E\cup F=V_\partial(D^\delta)$ is reachable in D^δ. Let $(q_\delta(0),\dots,q_\delta(\chi_\delta))$ be the LERW on D^δ started from δ conditioned to hit F before E_{-1}. So $q_\delta(0)=\delta$ and $q_\delta(\chi_\delta)=w_e$. Let $q_\delta(-1)=0$. Extend q_δ to be defined on $[-1,\chi_\delta]$ such that q_δ is linear on $[k-1,k]$ for each $k\in\mathbb{Z}_{[0,\chi_\delta]}$. Then q_δ is a simple curve in $D\cup\{0,w_e\}$ that connects 0 and w_e.

Let h_k be as in Proposition 2.1. Then for any fixed vertex v_0 on D^δ, $(h_k(v_0))$ is a martingale up to the time when $q_\delta(k)=w_e^\delta$ or $E_k=E_{-1}\cup\{q_\delta(0),\dots,q_\delta(k)\}$ disconnects v_0 from w_e. Let $D_k=D\setminus q_\delta([-1,k])$. Then $q_\delta(k)$ is a prime end of D_k. Note that h_k vanishes on $q_\delta(-1),\dots,q_\delta(k-1)$ and all boundary vertices of D^δ, is discrete harmonic at all interior vertices of D^δ except $q_\delta(0),\dots,q_\delta(k)$, and $h_k(w_e^\delta)=1$. So when δ is small, $\delta\cdot h_k$ is close to the generalized Poisson kernel P_k in D_k with the pole at $q_\delta(k)$ normalized by $\partial_{\mathbf{n}}P_k(w_e)=1$. Suppose $\gamma_0(t)$, $0\le t<S$, is an LERW$(D;0_+\to w_e)$ trace. Then Theorem 4.2 still holds for q_δ and γ_0 defined here if we replace "$\delta<\delta_0$" by "$\delta\in\mathcal{M}$ and $\delta<\delta_0$."

Now suppose I_e is a side arc of D that is bounded away from 0_+. Let I_e^δ be the set of boundary vertices of D^δ which determine prime ends that lie on I_e. If δ is small enough, I_e^δ is nonempty, and there is a lattice path on D^δ that connects δ with I_e^δ without passing through any boundary vertex not in I_e^δ. Then we let $F=I_e^\delta$ and $E_{-1}=V_\partial(D^\delta)\setminus F$. Let $(q_\delta(0),\dots,q_\delta(\chi_\delta))$ be the LERW on D^δ started from δ conditioned to hit F before E_{-1}. So $q_\delta(0)=\delta$ and $q_\delta(\chi_\delta)\in I_e$.

Let h_k be as in Proposition 2.1. Then for any fixed vertex v_0 on D^δ, $(h_k(v_0))$ is a martingale up to the time $q_\delta(k)$ is close to I_e or $E_k:=E_{-1}\cup\{q_\delta(0),\dots,q_\delta(k)\}$ disconnects v_0 from I_e. Note that h_k vanishes on $q_\delta(-1),\dots,q_\delta(k-1)$ and all boundary vertices of D^δ, h_k is discrete harmonic at every interior vertex of D^δ except $q_\delta(0),\dots,q_\delta(k)$, and $\sum_{v\in I_e^\delta}\Delta h_k(v)=1$. So when δ is small, the function h_k seems to be close to the generalized Poisson kernel P_k in D_k with the pole at $q_\delta(k)$ normalized by $\int_{I_e}\partial_{\mathbf{n}}P_k(z)\,ds(z)=1$.

If I_e is a whole side of D, then Theorem 4.2 still holds for q_δ and γ_0 defined here. If I_e is not a whole side, for the purpose of convergence, we may need some additional boundary conditions. Suppose the two ends of I_e correspond to $w_e^1, w_e^2\in\partial D$, near which ∂D is flat, and $w_e^1,w_e^2\in\delta_e\mathbb{Z}^2$ for some $\delta_e>0$. Let $\mathcal{M}$ be the set of $\delta>0$ such that $w_e^1,w_e^2\in\delta\mathbb{Z}^2$. Then Theorem 4.2 still holds for q_δ and γ_0 defined here if we replace "$\delta<\delta_0$" by "$\delta\in\mathcal{M}$ and $\delta<\delta_0$."

5. Existence and uniqueness. In this section we will prove Theorem 3.1. The proof is somehow similar to that of the existence and uniqueness of the solution of an ordinary differential equation.

5.1. *Convergence of domains.*

DEFINITION 5.1. Suppose D_n is a sequence of domains and D is a domain. We say that (D_n) converges to D, denoted by $D_n\xrightarrow{\text{Cara}}D$, if for every $z\in D$, $\operatorname{dist}^\#(z,\partial^\# D_n)\to\operatorname{dist}^\#(z,\partial^\# D)$. This is equivalent to the followings:

(i) every compact subset of D is contained in all but finitely many D_n's; and

(ii) for every point $z_0\in\partial^\# D$, $\operatorname{dist}^\#(z_0,\partial^\# D_n)\to 0$ as $n\to\infty$.

A sequence of domains may converge to two different domains. For example, let $D_n=\mathbb{C}\setminus((-\infty,n])$. Then $D_n\xrightarrow{\text{Cara}}\mathbb{H}$, and $D_n\xrightarrow{\text{Cara}}-\mathbb{H}$ as well. But two different limit domains of the same domain sequence must be disjoint from each other, because if they have nonempty intersection, then one contains some boundary point of the other, which implies a contradiction.

If only condition (i) in the definition is satisfied, then for any $z\in D$, $\operatorname{dist}^\#(z,\partial^\# D)\le\liminf\operatorname{dist}^\#(z,\partial^\# D_n)$. Thus $D_n\cap D\xrightarrow{\text{Cara}}D$. If $D_n\xrightarrow{\text{Cara}}D$, $E_n\xrightarrow{\text{Cara}}E$, and $z_0\in D\cap E$. Let F_n (resp. F) be the connected component of $D_n\cap E_n$ (resp. $D\cap E$) that contains z_0. Then for any $z\in F$, $\operatorname{dist}^\#(z,\partial^\# F_n)=\operatorname{dist}^\#(z,\partial^\# D_n)\wedge\operatorname{dist}^\#(z,\partial^\# E_n)$ for each n, and $\operatorname{dist}^\#(z,\partial^\# F)=\operatorname{dist}^\#(z,\partial^\# D)\wedge\operatorname{dist}^\#(z,\partial^\# E)$, which implies $F_n\xrightarrow{\text{Cara}}F$. Thus if $D_n\xrightarrow{\text{Cara}}D$, $E_n\xrightarrow{\text{Cara}}E$, $D_n\subset E_n$ for each n, and $D\cap E\ne\varnothing$, then we have $D\subset E$.

Suppose $D_n\xrightarrow{\text{Cara}}D$, and for each n, f_n is a $\widehat{\mathbb{C}}$-valued function on D_n, and f is a $\widehat{\mathbb{C}}$-valued function on D. We say that f_n converges to f locally uniformly in D, or $f_n\xrightarrow{\text{l.u.}}f$ in D, if for each compact subset F of D, f_n converges to f in the spherical metric uniformly on F. If every f_n is analytic (resp. harmonic), then f is also analytic (resp. harmonic).

LEMMA 5.1. *Suppose* $D_n \stackrel{\text{Cara}}{\longrightarrow} D$, f_n *maps* D_n *conformally onto some domain* E_n *for each* n, *and* $f_n \stackrel{\text{l.u.}}{\longrightarrow} f$ *in* D. *Then either* f *is constant on* D, *or* f *maps* D *conformally onto some domain* E. *And in the latter case,* $E_n \stackrel{\text{Cara}}{\longrightarrow} E$ *and* $f_n^{-1} \stackrel{\text{l.u.}}{\longrightarrow} f^{-1}$ *in* E.

This lemma is similar to Theorem 1.8, the Carathéodory kernel theorem, in [13], and the proof is also similar. When applying this lemma, we will usually first exclude the possibility that f is constant, and then obtain the convergence of the image domains and the inverse functions.

5.2. *Topology on the space of hulls.* If H is a nonempty hull in $\mathbb{H}$ w.r.t. ∞, then $\overline{H}\cap\mathbb{R}$ is nonempty. Let $a_H=\inf(\overline{H}\cap\mathbb{R})$ and $b_H=\sup(\overline{H}\cap\mathbb{R})$. Let

$$\Sigma_H=\mathbb{C}\setminus(H\cup\{\overline{z}:z\in H\}\cup[a_H,b_H]).$$

By the reflection principle, φ_H extends to Σ_H, and maps Σ_H conformally onto $\mathbb{C}\setminus[c_H,d_H]$ for some $c_H<d_H\in\mathbb{R}$. Moreover, φ_H is increasing on $(-\infty,a_H)$ and $(b_H,+\infty)$, and maps them onto $(-\infty,c_H)$ and $(d_H,+\infty)$, respectively. So φ_H^{-1} extends conformally to $\mathbb{C}\setminus[c_H,d_H]$. And $[c_H,d_H]$ is the smallest in the sense that if φ_H^{-1} extends conformally to $\mathbb{C}\setminus I$ for some closed interval I, then $[c_H,d_H]\subset I$. If $H=\varnothing$, we do not define a_H,b_H,c_H,d_H, but still use the notation $[a_\varnothing,b_\varnothing]$ and $[c_\varnothing,d_\varnothing]$ to denote empty sets. Then $\Sigma_\varnothing=\mathbb{C}$, so it is true that $\varphi_\varnothing$ maps $\Sigma_\varnothing$ conformally onto $\mathbb{C}\setminus[c_\varnothing,d_\varnothing]$.

If γ is a crosscut in $\mathbb{H}$, we define $H(\gamma)$ to be γ unions the bounded component of $\mathbb{H}\setminus\gamma$. Then $H(\gamma)$ is a hull in $\mathbb{H}$ w.r.t. ∞. We call it the hull bounded by γ. If $A\subset\overline{H(\gamma)}$, then we say γ encloses A. If $A\subset\overline{H(\gamma)}$ and $\overline{A}\cap\overline{\gamma}=\varnothing$, then we say γ strictly encloses A. For simplicity, we write x_γ instead of $x_{H(\gamma)}$ when x is one of the following symbols: a,b,c,d,Σ,φ.

Since $\widehat{\partial}(\mathbb{H}\setminus H(\gamma))=(\widehat{\mathbb{R}}\setminus(a_\gamma,b_\gamma))\cup\gamma$ is a simple curve, so φ_γ extends to a homeomorphism of $\overline{\mathbb{H}\setminus H(\gamma)}$, and maps γ onto $[c_\gamma,d_\gamma]$. So $\varphi_{H(\gamma)}^{-1}$ has a continuous extension to $\mathbb{H}\cup\mathbb{R}$, and maps (c_γ,d_γ) onto γ. From the results about Poisson kernel, we have

$$\varphi_\gamma^{-1}(z)-z=\int_{c_\gamma}^{d_\gamma}\frac{-1}{z-x}\frac{\operatorname{Im}\varphi_\gamma^{-1}(x)}{\pi}\,dx,$$

for any $z\in\Sigma_\gamma$. From the behavior of φ_γ near ∞, we have $\int_{c_\gamma}^{d_\gamma}\operatorname{Im}\varphi_\gamma^{-1}(x)/\pi\,dx=\operatorname{hcap}(H(\gamma))$. If H is a general nonempty hull in $\mathbb{H}$ w.r.t. ∞, then φ_H^{-1} may not have continuous extension to $[c_H,d_H]$. We may use a sequence of hulls bounded by crosscuts to approximate H. Then we conclude that there is a positive measure μ_H supported by $[c_H,d_H]$ with total mass $|\mu_H|=\operatorname{hcap}(H)$ such that for any $z\in\Sigma_H$,

$$\varphi_H^{-1}(z)-z=\int_{c_H}^{d_H}\frac{-1}{z-x}\,d\mu_H(x). \tag{5.1}$$

EXAMPLE. Suppose $x_0 \in \mathbb{R}$ and $r_0 > 0$. Let $\alpha = \{z \in \mathbb{H} : |z - x_0| = r_0\}$. Then α is a crosscut in $\mathbb{H}$, $H(\alpha) = \{z \in \mathbb{H} : |z - x_0| \le r_0\}$ and $[a_\alpha, b_\alpha] = [x_0 - r_0, x_0 + r_0]$. It is clear that $\varphi_\alpha(z) = z + \frac{r_0^2}{z - x_0}$. Thus $\mathrm{hcap}(H(\alpha)) = r_0^2$ and $[c_\alpha, d_\alpha] = [x_0 - 2r_0, x_0 + 2r_0]$.

LEMMA 5.2. *If H is a nonempty hull in $\mathbb{H}$ w.r.t. ∞, then $\varphi_H^{-1}(x) > x$ for any $x \in (-\infty, c_H)$; $\varphi_H^{-1}(x) < x$ for any $x \in (d_H, +\infty)$; $\varphi_H(x) < x$ for any $x \in (-\infty, a_H)$; $\varphi_H(x) > x$ for any $x \in (b_H, +\infty)$. So if H is any hull in $\mathbb{H}$ w.r.t. ∞, then $[a_H, b_H] \subset [c_H, d_H]$.*

PROOF. This follows from (5.1) and that φ_H maps $(-\infty, a_H)$ and $(b_H, +\infty)$ onto $(-\infty, c_H)$ and $(d_H, +\infty)$, respectively. $\square$

If $H_1 \subset H_2$ are two hulls in $\mathbb{H}$ w.r.t. ∞, we call H_1 a sub-hull of H_2. Then $H_2/H_1 := \varphi_{H_1}(H_2 \setminus H_1)$ is also a hull in $\mathbb{H}$ w.r.t. ∞. We call H_2/H_1 a quotient-hull of H_2. It is clear that $\varphi_{H_2} = \varphi_{H_2/H_1} \circ \varphi_{H_1}$. Thus $\mathrm{hcap}(H_2) = \mathrm{hcap}(H_2/H_1) + \mathrm{hcap}(H_1)$, and so $\mathrm{hcap}(H_1), \mathrm{hcap}(H_2/H_1) \le \mathrm{hcap}(H_2)$.

LEMMA 5.3. *If $H_1 \subset H_2$ are two hulls in $\mathbb{H}$ w.r.t. ∞, then $[c_{H_1}, d_{H_1}] \subset [c_{H_2}, d_{H_2}]$ and $[c_{H_2/H_1}, d_{H_2/H_1}] \subset [c_{H_2}, d_{H_2}]$.*

PROOF. If $H_1 = \varnothing$ or $H_1 = H_2$, then $H_2/H_1 = H_2$ or $H_2/H_1 = \varnothing$, so it is trivial. Now suppose $\varnothing \subsetneqq H_1 \subsetneqq H_2$. Then $H_2/H_1 \ne \varnothing$. Since $\varphi_{H_2/H_1}^{-1}(z) = \varphi_{H_1} \circ \varphi_{H_2}^{-1}(z)$ for $z \in \mathbb{H}$, $\varphi_{H_2}^{-1}$ maps $\mathbb{C} \setminus [c_{H_2}, d_{H_2}]$ onto Σ_{H_2}, and φ_{H_1} extends conformally to $\Sigma_{H_1} \supset \Sigma_{H_2}$, so φ_{H_2/H_1}^{-1} extends conformally to $\mathbb{C} \setminus [c_{H_2}, d_{H_2}]$. From the minimum property of $[c_{H_2/H_1}, d_{H_2/H_1}]$, we have $[c_{H_2/H_1}, d_{H_2/H_1}] \subset [c_{H_2}, d_{H_2}]$.

If $x \in (-\infty, a_{H_2})$, then $\varphi_{H_2}(x) \in (-\infty, c_{H_2}) \subset (-\infty, c_{H_2/H_1})$. Since $\varphi_{H_2/H_1}^{-1}(x) > x$ on $(-\infty, c_{H_2/H_1})$, so $\varphi_{H_1}(x) = \varphi_{H_2/H_1}^{-1} \circ \varphi_{H_2}(x) > \varphi_{H_2}(x)$ on $(-\infty, a_{H_2})$. Thus

$$\begin{aligned} c_{H_1} &= \sup \varphi_{H_1}((-\infty, a_{H_1})) \ge \sup \varphi_{H_1}((-\infty, a_{H_2})) \\ &\ge \sup \varphi_{H_2}((-\infty, a_{H_2})) = c_{H_2}. \end{aligned}$$

Similarly, we have $d_{H_1} \le d_{H_2}$. Thus $[c_{H_1}, d_{H_1}] \subset [c_{H_2}, d_{H_2}]$. $\square$

COROLLARY 5.1. *If $H_1 \subset H_2 \subset H_3$ are hulls in $\mathbb{H}$ w.r.t. ∞, then $\mathrm{hcap}(H_2/H_1) \le \mathrm{hcap}(H_3)$ and $[c_{H_2/H_1}, d_{H_2/H_1}] \subset [c_{H_3}, d_{H_3}]$. We call H_2/H_1 a sub-quotient-hull of H_3.*

Let H be a nonempty hull in $\mathbb{H}$ w.r.t. ∞. Let $\mathcal{H}(H)$ denote the set of all sub-hulls of H. Let $\mathcal{H}_{\mathrm{sq}}(H)$ denote the set of all sub-quotient-hulls of H. If α

is a crosscut in $\mathbb{H}$, we write $\mathcal{H}(\alpha)$ for $\mathcal{H}(H(\alpha))$, and $\mathcal{H}_{\rm sq}(\alpha)$ for $\mathcal{H}_{\rm sq}(H(\alpha))$. Choose $d>0$. Let $\alpha=\{z\in\mathbb{C}:|z-(c_H+d_H)/2|=|d_H-c_H|/2+d\}$. Then α is a Jordan curve that encloses $[c_H,d_H]$, and d is the distance between α and $[c_H,d_H]$. Suppose $K\in\mathcal{H}_{\rm sq}(H)$. Then $[c_K,d_K]\subset[c_H,d_H]$. If z lies on or outside α, from (5.1),

$$|\varphi_K^{-1}(z)-z|\le|\mu_K|/d=\operatorname{hcap}(K)/d\le\operatorname{hcap}(H)/d.$$

If $z\in\mathbb{C}\setminus[c_K,d_K]$ lies inside α, then $\varphi_K^{-1}(z)$ lies inside $\varphi_K^{-1}(\alpha)$. Choose $w\in\alpha$; then

$$\begin{aligned}|\varphi_K^{-1}(z)-z|&\le|z-w|+|w-\varphi_K^{-1}(w)|+|\varphi_K^{-1}(w)-\varphi_K^{-1}(z)|\\&\le\operatorname{diam}(\alpha)+\operatorname{hcap}(H)/d+\operatorname{diam}(\varphi_K^{-1}(\alpha))\\&\le 2|d_H-c_H|+4d+3\operatorname{hcap}(H)/d.\end{aligned}$$

Let $d=\sqrt{\operatorname{hcap}(H)}$ and $M_H=2|d_H-c_H|+7\sqrt{\operatorname{hcap}(H)}$. Then for any $z\in\mathbb{C}\setminus[c_K,d_K]$, $|\varphi_K^{-1}(z)-z|\le M_H$. Since φ_K^{-1} maps $\mathbb{C}\setminus[c_K,d_K]$ onto Σ_K, so for any $z\in\Sigma_K$, $|\varphi_K(z)-z|\le M_H$. Since $\mathbb{C}\setminus[c_K,d_K]\supset\mathbb{C}\setminus[c_H,d_H]$, so $\{\varphi_K^{-1}(z)-z:K\in\mathcal{H}_{\rm sq}(H)\}$ is uniformly bounded in $\mathbb{C}\setminus[c_H,d_H]$ by M_H, and so is a normal family.

Let $\mathcal{H}$ denote the set of all hulls in $\mathbb{H}$ w.r.t. ∞. Choose a sequence of compact subsets (F_n) of $\mathbb{H}$ such that $F_n\subset\operatorname{int}F_{n+1}$ for each $n\in\mathbb{N}$, and $\bigcup_n F_n=\mathbb{H}$. We may define a distant function $d_{\mathcal{H}}$ on $\mathcal{H}$ such that

$$d_{\mathcal{H}}(H_1,H_2)=\sum_{n=1}^{\infty}\frac{1}{2^n}\Big(1\wedge\sup_{z\in F_n}\{|\varphi_{H_1}^{-1}(z)-\varphi_{H_2}^{-1}(z)|\}\Big).$$

We use $\xrightarrow{\mathcal{H}}$ to denote the convergence w.r.t. $d_{\mathcal{H}}$. It is clear that $H_n\xrightarrow{\mathcal{H}}H$ iff $\varphi_{H_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_H^{-1}$ in $\mathbb{H}$. So the topology does not depend on the choice of (F_n).

From Lemma 5.1, if $H_n\xrightarrow{\mathcal{H}}H$, then $\mathbb{H}\setminus H_n\xrightarrow{\text{Cara}}\mathbb{H}\setminus H$ and $\varphi_{H_n}\xrightarrow{\text{l.u.}}\varphi_H$ in $\mathbb{H}\setminus H$. However, $\mathbb{H}\setminus H_n\xrightarrow{\text{Cara}}\mathbb{H}\setminus H$ does not imply $H_n\xrightarrow{\mathcal{H}}H$. For example, let $H_n=\{z\in\mathbb{H}:|z-2n|\le n\}$ for $n\in\mathbb{N}$. Then $\mathbb{H}\setminus H_n\xrightarrow{\text{Cara}}\mathbb{H}=\mathbb{H}\setminus\varnothing$, but $\varphi_{H_n}(z)=z+n^2/(z-2n)\not\to z=\varphi_\varnothing(z)$. And $H_n\xrightarrow{\mathcal{H}}H$ does not imply $\Sigma_{H_n}\xrightarrow{\text{Cara}}\Sigma_H$. For example, let $H_n=\{z\in\mathbb{H}:|\operatorname{Re}z|\le1,\operatorname{Im}z\le1/n\}$ for $n\in\mathbb{N}$. Then $H_n\xrightarrow{\mathcal{H}}\varnothing$, but $\Sigma_{H_n}\xrightarrow{\text{Cara}}\mathbb{C}\setminus[-1,1]\ne\mathbb{C}=\Sigma_\varnothing$.

Suppose $H_n\xrightarrow{\mathcal{H}}H$, $K_n\xrightarrow{\mathcal{H}}K$ and $K_n\subset H_n$ for each n. Then $\mathbb{H}\setminus H_n\xrightarrow{\text{Cara}}\mathbb{H}\setminus H$, $\mathbb{H}\setminus K_n\xrightarrow{\text{Cara}}\mathbb{H}\setminus K$ and $\mathbb{H}\setminus H_n\subset\mathbb{H}\setminus K_n$ for each n. Since $(\mathbb{H}\setminus H)\cap(\mathbb{H}\setminus K)=\mathbb{H}\setminus(H\cup K)\ne\varnothing$, so $\mathbb{H}\setminus H\subset\mathbb{H}\setminus K$. Thus $K\subset H$. Let $L_n=H_n/K_n$ for each n and $L=K/H$. Then $\varphi_{L_n}^{-1}=\varphi_{K_n}\circ\varphi_{H_n}^{-1}$ and $\varphi_L^{-1}=\varphi_K\circ\varphi_H^{-1}$. Since $\varphi_{H_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_H^{-1}$ in $\mathbb{H}$, and $\varphi_{K_n}\xrightarrow{\text{l.u.}}\varphi_K$ in $\mathbb{H}\setminus K\supset\mathbb{H}\setminus H=\varphi_H(\mathbb{H})$, so $\varphi_{L_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_L^{-1}$ in $\mathbb{H}$. Thus $L_n\xrightarrow{\mathcal{H}}L$, that is, $H_n/K_n\xrightarrow{\mathcal{H}}H/K$.

LEMMA 5.4 (Compactness). *$\mathcal{H}(H)$ and $\mathcal{H}_{\rm sq}(H)$ are compact. Moreover, we have:*

(i) *Suppose (K_n) is a sequence in $\mathcal{H}(H)$, then it has a subsequence (L_n) that converges to some $K\in\mathcal{H}(H)$ w.r.t. $d_{\mathcal{H}}$, and $\varphi_{L_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_K^{-1}$ in $\mathbb{C}\setminus[c_H,d_H]$, $\Sigma_{L_n}\setminus[a_H,b_H]\xrightarrow{\text{Cara}}\Sigma_K\setminus[a_H,b_H]$, and $\varphi_{L_n}\xrightarrow{\text{l.u.}}\varphi_K$ in $\Sigma_K\setminus[a_H,b_H]$.*

(ii) *Suppose (K_n) is a sequence in $\mathcal{H}_{sq}(H)$; then it has a subsequence (L_n) that converges to some $K\in\mathcal{H}_{sq}(H)$ w.r.t. $d_{\mathcal{H}}$, and $\varphi_{L_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_K^{-1}$ in $\mathbb{C}\setminus[c_H,d_H]$, $\Sigma_{L_n}\setminus[c_H,d_H]\xrightarrow{\text{Cara}}\Sigma_K\setminus[c_H,d_H]$, and $\varphi_{L_n}\xrightarrow{\text{l.u.}}\varphi_K$ in $\Sigma_K\setminus[c_H,d_H]$.*

PROOF. (i) Since $\{\varphi_{K_n}^{-1}(z)-z:n\in\mathbb{N}\}$ is uniformly bounded in $\mathbb{C}\setminus[c_H,d_H]$, so (K_n) has a subsequence (L_n) such that $\varphi_{L_n}^{-1}(z)-z$ converges to some function f locally uniformly in $\mathbb{C}\setminus[c_H,d_H]$. Then $|f(z)|\le M$ for any $z\in\mathbb{C}\setminus[c_H,d_H]$. Let $g(z)=f(z)+z$ for $z\in\mathbb{C}\setminus[c_H,d_H]$. Then $\varphi_{L_n}^{-1}\xrightarrow{\text{l.u.}}g$ in $\mathbb{C}\setminus[c_H,d_H]$. There are $z_1,z_2\in\mathbb{C}\setminus[c_H,d_H]$ with $|z_1-z_2|>2M$. Then $|g(z_1)-g(z_2)|\ge|z_1-z_2|-|g(z_1)-z_1|-|g(z_2)-z_2|>2M-M-M=0$. So g is not constant. From Lemma 5.1, g is a conformal map. Since for each n, $\mathbb{H}\supset\varphi_{L_n}^{-1}(\mathbb{H})=\mathbb{H}\setminus L_n\supset\mathbb{H}\setminus H$, so $\mathbb{H}\supset g(\mathbb{H})\supset\mathbb{H}\setminus H$. Let $K=\mathbb{H}\setminus g(\mathbb{H})$. Then $K\in\mathcal{H}(H)$, and g maps $\mathbb{H}$ conformally onto $\mathbb{H}\setminus K$. Since $\varphi_{L_n}^{-1}(z)-z=O(1/z)$ as $z\to\infty$, so $g(z)-z=O(1/z)$ as $z\to\infty$. Thus $g(z)=\varphi_K^{-1}(z)$ for $z\in\mathbb{C}\setminus[c_H,d_H]$. So $\varphi_{L_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_K^{-1}$ in $\mathbb{C}\setminus[c_H,d_H]$. Especially, $\varphi_{L_n}^{-1}\xrightarrow{\text{l.u.}}\varphi_K^{-1}$ in $\mathbb{H}$. So K is a subsequential limit of (K_n). Thus $\mathcal{H}(H)$ is compact.

For $L\in\mathcal{H}(H)$, let $\Sigma_L^1:=\Sigma_L\setminus[a_H,b_H]$, $\Sigma_L^2:=\Sigma_L\setminus[c_H,d_H]$. Then $\Sigma_L^2\subset\Sigma_L^1$, and

$$\Sigma_L^1=(\mathbb{H}\setminus L)\cup\{z\in\mathbb{C}:\overline{z}\in\mathbb{H}\setminus L\}\cup(-\infty,a_H)\cup(b_H,+\infty),\tag{5.2}$$

$$\Sigma_L^2=(\mathbb{H}\setminus L)\cup\{z\in\mathbb{C}:\overline{z}\in\mathbb{H}\setminus L\}\cup(-\infty,c_H)\cup(d_H,+\infty),\tag{5.3}$$

because $(\mathbb{C}\setminus\Sigma_L)\cap\mathbb{R}\subset[a_L,b_L]\subset[a_H,b_H]\subset[c_H,d_H]$. So from $\mathbb{H}\setminus L_n\xrightarrow{\text{Cara}}\mathbb{H}\setminus K$, we have $\Sigma_{L_n}^j\xrightarrow{\text{Cara}}\Sigma_K^j$ for $j=1,2$. From Lemma 5.1, $\varphi_{L_n}^{-1}(\mathbb{C}\setminus[c_H,d_H])\xrightarrow{\text{Cara}}\varphi_K^{-1}(\mathbb{C}\setminus[c_H,d_H])$ and $\varphi_{L_n}\xrightarrow{\text{l.u.}}\varphi_K$ in $\varphi_K^{-1}(\mathbb{C}\setminus[c_H,d_H])$. Note that $\varphi_K^{-1}(\mathbb{C}\setminus[c_H,d_H])\supset\Sigma_K^2$, where the inclusion follows from Lemma 5.2. Thus $\varphi_{L_n}\xrightarrow{\text{l.u.}}\varphi_K$ in Σ_K^2.

Since $|\varphi_{L_n}(z)-z|\le M$ for all $n\in\mathbb{N}$ and $z\in\Sigma_{L_n}$, and $\Sigma_{L_n}^1\subset\Sigma_{L_n}$, so every subsequence of (φ_{L_n}) has a subsequence that converges to some analytic function h locally uniformly in Σ_K^1. Since $\varphi_{L_n}\xrightarrow{\text{l.u.}}\varphi_K$ in $\Sigma_K^2\subset\Sigma_K^1$, so h agrees with φ_K on Σ_K^2. Since they are both analytic, so h agrees with φ_K on Σ_K^1. Since all subsequential limits of φ_{L_n} in Σ_K^1 are the same function φ_K, so $\varphi_{L_n}\xrightarrow{\text{l.u.}}\varphi_K$ in $\Sigma_K^1=\Sigma_K\setminus[a_H,b_H]$.

(ii) Suppose $K_n = K_n^2/K_n^1$ with $K_n^1 \subset K_n^2 \subset H$. From (i), (K_n) has a subsequence $(L_n = L_n^2/L_n^1)$ such that $L_n^j \xrightarrow{\mathcal{H}} K^j$ for some $K^j \in \mathcal{H}(H)$, $j = 1, 2$. Since $L_n^1 \subset L_n^2$ for each n, so $K^1 \subset K^2$. Let $K = K^2/K^1$. Then $K \in \mathcal{H}_{sq}(H)$, and $L_n = L_n^2/L_n^1 \xrightarrow{\mathcal{H}} K^2/K^1 = K$. So K is a subsequential limit of (K_n). Thus $\mathcal{H}_{sq}(H)$ is compact.

Since $\{\varphi_{L_n}^{-1}(z) - z : n \in \mathbb{N}\}$ is uniformly bounded in $\mathbb{C} \setminus [c_H, d_H]$, so every subsequence of $(\varphi_{L_n}^{-1})$ has a subsequence which converges to some h locally uniformly in $\mathbb{C} \setminus [c_H, d_H]$. Then h agrees with φ_K^{-1} on $\mathbb{H}$. Since they are both analytic in $\mathbb{C} \setminus [c_H, d_H]$, so h agrees with φ_K^{-1} on $\mathbb{C} \setminus [c_H, d_H]$. Thus $(\varphi_{L_n}^{-1}) \xrightarrow{\text{l.u.}} \varphi_K^{-1}$ in $\mathbb{C} \setminus [c_H, d_H]$.

For $L \in \mathcal{H}_{\text{sq}}(H)$, we define Σ_L^j, $j = 1, 2$, as in (i). Then (5.3) still holds because $[a_L, b_L] \subset [c_L, d_L] \subset [c_H, d_H]$, but (5.2) does not because $[a_L, b_L] \subset [a_H, b_H]$ may not be true. A similar argument gives that $\varphi_{L_n} \xrightarrow{\text{l.u.}} \varphi_K$ in $\Sigma_K^2 = \Sigma_K \setminus [c_H, d_H]$. $\square$

5.3. *Lipschitz conditions.* Suppose $\xi \in C([0, a])$ for some $a > 0$, and $K_a^\xi \in \mathcal{H}(\alpha)$. Then for each $t \in [0, a]$, $\varphi_t^\xi = \varphi_{K_t^\xi}$, and $K_t^\xi \in \mathcal{H}(\alpha)$. For $0 \le t_1 < t_2 \le a$, let $K_{t_1,t_2}^\xi = K_{t_2}^\xi / K_{t_1}^\xi$. Then $K_{t_1,t_2}^\xi \in \mathcal{H}_{\text{sq}}(\alpha)$ and $\varphi_{K_{t_1,t_2}^\xi} = \varphi_{t_2}^\xi \circ (\varphi_{t_1}^\xi)^{-1}$, $\varphi_{K_{t_1,t_2}^\xi}^{-1} = \varphi_{t_1}^\xi \circ (\varphi_{t_2}^\xi)^{-1}$. Since $\xi(t_1) \in \overline{K_{t_1,t_2}^\xi}$, so

$$\xi(t_1) \in [a_{K_{t_1,t_2}^\xi}, b_{K_{t_1,t_2}^\xi}] \subset [c_{K_{t_1,t_2}^\xi}, d_{K_{t_1,t_2}^\xi}] \subset [c_\alpha, d_\alpha].$$

This holds for any $t_1 \in [0, a)$. Since ξ is continuous, so we also have $\xi(a) \in [c_\alpha, d_\alpha]$.

Lemma 5.5. *Suppose α_0 and α_1 are crosscuts in $\mathbb{H}$, and α_0 is strictly enclosed by α_1. Then there are $\delta, C > 0$ such that if $\zeta, \eta \in C([0, a])$, $\|\zeta - \eta\|_a < \delta$, and $K_a^\zeta \subset H(\alpha_0)$, then $K_a^\eta \subset H(\alpha_1)$, and for any $z \in \overline{\mathbb{H} \setminus H(\alpha_1)}$,*

$$|\varphi_a^\zeta(z) - \varphi_a^\eta(z)| \le Ca\|\zeta - \eta\|_a.$$

Proof. Suppose $\zeta, \eta \in C([0, a])$ and $K_a^\zeta \subset H(\alpha_0)$. Choose a crosscut $\alpha_{0.5}$ in $\mathbb{H}$ that strictly encloses α_0, and is strictly enclosed by α_1. Then $\overline{\alpha_{0.5}}$ and $\overline{\alpha_1}$ are disjoint compact subsets of Σ_{α_0}, which contains $\Sigma_K \setminus [a_{\alpha_0}, b_{\alpha_0}]$ for any $K \in \mathcal{H}(\alpha_0)$. From the compactness of $\mathcal{H}(\alpha_0)$, there is $d > 0$, such that the distance between $\varphi_K(\alpha_{0.5})$ and $\varphi_K(\alpha_1)$ is at least d for any $K \in \mathcal{H}(\alpha_0)$. For $t \in [0, a]$, since $K_t^\zeta \in \mathcal{H}(\alpha_0)$, so the distance between $\varphi_t^\zeta(\alpha_{0.5})$ and $\varphi_t^\zeta(\alpha_1)$ is at least d. Since K_a^ζ is enclosed by $\alpha_{0.5}$, so $K_{t,a}^\zeta = \varphi_t^\zeta(K_a^\zeta \setminus K_t^\zeta)$ is enclosed by $\varphi_t^\zeta(\alpha_{0.5})$, which implies that $\zeta(t) \in \overline{K_{t,a}^\zeta}$ is enclosed by $\varphi_t^\zeta(\alpha_{0.5})$. Thus the distance between $\zeta(t)$ and $\varphi_t^\zeta(z)$ is at least d for any $z\in\overline{\mathbb{H}\setminus H(\alpha_1)}$ and $t\in[0,a]$. Fix $z\in\overline{\mathbb{H}\setminus H(\alpha_1)}$ and $\delta\in(0,d/3]$. Then $|\varphi_t^\zeta(z)-\zeta(t)|\ge d$ for any $t\in[0,a]$. Suppose $\|\zeta-\eta\|_a<\delta$. Note that $\varphi_0^\zeta(z)=z=\varphi_0^\eta(z)$. Let $[0,b)$ be the maximal subinterval of $[0,a)$ on which $\varphi_t^\eta(z)$ is defined and $|\varphi_t^\zeta(z)-\varphi_t^\eta(z)|\le d/3$. Then for any $t\in[0,b)$,

$$|\varphi_t^\eta(z)-\eta(t)|\ge|\varphi_t^\zeta(z)-\zeta(t)|-|\varphi_t^\zeta(z)-\varphi_t^\eta(z)|-|\zeta(t)-\eta(t)|\ge d/3.$$

Thus $\varphi_b^\eta(z)$ is also defined. From the chordal Loewner equation, for $t\in[0,b]$,

$$\begin{aligned}
|\varphi_t^\zeta(z)-\varphi_t^\eta(z)| &\le \int_0^t \left|\frac{2}{\varphi_s^\zeta(z)-\zeta(s)}-\frac{2}{\varphi_s^\eta(z)-\eta(s)}\right| ds\\
&\le \int_0^t \left|\frac{2(\zeta(s)-\eta(s))}{(\varphi_s^\zeta(z)-\zeta(s))(\varphi_s^\eta(z)-\eta(s))}\right| ds\\
&\quad+\int_0^t \left|\frac{2(\varphi_s^\eta(z)-\varphi_s^\zeta(z))}{(\varphi_s^\zeta(z)-\zeta(s))(\varphi_s^\eta(z)-\eta(s))}\right| ds\\
&\le \frac{6t}{d^2}\|\zeta-\eta\|_a+\frac{6}{d^2}\int_0^t|\varphi_s^\zeta(z)-\varphi_s^\eta(z)|\,ds \qquad (5.4)\\
&\le \frac{6\delta t}{d^2}+\frac{6}{d^2}\int_0^t|\varphi_s^\zeta(z)-\varphi_s^\eta(z)|\,ds. \qquad (5.5)
\end{aligned}$$

Solving inequality (5.5), we get

$$|\varphi_b^\zeta(z)-\varphi_b^\eta(z)|\le\delta(e^{3b/d^2}-1)\le\delta(e^{3a/d^2}-1).$$

Let $h=\mathrm{hcap}(H(\alpha_0))$. Then $a=\mathrm{hcap}(K_a^\zeta)/2\le h/2$. Choose $\delta=\min\{d/3,\frac{d/6}{e^{3h/d^2}-1}\}$. Then $|\varphi_b^\zeta(z)-\varphi_b^\eta(z)|\le d/6$. So we have $b=a$, which implies that $\varphi_t^\eta(z)$ is defined on $[0,a]$, that is, $z\notin K_a^\eta$. Since this is true for any $z\in\mathbb{H}\setminus H(\alpha_1)$, so $K_a^\eta\subset H(\alpha_1)$. Finally, let $C=(\exp(\frac{3h}{d^2})-1)/(h/2)$. Solving inequality (5.4) for $t\in[0,a]$, we get

$$|\varphi_a^\zeta(z)-\varphi_a^\eta(z)|\le(e^{6a/d^2}-1)\|\zeta-\eta\|_a\le Ca\|\zeta-\eta\|_a$$

for any $z\in\overline{\mathbb{H}\setminus H(\alpha_1)}$, where the second "$\le$" holds because $a\le h/2$. □

LEMMA 5.6. *Suppose α and ρ are crosscuts in $\mathbb{H}$, and $[c_\alpha,d_\alpha]$ is strictly enclosed by ρ. Then there are $\delta,C>0$ such that if $\zeta,\eta\in C([0,a])$, $\|\zeta-\eta\|_a<\delta$, and $K_a^\zeta\subset H(\alpha)$, then K_a^η is enclosed by $(\varphi_a^\zeta)^{-1}(\rho)$, and for any $w\in\overline{\mathbb{H}\setminus H(\rho)}$,*

$$|w-\varphi_a^\eta\circ(\varphi_a^\zeta)^{-1}(w)|\le Ca\|\zeta-\eta\|_a.$$

Proof. Suppose $\zeta,\eta\in C([0,a])$ and $K_a^\zeta\subset H(\alpha)$. Choose ρ_0 that strictly encloses $[c_\alpha,d_\alpha]$, and is strictly enclosed by ρ. Then for any $t\in[0,a)$, $\zeta(t)\in \overline{K_{t,a}^\zeta}$ is enclosed by $\varphi_{K_{t,a}^\zeta}^{-1}(\rho_0)$. Note that $K_{t,a}^\zeta\in\mathcal{H}_{sq}(\alpha)$ and $\varphi_{K_{t,a}^\zeta}^{-1}=\varphi_t^\zeta\circ(\varphi_a^\zeta)^{-1}$. From the compactness of $\mathcal{H}_{\rm sq}(\alpha)$ and an argument that is similar to the first paragraph of the last proof, we see that there is $d>0$ depending only on α and ρ such that $|\varphi_t^\zeta\circ(\varphi_a^\zeta)^{-1}(w)-\zeta(t)|\ge d$ for any $t\in[0,a]$ and $w\in\overline{\mathbb{H}\setminus H(\rho)}$. Fix $w\in\overline{\mathbb{H}\setminus H(\rho)}$. Applying the argument of the proof of the last lemma to $z=(\varphi_a^\zeta)^{-1}(w)$, we have $\delta,C>0$ depending only on α and ρ such that if $\|\zeta-\eta\|_a<\delta$, then $\varphi_a^\eta(z)$ is well defined, and

$$|w-\varphi_a^\eta\circ(\varphi_a^\zeta)^{-1}(w)|=|\varphi_a^\zeta(z)-\varphi_a^\eta(z)|\le Ca\|\zeta-\eta\|_a.$$

That $\varphi_a^\eta(z)$ is well defined implies that $(\varphi_a^\zeta)^{-1}(w)=z\notin K_a^\eta$. Since this holds for any $w\in\mathbb{H}\setminus H(\rho)$, so K_a^η is enclosed by $(\varphi_a^\zeta)^{-1}(\rho)$. □

Now suppose Ω is an almost $\mathbb{H}$ domain, and $p\in\Omega$. Suppose α is a crosscut in $\mathbb{H}$ such that $H(\alpha)\subset\Omega\setminus\{p\}$. From the compactness of $\mathcal{H}(\alpha)$, there is $\mathbf{h}>0$ depending only on Ω,p,α, such that if $K\in\mathcal{H}(\alpha)$, then $\operatorname{dist}(\varphi_K(\{p\}\cup\partial\Omega\setminus\mathbb{R}),\mathbb{R})\ge\mathbf{h}$. Let ρ_1 and ρ_2 be crosscuts in $\mathbb{H}$ with height smaller than $\mathbf{h}/2$ such that ρ_1 strictly encloses ρ_2, and ρ_2 strictly encloses $[c_\alpha,d_\alpha]$. Then for any $K\in\mathcal{H}(\alpha)$, $H(\varphi_K^{-1}(\rho))\subset\Omega_K\setminus\{\varphi_K(p)\}$.

Lemma 5.7. *There are $\delta,C>0$ such that if $\zeta,\eta\in C([0,a])$, $\|\zeta-\eta\|_a<\delta$, and $K_a^\zeta\subset H(\alpha)$, then for any $z\in\rho_1$,*

$$|J_a^\zeta(z)-J_a^\eta(z)|\le Ca\|\zeta-\eta\|_a. \tag{5.6}$$

Proof. Choose a crosscut α_1 in $\mathbb{H}$ that strictly encloses α such that $H(\alpha_1)\subset\Omega\setminus\{p\}$. Suppose $\zeta,\eta\in C([0,a])$ and $K_a^\zeta\subset H(\alpha)$. From Lemma 5.5, there is $\delta_0>0$ depending only on α and α_1 such that if $\|\zeta-\eta\|_a<\delta_0$, then $K_a^\eta\subset H(\alpha_1)$.

From Lemma 5.6, there are $\delta_1,C_1>0$ depending only on α,ρ_1,ρ_2, such that if $\|\zeta-\eta\|_a<\delta_1$, then K_a^η is enclosed by $(\varphi_a^\zeta)^{-1}(\rho_2)$, and for any $z\in\rho_1\cup\rho_2$,

$$|z-\varphi_a^\eta\circ(\varphi_a^\zeta)^{-1}(z)|\le C_1a\|\zeta-\eta\|_a. \tag{5.7}$$

Let $F=\{z\in\overline{\mathbb{H}}:\operatorname{dist}(z,H(\rho_1))\le\mathbf{h}/4\}$. From the compactness of $\mathcal{H}(\alpha_1)$, there is $D>0$ depending only on Ω,p,α_1,F, such that if $K_t^\xi\in\mathcal{H}(\alpha_1)$, then for any $z\in F$,

$$|\nabla J_t^\xi(z)|\le D. \tag{5.8}$$

Let $h_0=\operatorname{hcap}(H(\alpha))$. Then $a=\operatorname{hcap}(K_a^\zeta)/2\le h_0/2$. Let $\delta=\min\{\delta_0,\delta_1,\mathbf{h}/(2C_1h_0)\}$. Suppose $\|\zeta-\eta\|_a<\delta$. Then for any $z\in\rho_1\cup\rho_2$,

$$|z-\varphi_a^\eta\circ(\varphi_a^\zeta)^{-1}(z)|\le C_1a\delta\le C_1h_0\delta/2\le\mathbf{h}/4,$$

which implies that $[z, \varphi_a^\eta \circ (\varphi_a^\zeta)^{-1}(z)] \subset F$.

Define $G_t^\xi = G(\Omega \setminus K_t^\xi, p; \cdot)$ if $K_t^\xi \subset \Omega \setminus \{p\}$. For $j = 1, 2$, let

$$N_j = \sup_{z\in\rho_j}\{|J_a^\zeta(z) - J_a^\eta(z)|\} = \sup_{z\in\rho_j}\{|G_a^\zeta \circ (\varphi_a^\zeta)^{-1}(z) - G_a^\eta \circ (\varphi_a^\eta)^{-1}(z)|\};$$

$$N_j' = \sup_{z\in(\varphi_a^\zeta)^{-1}(\rho_j)}\{|G_a^\zeta(z) - G_a^\eta(z)|\} = \sup_{z\in\rho_j}\{|G_a^\zeta \circ (\varphi_a^\zeta)^{-1}(z) - G_a^\eta \circ (\varphi_a^\zeta)^{-1}(z)|\}.$$

There is $q \in (0, 1)$ depending only on ρ_1 and ρ_2 such that for any $z \in \rho_2$, the probability that a plane Brownian motion started from z hits ρ_1 before $\mathbb{R}$ is less than q. Since both J_a^ζ and J_a^η are harmonic in $H(\rho_1)$, have continuations to $\overline{H(\rho_1)}$, and vanish on $\mathbb{R}$, so $J_a^\zeta - J_a^\eta$ also has these properties. Since $\rho_2 \subset H(\rho_1)$, so

$$N_2 \le qN_1. \tag{5.9}$$

Since K_a^ζ and K_a^η are enclosed by $(\varphi_a^\zeta)^{-1}(\rho_2)$, so G_a^ζ and G_a^η are harmonic in $\Omega\setminus\{p\}\setminus H((\varphi_a^\zeta)^{-1}(\rho_2))$. Since they both behave like $-\ln(z-p)/(2\pi)+O(1)$ near p, so $G_a^\zeta - G_a^\eta$ has a harmonic extension in $\Omega \setminus H((\varphi_a^\zeta)^{-1}(\rho_2))$. Since $G_a^\zeta - G_a^\eta$ vanishes at every boundary point of $\Omega \setminus H((\varphi_a^\zeta)^{-1}(\rho_2))$ including ∞, except on $(\varphi_a^\zeta)^{-1}(\rho_2)$, and $(\varphi_a^\zeta)^{-1}(\rho_1) \subset \Omega \setminus H((\varphi_a^\zeta)^{-1}(\rho_2))$, so from the maximum principle for harmonic functions,

$$N_1' \le N_2'. \tag{5.10}$$

Fix $j \in \{1, 2\}$. From $[z, \varphi_a^\eta \circ (\varphi_a^\zeta)^{-1}(z)] \subset F$ for $z \in \rho_j$, $K_a^\eta \in \mathcal{H}(\alpha_1)$, and (5.7) and (5.8), we have

$$\begin{aligned}|N_j - N_j'| &\le \sup_{z\in\rho_j}\{|G_a^\eta \circ (\varphi_a^\eta)^{-1}(z) - G_a^\eta \circ (\varphi_a^\zeta)^{-1}(z)|\}\\ &= \sup_{z\in\rho_j}\{|J_a^\eta(z) - J_a^\eta(\varphi_a^\eta \circ (\varphi_a^\zeta)^{-1}(z))|\}\\ &\le \sup_{w\in F}|\nabla J_a^\eta(w)| \sup_{z\in\rho_j}\{|z - \varphi_a^\eta \circ (\varphi_a^\zeta)^{-1}(z)|\}\\ &\le DC_1 a\|\zeta - \eta\|_a.\end{aligned}$$

From (5.9), (5.10) and the above inequality, we have

$$\begin{aligned}N_1 &\le N_1' + DC_1a\|\zeta-\eta\|_a \le N_2' + DC_1a\|\zeta-\eta\|_a\\ &\le N_2 + 2DC_1a\|\zeta-\eta\|_a \le qN_1 + 2DC_1a\|\zeta-\eta\|_a,\end{aligned}$$

which implies that $N_1 \le Ca\|\zeta - \eta\|_a$, where $C = 2DC_1/(1 - q)$. So we get (5.6). $\square$

LEMMA 5.8. *There are $\delta, C > 0$ such that if $\zeta, \eta \in C([0, a])$, $\|\zeta - \eta\|_a < \delta$, and $K_a^\zeta \subset H(\alpha)$, then $|X_a^\zeta - X_a^\eta| \le C\|\zeta - \eta\|_a$.*

Proof. Suppose $\zeta,\eta\in C([0,a])$ and $K_a^\zeta\subset H(\alpha)$. Choose a crosscut α_1 in $\mathbb{H}$ that strictly encloses α such that $H(\alpha_1)\subset\Omega\setminus\{p\}$. Let ρ be a crosscut in $\mathbb{H}$ with height smaller than $\mathbf{h}/2$ that strictly encloses $[c_\alpha,d_\alpha]$. From Lemmas 5.5 and 5.7, there are $\delta_0,C_0>0$ depending only on $\Omega,p,\alpha,\alpha_1,\rho$, such that if $\|\zeta-\eta\|_a<\delta_0$, then $K_a^\eta\subset H(\alpha_1)$ and for any $z\in\rho$, $|J_a^\zeta(z)-J_a^\eta(z)|\le C_0a\|\zeta-\eta\|_a$. Let $d_0=\operatorname{dist}([c_\alpha,d_\alpha],\rho)/2>0$, and $\delta=\delta_0\wedge d_0$.

Suppose $\|\zeta-\eta\|_a<\delta$. Then $K_a^\zeta,K_a^\eta\subset H(\alpha_1)$. From the compactness of $\mathcal{H}(\alpha_1)$, there are $m,M_1,M_2,M_3>0$ depending only on $\Omega,p,\alpha,\alpha_1,\rho$, such that for any $x\in[c_\alpha-d_0,d_\alpha+d_0]$,

$$m\le\partial_yJ_a^\zeta(x),\partial_yJ_a^\eta(x)\le M_1\quad\text{and}\quad|\partial_x^{j-1}\partial_yJ_a^\zeta(x)|,|\partial_x^{j-1}\partial_yJ_a^\eta(x)|\le M_j,$$

for $j=2,3$. Let $C_1=M_3/m+M_2^2/m^2$. So for any $x\in[c_\alpha-d_0,d_\alpha+d_0]$,

$$\begin{aligned}&|\partial_x(\partial_x\partial_y/\partial_y)J_a^\zeta(x)|\\ &\quad=|(\partial_x^2\partial_y/\partial_y-((\partial_x\partial_y\cdot\partial_x\partial_y)/(\partial_y\cdot\partial_y)))J_a^\zeta(x)|\le C_1.\end{aligned}\tag{5.11}$$

Since $\operatorname{dist}([c_\alpha-d_0,d_\alpha+d_0],\rho)\ge d_0$, so for any $x\in[c_\alpha-d_0,d_\alpha+d_0]$,

$$|\partial_x^{j-1}\partial_y(J_a^\zeta-J_a^\eta)(x)|\le\frac{2j\,!}{d_0^j}\sup_{z\in\rho}|J_a^\zeta(z)-J_a^\eta(z)|\le\frac{2j\,!}{d_0^j}C_0a\|\zeta-\eta\|_a,$$

for $j=1,2$, from which follows that

$$\begin{aligned}&|(\partial_x\partial_y/\partial_y)J_a^\zeta(x)-(\partial_x\partial_y/\partial_y)J_a^\eta(x)|\\ &\quad=|\partial_x\partial_yJ_a^\zeta(x)\,\partial_yJ_a^\eta(x)-\partial_x\partial_yJ_a^\eta(x)\,\partial_yJ_a^\zeta(x)|/|\partial_yJ_a^\zeta(x)\partial_yJ_a^\eta(x)|\\ &\quad\le|\partial_x\partial_yJ_a^\zeta(x)\,\partial_yJ_a^\eta(x)-\partial_x\partial_yJ_a^\zeta(x)\,\partial_yJ_a^\zeta(x)|/m^2\\ &\qquad+|\partial_x\partial_yJ_a^\zeta(x)\,\partial_yJ_a^\zeta(x)-\partial_x\partial_yJ_a^\eta(x)\,\partial_yJ_a^\zeta(x)|/m^2\\ &\quad\le M_2|\partial_y(J_a^\zeta-J_a^\eta)(x)|/m^2+M_1|\partial_x\partial_y(J_a^\zeta-J_a^\eta)(x)|/m^2\\ &\quad\le(2M_2/d_0+4M_1/d_0^2)C_0a\|\zeta-\eta\|_a/m^2\le C_2\|\zeta-\eta\|_a,\end{aligned}\tag{5.12}$$

if we let $C_2=(M_2/d_0+2M_1/d_0^2)C_0\operatorname{hcap}(H(\alpha))/m^2$.

Since $K_a^\zeta\in\mathcal{H}(\alpha)$, so $\zeta(a)\in[c_\alpha,d_\alpha]$. From $|\eta(a)-\zeta(a)|\le\delta\le d_0$, we have $\eta(a)\in[c_\alpha-d_0,d_\alpha+d_0]$. Let $C=C_1+C_2$. From (5.11) and (5.12), we have

$$\begin{aligned}|X_a^\zeta-X_a^\eta|&=|(\partial_x\partial_y/\partial_y)J_a^\zeta(\zeta(a))-(\partial_x\partial_y/\partial_y)J_a^\eta(\eta(a))|\\ &\le|(\partial_x\partial_y/\partial_y)J_a^\zeta(\zeta(a))-(\partial_x\partial_y/\partial_y)J_a^\zeta(\eta(a))|\\ &\quad+|(\partial_x\partial_y/\partial_y)J_a^\zeta(\eta(a))-(\partial_x\partial_y/\partial_y)J_a^\eta(\eta(a))|\\ &\le C_1|\zeta(a)-\eta(a)|+C_2\|\zeta-\eta\|_a\le C\|\zeta-\eta\|_a.\end{aligned}$$

□

Proof of Theorem 3.1. Let $\xi_0(t)=A(0)$, $t\in[0,\infty)$. We may have $a_0>0$ such that $K_{a_0}^{\xi_0}\subset\Omega\setminus\{p\}$. Choose crosscuts α_0 and α_1 in $\mathbb{H}$ such that

$K_{a_0}^{\xi_0}$ is enclosed by α_0, α_0 is strictly enclosed by α_1, and $H(\alpha_1)\subset\Omega\setminus\{p\}$. From Lemma 5.5, there is $\delta_1>0$ such that for any $t\in[0,a_0]$, if $\eta\in C([0,t])$ satisfies $\|\eta-\xi_0\|_t<\delta_1$, then $K_t^\eta\subset H(\alpha_1)$. Let $\delta_2, C>0$ be the constants given by Lemma 5.8 with $\alpha=\alpha_1$. Let $\delta=\delta_1\wedge(\delta_2/2)$. Then for any $t\in(0,a_0]$, if $\eta_j\in C([0,t])$ and $\|\eta_j-\xi_0\|_t<\delta$, $j=1,2$, then

$$|X_t^{\eta_1}-X_t^{\eta_2}|\le C\|\eta_1-\eta_2\|_t. \tag{5.13}$$

Define a sequence of functions $(\xi_n(t))$ by induction:

$$\xi_{n+1}(t)=A(t)+\lambda\int_0^t X_s^{\xi_n}\,ds, \tag{5.14}$$

as long as $X_s^{\xi_n}$, $0\le s\le t$, are defined. From Lemma 6.3, we see that X_t^ξ is continuous, and so the integral makes sense. We may choose $a\in(0,a_0]$ such that $|\lambda|Ca<1/2$ and $\|\xi_1-\xi_0\|_a<\delta/2$. For $n=1$, we have $\|\xi_n-\xi_0\|_a<(1-1/2^n)\delta$ and $\|\xi_n-\xi_{n-1}\|_a<\delta/2^n$. Suppose this is true for some $n\in\mathbb{N}$. Then from (5.13) and (5.14), we have

$$\begin{aligned}|\xi_{n+1}(t)-\xi_n(t)|&\le|\lambda|\int_0^t|X_s^{\xi_n}-X_s^{\xi_{n-1}}|\,ds\le|\lambda|\int_0^t C\|\xi_n-\xi_{n-1}\|_a\,ds\\&\le|\lambda|Ca\|\xi_n-\xi_{n-1}\|_a<\|\xi_n-\xi_{n-1}\|_a/2<\delta/2^{n+1},\end{aligned}$$

for $t\in[0,a]$. Thus $\|\xi_{n+1}-\xi_n\|_a<\delta/2^{n+1}$, and $\|\xi_{n+1}-\xi_0\|_b<\delta/2^{n+1}+\|\xi_n-\xi_0\|_b<(1-1/2^{n+1})\delta$. From induction, we have $\|\xi_{n+1}-\xi_n\|_a<\delta/2^{n+1}$ for any $n\in\mathbb{N}$. Thus (ξ_n) restricted to $[0,a]$ is a Cauchy sequence in $C([0,a])$. Let $\xi_\infty=\lim_{n\to\infty}\xi_n|_{[0,a]}\in C([0,a])$. Let $n\to\infty$ in (5.14); we see that ξ_∞ solves (3.2) for $t\in[0,a]$.

Let $\mathcal{S}$ be the set of all couples (ξ,T) such that $T>0$ and ξ solves (3.2) for $t\in[0,T]$. We have proved that $\mathcal{S}$ is nonempty. We claim that if $(\xi,T)\in\mathcal{S}$, then there is $(\xi_e,T_e)\in\mathcal{S}$ such that $T_e>T$ and $\xi_e(t)=\xi(t)$ for $t\in[0,T]$. To prove this claim, let $\breve\Omega=\varphi_T^\xi(\Omega\setminus K_T^\xi)$ and $\breve p=\varphi_T^\xi(p)$. If $K_t^{\breve\xi}\subset\breve\Omega\setminus\{\breve p\}$, let $\breve J_t^{\breve\xi}=G(\breve\Omega\setminus K_t^{\breve\xi},\breve p;\cdot)\circ(\varphi_t^{\breve\xi})^{-1}$, and $\breve X_t^{\breve\xi}=(\partial_x\partial_y/\partial_y)\breve J_t^{\breve\xi}(\breve\xi(t))$. From the first part of the proof, the solution to

$$\breve\xi(t)=\xi(T)+A(T+t)-A(T)+\lambda\int_0^t\breve X_s^{\breve\xi}\,ds \tag{5.15}$$

exists on $[0,\breve a]$ for some $\breve a>0$. Let $T_e=T+\breve a>T$. Define $\xi_e(t)=\xi(t)$ for $t\in[0,T]$ and $\xi_e(t)=\breve\xi(t-T)$ for $t\in[T,T_e]$. It is clear that $\xi_e\in C([0,T_e])$. Since ξ_e agrees with ξ on $[0,T]$, so ξ_e solves (3.2) for $t\in[0,T]$. For $t\in[0,T_e-T]$, we have $\varphi_{T+t}^{\xi_e}=\varphi_t^{\breve\xi}\circ\varphi_T^\xi$ and $K_{T+t}^{\xi_e}=K_T^\xi\cup(\varphi_T^\xi)^{-1}(K_t^{\breve\xi})$. Since φ_T^ξ maps p to $\breve p$, and $\Omega\setminus K_{T+t}^{\xi_e}$ onto $\breve\Omega\setminus K_t^{\breve\xi}$, so

$$\begin{aligned}\breve J_t^{\breve\xi}&=G(\breve\Omega\setminus K_t^{\breve\xi},\breve p;\cdot)\circ(\varphi_t^{\breve\xi})^{-1}=G(\Omega\setminus K_{T+t}^{\xi_e},p;\cdot)\circ(\varphi_T^\xi)^{-1}\circ(\varphi_t^{\breve\xi})^{-1}\\&=G(\Omega\setminus K_{T+t}^{\xi_e},p;\cdot)\circ(\varphi_{T+t}^{\xi_e})^{-1}=J_{T+t}^{\xi_e}.\end{aligned}$$

So $\breve X^{\breve\xi}_t = X^{\xi_e}_{T+t}$. Thus for $t\in[0,T_e-T]$,

$$\begin{aligned}\xi_e(T+t)=\breve\xi(t)&=\xi(T)+A(T+t)-A(T)+\lambda\int_0^t \breve X^{\breve\xi}_s\,ds\\ &=A(T+t)+\lambda\int_0^T X^{\xi}_s\,ds+\lambda\int_0^t X^{\xi_e}_{T+s}\,ds\\ &=A(T+t)+\lambda\int_0^{T+t} X^{\xi_e}_s\,ds.\end{aligned}$$

So ξ_e solves (3.2) for $t\in[T,T_e]$. Thus $(\xi_e,T_e)\in\mathcal S$. So the claim is justified.

Suppose $(\xi_1,T_1),(\xi_2,T_2)\in\mathcal S$. For $j=1,2$, since $\xi_j(0)=A(0)=\xi_0(0)$, so there is $S_j\in(0,T_j\wedge a_0]$ such that $\|\xi_j-\xi_0\|_{S_j}<\delta$. Choose $S_3\in(0,S_1\wedge S_2]$ such that $C|\lambda|S_3<1$. From (3.2) and (5.13), we have $\|\xi_1-\xi_2\|_{S_3}\le|\lambda|CS_3\|\xi_1-\xi_2\|_{S_3}$, so $\|\xi_1-\xi_2\|_{S_3}=0$, which means that $\xi_1(t)=\xi_2(t)$ for $0\le t\le S_3$.

Let $T_0=T_1\wedge T_2$. We claim that $\xi_1(t)=\xi_2(t)$ for $t\in[0,T_0]$. Let $T\in[0,T_0]$ be the maximum such that $\xi_1(t)=\xi_2(t)$ for $t\in[0,T]$. Suppose $T<T_0$. Let $\breve\xi_1(t)=\xi_1(T+t)$, $\breve\xi_2(t)=\xi_2(T+t)$ for $t\in[0,T_0-T]$. Then $\breve\xi_1$ and $\breve\xi_2$ both solve (5.15) for $t\in[0,T_0-T]$. From the last paragraph, there is $S_3\in(0,T_0-T]$ such that $\breve\xi_1(t)=\breve\xi_2(t)$ for $0\le t\le S_3$, which implies that $\xi_1(t)=\xi_2(t)$ for $0\le t\le T+S_3$. This contradicts the maximum property of T. So $T=T_0$, and $\xi_1(t)=\xi_2(t)$ for $t\in[0,T_0]$.

Let $T_A=\sup\{T:(\xi,T)\in\mathcal S\}$. Define ξ_A on $[0,T_A)$ as follows. For any $t\in[0,T_A)$, choose $(\xi,T)\in\mathcal S$ such that $t\le T$, and let $\xi_A(t)=\xi(t)$. From the last paragraph, ξ_A is well defined, and solves (3.2) for $t\in[0,T_A)$. The uniqueness of ξ_A also follows from the last paragraph. There is no solution to (3.2) defined on $[0,T_A]$. Otherwise, there exists some solution on $[0,T_A+\varepsilon]$ for some $\varepsilon>0$, which contradicts the definition of T_A.

(i) Suppose $A_0\in C([0,\infty))$, $a\in(0,\infty)$, and $T_{A_0}>a$. Then $K_a^{\xi_{A_0}}\subset\Omega\setminus\{p\}$. Choose a crosscut α in $\mathbb H$ such that $K_a^{\xi_{A_0}}\subset H(\alpha)\subset\Omega\setminus\{p\}$. Let $\delta_0,C_0>0$ be the δ,C given by Lemma 5.8 with $\zeta=\xi_{A_0}$. Let $C=\exp(C_0|\lambda|a)$ and $\delta=\delta_0/C$. Suppose $A\in C([0,\infty))$ and $\|A-A_0\|_a<\delta$. Then $|\xi_A(0)-\xi_{A_0}(0)|=|A(0)-A_0(0)|<\delta\le\delta_0$. Let $b\in[0,a\wedge T_A)$ be the maximal such that $|\xi_A(t)-\xi_{A_0}(t)|<\delta_0$ for $t\in[0,b)$. From the properties of δ_0 and C_0, for $0\le t<b$,

$$|X^{\xi_A}_t-X^{\xi_{A_0}}_t|\le C_0\|\xi-\xi_{A_0}\|_t. \tag{5.16}$$

So $X^{\xi_A}_t$ is bounded on $[0,b)$. From (3.2), $\lim_{t\to b^-}\xi_A(t)$ exists. If $T_A=b$, we define $\xi_A(T)=\lim_{t\to b^-}\xi_A(t)$, then ξ_A solves (3.2) for $t\in[0,T]$, which is a contradiction. Thus $T_A>b$. From (3.2) and (5.16), we have that for any $0\le t<b$,

$$|\xi_A(t)-\xi_{A_0}(t)|\le\|A-A_0\|_a+C_0|\lambda|\int_0^t|\xi_A(s)-\xi_{A_0}(s)|\,ds.$$

Solving this inequality, we have that for any $0 \le t < b$,

$$|\xi_A(t) - \xi_{A_0}(t)| \le \exp(C_0|\lambda|t)\|A - A_0\|_a \le C\|A - A_0\|_a.$$

So $|\xi_A(b) - \xi_{A_0}(b)| \le C\|A - A_0\|_a < C\delta = \delta_0$. From the definition of b, we have $b = a$. Thus $T_A > a$ and $\|\xi_A - \xi_{A_0}\|_a \le C\|A - A_0\|_a$ if $\|A - A_0\|_a < \delta$. So $\{T_A > a\}$ is open w.r.t. $\|\cdot\|_a$, and $A \mapsto \xi_A$ is $(\|\cdot\|_a, \|\cdot\|_a)$ continuous on $\{T_A > a\}$.

(ii) Suppose α is a crosscut in $\mathbb{H}$ such that $\bigcup_{0\le t<T} K_t^\xi \subset H(\alpha) \subset \Omega \setminus \{p\}$. Then $T \le \mathrm{hcap}(H(\alpha))/2 < +\infty$. From the compactness of $\mathcal{H}(\alpha)$, X_t^ξ is bounded on $[0,T)$. So from (3.2), $\xi(t) \to x$ for some $x \in \mathbb{R}$ as $t \to T$. Define $\xi(T) = x$. Then $\xi \in C([0,T])$, $K_T^\xi \subset H(\alpha) \subset \Omega \setminus \{p\}$, and so J_t^ξ is defined for $t \in [0,T]$. Then $\xi(t)$ solves (3.2) for $0 \le t \le T$, which is a contradiction. □

6. Convergence of the driving functions. From now on, we begin proving Theorem 4.2. We first study the case that the target is an interior point. In this section, we will show that the driving functions for the discrete LERW converge to those for the continuous LERW.

6.1. *Some estimates.* Suppose Ω is an almost $\mathbb{H}$ domain and $p \in \Omega$. We now use the notation in Sections 3 and 4 in the case that the target is an interior point. Let α be a crosscut in $\mathbb{H}$ such that $H(\alpha) \subset \Omega \setminus \{p\}$; and let F be a compact subset of $\Omega \setminus H(\alpha)$. In the lemmas in this subsection, a uniform constant is a number that depends only on Ω, p, α, F. From the compactness of $\mathcal{H}(\alpha)$ (Lemma 5.4), there is a uniform constant $\mathbf{h} > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t \in [0,a]$, $\mathrm{dist}(\varphi_t^\xi(\partial\Omega \setminus \mathbb{R}) \cup \varphi_t^\xi(F), \mathbb{R}) \wedge \mathrm{dist}(\varphi_t^\xi(F), \varphi_t^\xi(\partial\Omega \setminus \mathbb{R})) \ge \mathbf{h}$.

LEMMA 6.1. *There are uniform constants $C_1, C_2 > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t_1 \le t_2 \in [0,a]$ and $z \in F$,*

$$|\varphi_{t_2}^\xi(z) - \varphi_{t_1}^\xi(z)| \le C_1|t_2 - t_1|;$$

$$\left|\varphi_{t_2}^\xi(z) - \varphi_{t_1}^\xi(z) - \frac{2(t_2 - t_1)}{\varphi_{t_1}^\xi(z) - \xi(t_1)}\right| \le C_2|t_2 - t_1|\left(|t_2 - t_1| + \sup_{t\in[t_1,t_2]}\{|\xi(t) - \xi(t_1)|\}\right).$$

PROOF. Suppose $K_a^\xi \subset H(\alpha)$. Then $|\varphi_t^\xi(z) - \xi(t)| \ge \mathbf{h}$ for any $t \in [0,a]$ and $z \in F$. Since $\varphi_{t_2}^\xi(z) - \varphi_{t_1}^\xi(z) = \int_{t_1}^{t_2} \frac{2}{\varphi_t^\xi(z) - \xi(t)}\,dt$, so $|\varphi_{t_2}^\xi(z) - \varphi_{t_1}^\xi(z)| \le$

$C_1|t_2-t_1|$ for any $t_1\le t_2\in[0,a]$ and $z\in F$, where $C_1=2/\mathbf{h}>0$. Thus for $t_1\le t_2\in[0,a]$ and $z\in F$,

$$\Big|\frac{2}{\varphi^\xi_{t_2}(z)-\xi(t_2)}-\frac{2}{\varphi^\xi_{t_1}(z)-\xi(t_1)}\Big|\le\frac{2}{\mathbf{h}^2}(|\varphi^\xi_{t_2}(z)-\varphi^\xi_{t_1}(z)|+|\xi(t_2)-\xi(t_1)|)$$
$$\le 2C_1/\mathbf{h}^2|t_2-t_1|+2/\mathbf{h}^2|\xi(t_2)-\xi(t_1)|.$$

Let $C_2:=2(C_1\vee 1)/\mathbf{h}^2>0$. Then for $t_1\le t_2\in[0,a]$ and $z\in F$,

$$\begin{aligned}&\Big|\varphi^\xi_{t_2}(z)-\varphi^\xi_{t_1}(z)-\frac{2(t_2-t_1)}{\varphi^\xi_{t_1}(z)-\xi(t_1)}\Big|\\ &=\Big|\int_{t_1}^{t_2}\Big(\frac{2}{\varphi^\xi_{t}(z)-\xi(t)}-\frac{2}{\varphi^\xi_{t_1}(z)-\xi(t_1)}\Big)\,dt\Big|\\ &\le\int_{t_1}^{t_2}\Big|\frac{2}{\varphi^\xi_{t}(z)-\xi(t)}-\frac{2}{\varphi^\xi_{t_1}(z)-\xi(t_1)}\Big|\,dt\\ &\le C_2|t_2-t_1|\Big(|t_2-t_1|+\sup_{t\in[t_1,t_2]}\{|\xi(t)-\xi(t_1)|\}\Big).\end{aligned}$$
□

LEMMA 6.2. *For each $n_1\in\{0,1\}$, $n_2,n_3\in\mathbb{Z}_{\ge0}$, there is a uniform constant $C>0$ depending on n_1,n_2,n_3, such that if $K^\xi_a\subset H(\alpha)$, then for any $t\in[0,a]$, $x\in[c_\alpha,d_\alpha]$, and $z\in F$, we have*

$$|\partial_1^{n_1}\partial_2^{n_2}\partial_{3,z}^{n_3}P^\xi(t,x,\varphi^\xi_t(z))|\le C.$$

PROOF. For $K\in H(\alpha)$, $x\in\mathbb{R}$ and $z\in\Omega_K$, let $P(K,x,z)$ be as in Section 4. Since $\partial\Omega_K$ is analytic, so $P(K,x,\cdot)$ extends harmonically across $\partial\Omega_K$. For $K\in H(\alpha)$ and $x,y\in\mathbb{R}$, let $Q_y(K,x,\cdot)$ be defined on $\overline{\Omega_K}\setminus\{x\}$ such that $Q_y(K,x,\cdot)$ is harmonic in Ω_K; vanishes on $\mathbb{R}\setminus\{x\}$; behaves like $\operatorname{Im}\frac{c}{z-x}+O(1)$ near x for some $c\in\mathbb{R}$; $Q_y(K,x,z)=-2\operatorname{Re}(\partial_{3,z}P(K,x,z)\cdot\frac{2}{z-y})$ for $z\in\partial\Omega_K\setminus\mathbb{R}$ and $z=\varphi_K(p)$. From the compactness of $H(\alpha)$, for any $n_2,n_3\in\mathbb{Z}_{\ge0}$, there is a uniform constant $C>0$ depending on n_2,n_3, such that for any $K\in H(\alpha)$, $x,y\in[c_\alpha,d_\alpha]$, and $z\in F$, we have

$$|\partial_2^{n_2}\partial_{3,z}^{n_3}P(K,x,\varphi_K(z))|,\ |\partial_2^{n_2}\partial_{3,z}^{n_3}Q_y(K,x,\varphi_K(z))|\le C.$$

Note $P^\xi(t,x,z)=P(K^\xi_t,x,z)$ and $\partial_1P^\xi(t,x,z)=Q_{\xi(t)}(K^\xi_t,x,z)$, so we are done. □

LEMMA 6.3. *There is a uniform constant $C>0$ such that if $K^\xi_a\subset H(\alpha)$, then for any $t,t'\in[0,a]$, $|X^\xi_t|\le C$ and $|X^\xi_t-X^\xi_{t'}|\le C(|t-t'|+|\xi(t)-\xi(t')|)$.*

PROOF. Suppose $K_a^\xi \subset H(\alpha)$. Let $J^\xi(t,x) = J_t^\xi(x)$. Note that $X_t^\xi = (\partial_{2,z}^2/\partial_{2,z})J^\xi(t,x)$. Since $\xi(t) \in [c_\alpha, d_\alpha]$ for $t \in [0,a]$, so it suffices to prove that there is a uniform constant $C > 0$ such that for any $t \in [0,a]$ and $x \in [c_\alpha, d_\alpha]$, $|\partial_1^{n_1}\partial_{2,z}^{n_2}(\partial_{2,z}^2/\partial_{2,z})J^\xi(t,x)| \le C$ for $n_1, n_2 \in \{0,1\}$. We need to show that $|\partial_{2,z}J^\xi(t,x)|$ is bounded from below by a positive uniform constant, and $|\partial_1^{n_1}\partial_{2,z}^{n_2+1}J^\xi(t,x)|$ is bounded from above by a positive uniform constant. The proof is similar to that of the above lemma. □

LEMMA 6.4. *There is a uniform constant $C > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t_1 \le t_2 \in [0,a]$ and $z \in F$, we have*

$$|\partial_1 P^\xi(t_2, \xi(t_2), \varphi_{t_2}^\xi(z)) - \partial_1 P^\xi(t_1, \xi(t_1), \varphi_{t_1}^\xi(z))| \le C(|t_2 - t_1| + |\xi(t_2) - \xi(t_1)|).$$

PROOF. This follows from Lemma 4.1, and the above three lemmas. □

LEMMA 6.5. *There is a uniform constant $d_1 > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t_1 < t_2 \in [0,a]$ that satisfy $|t_2 - t_1| \le d_1$, and for any $z \in F$, we have*

$$\begin{aligned} &P^\xi(t_2, \xi(t_2), \varphi_{t_2}^\xi(z)) - P^\xi(t_1, \xi(t_1), \varphi_{t_1}^\xi(z)) \\ &\quad = \partial_2 P^\xi(t_1, \xi(t_1), \varphi_{t_1}^\xi(z)) \cdot [(\xi(t_2) - \xi(t_1)) - (t_2 - t_1)X_{t_1}^\xi] \\ &\qquad + 1/2\partial_2^2 P^\xi(t_1, \xi(t_1), \varphi_{t_1}^\xi(z)) \cdot [(\xi(t_2) - \xi(t_1))^2 - 2(t_2 - t_1)] \\ &\qquad + O(A^2) + O(AB) + O(AB^2) + O(B^3), \end{aligned}$$

where $A := |t_2 - t_1|$, $B := \sup_{s,t \in [t_1,t_2]}\{|\xi(s) - \xi(t)|\}$, and $O(X)$ is some number whose absolute value is bounded by $C|X|$ for some uniform constant $C > 0$.

PROOF. We may choose a compact subset F' of $\Omega \setminus H(\rho)$ such that F is contained in the interior of F'. So from the compactness of $\mathcal{H}(\alpha)$, there is a uniform constant $d_0 > 0$ such that for any $K \in H(\alpha)$, $\mathrm{dist}(\varphi_K(F), \partial\varphi_K(F')) \ge d_0$. Suppose $K_a^\xi \subset H(\alpha)$. From Lemma 6.1, there is a uniform constant $d_1 > 0$ such that if $s, t \in [0,a]$ satisfy $|s - t| \le d_1$, then for any $z \in F$, $[\varphi_s^\xi(z), \varphi_t^\xi(z)] \subset \varphi_s^\xi(F')$.

Fix $z \in F$ and $t_1 < t_2 \in [0,a]$ with $|t_2 - t_1| \le d_1$. Let $P_1 = P^\xi(t_2, \xi(t_2), \varphi_{t_2}^\xi(z))$, $P_2 = P^\xi(t_1, \xi(t_2), \varphi_{t_2}^\xi(z))$, $P_3 = P^\xi(t_1, \xi(t_1), \varphi_{t_2}^\xi(z))$, $P_4 = P^\xi(t_1, \xi(t_1), \varphi_{t_1}^\xi(z))$. Then

$$P^\xi(t_2, \xi(t_2), \varphi_{t_2}^\xi(z)) - P^\xi(t_1, \xi(t_1), \varphi_{t_1}^\xi(z)) = (P_1 - P_2) + (P_2 - P_3) + (P_3 - P_4).$$

Now $P_1 - P_2 = \int_{t_1}^{t_2} \partial_1 P^\xi(t, \xi(t_2), \varphi_{t_2}^\xi(z))\, dt$. Fix any $t \in [t_1, t_2]$. Applying Lemmas 6.1 and 6.2 to F', since $\xi(t), \xi(t_2) \in [c_\alpha, d_\alpha]$ and $[\varphi_t^\xi(z), \varphi_{t_2}^\xi(z)] \subset$

$\varphi^\xi_t(F')$, so we have

$$\partial_1 P^\xi(t,\xi(t_2),\varphi^\xi_{t_2}(z)) - \partial_1 P^\xi(t,\xi(t),\varphi^\xi_t(z)) = O(A)+O(B).$$

Applying Lemma 6.4 to F, we have

$$\partial_1 P^\xi(t,\xi(t),\varphi^\xi_t(z)) - \partial_1 P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z)) = O(A)+O(B).$$

So we get

$$P_1 - P_2 = \partial_1 P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z))(t_2-t_1) + O(A^2)+O(AB).$$

Applying Lemma 6.2 to F', since $\varphi^\xi_{t_2}(z)\in\varphi^\xi_{t_1}(F')$, so we have

$$\begin{aligned} P_2 - P_3 = {} & \partial_2 P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_2}(z))(\xi(t_2)-\xi(t_1)) \\ & + 1/2\partial_2^2 P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_2}(z))(\xi(t_2)-\xi(t_1))^2 + O(B^3). \end{aligned}$$

Applying Lemmas 6.1 and 6.2 to F', since $[\varphi^\xi_{t_1}(z),\varphi^\xi_{t_2}(z)]\subset\varphi^\xi_{t_1}(F')$, so we have

$$\partial_2^j P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_2}(z)) - \partial_2^j P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z)) = O(A),$$

for $j=1,2$. Thus

$$\begin{aligned} P_2 - P_3 = {} & \partial_2 P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z))(\xi(t_2)-\xi(t_1)) \\ & + 1/2\partial_2^2 P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z))(\xi(t_2)-\xi(t_1))^2 \\ & + O(AB)+O(AB^2)+O(B^3). \end{aligned}$$

Applying Lemmas 6.1 and 6.2 to F', since $[\varphi^\xi_{t_1}(z),\varphi^\xi_{t_2}(z)]\subset\varphi^\xi_{t_1}(F')$, so we have

$$\begin{aligned} P_3 - P_4 &= 2\,\mathrm{Re}(\partial_{3,z}P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z))(\varphi^\xi_{t_2}(z)-\varphi^\xi_{t_1}(z))) + O(A^2) \\ &= 2\,\mathrm{Re}(\partial_{3,z}P^\xi(t_1,\xi(t_1),\varphi^\xi_{t_1}(z))\frac{2(t_2-t_1)}{\varphi^\xi_{t_1}(z)-\xi(t_1)}) + O(AB)+O(A^2). \end{aligned}$$

The conclusion follows from Lemma 4.1 and the equalities for $P_j - P_{j+1}$, $j=1,2,3$. □

6.2. *Convergence.* We use the notation in Section 4.2. We may choose crosscuts ρ_j, $j=0,1,2$, in D such that $H(\rho_0)$ is a neighborhood of 0_+ in D, $H(\rho_0)\subset H(\rho_1)\subset H(\rho_2)\subset D\setminus\{z_e,\infty\}$, and

$$d_0 := \min\{\mathrm{dist}(0,\rho_0),\mathrm{dist}(\rho_0,\rho_1),\mathrm{dist}(\rho_1,\rho_2),\mathrm{dist}(\rho_2,z_e)\} > 0.$$

Now suppose $\delta<d_0$. Then $w_e^\delta\notin H(\rho_2)$ as $|w_e^\delta - z_e|<\delta$, any edge of D^δ can intersect at most one of ρ_j's, and $(0,\delta]\subset H(\rho_0)$. Thus the LERW curve q_δ

must cross all of these ρ_j's. Let F_D be a compact subset of $D\setminus\{\infty\}\setminus H(\rho_2)$ with nonempty interior. Suppose f maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(0_+)=0$. Let $p=f(z_e)$, $F_\Omega=f(F_D)$ and $\alpha_j=f(\rho_j)$, $j=0,1,2$. Then F_Ω is a compact subset of Ω with nonempty interior; α_j's are crosscuts in $\mathbb{H}$; α_0 strictly encloses 0; α_{j+1} strictly encloses α_j; and $\{p\},F_\Omega\subset\Omega\setminus H(\alpha_2)$.

In this subsection, a uniform constant is a number that depends only on D, z_e, ρ_0, ρ_1, ρ_2, F_D, f, and some other variables we will specify. We use $O(X)$ to denote a number whose absolute value is bounded by $C|X|$ for some positive uniform constant C. We use $o_\delta(X)$ to denote a number whose absolute value is bounded by $C(\delta)|X|$ for some positive uniform constant $C(\delta)$ depending on δ, such that $C(\delta)\to 0$ as $\delta\to 0$.

Let L^δ denote the set of finite simple lattice paths $X=(X(-1),X(0),\dots,X(s))$, $s\in\mathbb{N}$, on D^δ, such that $X(-1)=0$, $X(0)=\delta$, $X(k)\in D$ for $0\le k\le s$, and $\bigcup_{k=0}^s(X(k-1),X(k)]\subset H(\rho_1)$. Let $\operatorname{Set}(X)=\{X(0),\dots,X(s)\}$, $\operatorname{Tip}(X)=X(s)$, $D_X=D\setminus\bigcup_{k=0}^s(X(k-1),X(k)]$; P_X be the generalized Poisson kernel in D_X with the pole at $\operatorname{Tip}(X)$, normalized by $P_X(z_e)=1$; and g_X be defined on $V(D^\delta)$ such that $g_X\equiv 0$ on $V_\partial(D^\delta)\cup\operatorname{Set}(X)\setminus\{\operatorname{Tip}(X)\}$, $\Delta_{D^\delta}g_X\equiv 0$ on $V_I(D^\delta)\setminus\operatorname{Set}(X)$, and $g_X(w_e^\delta)=1$.

LEMMA 6.6. *Suppose $G=(V,E)$ is a connected locally finite graph. Suppose $A,B\subset V$ are such that B is finite and $A\cup B$ is reachable. Suppose h is a nonnegative bounded function on V such that h vanishes on A, and is discrete harmonic on $V\setminus(A\cup B)$. Then we have $\sum_{w\in A}\Delta_G h(w)=-\sum_{w\in B}\Delta_G h(w)$.*

PROOF. For $w_0\in B$, let H_{w_0} be the bounded function on V, which is discrete harmonic in $V\setminus(A\cup B)$, vanishes on $A\cup B\setminus\{w_0\}$, and equals 1 at w_0. Then the lemma holds if $h=H_{w_0}$. Since $h(w)=\sum_{w_0\in B}h(w_0)H_{w_0}(w)$, so we are done. □

PROPOSITION 6.1. *For any $\varepsilon>0$, there is $\delta_0>0$ such that if $0<\delta<\delta_0$, then for any $X\in L^\delta$, and any $w\in V(D^\delta)\cap(D\setminus H(\rho_2))$, we have $|g_X(w)-P_X(w)|<\varepsilon$.*

SKETCH OF THE PROOF. Suppose the proposition is not true. Then we can find $\varepsilon_0>0$, a sequence of lattice paths $X_n\in L^{\delta_n}$ with $\delta_n\to 0$, and a sequence of points $w_n\in V^{\delta_n}\cap(D\setminus H(\rho_2))$, such that $|g_{X_n}(w_n)-P_{X_n}(w_n)|>\varepsilon_0$ for all $n\in\mathbb{N}$. For simplicity of notation, we write g_n for g_{X_n}, P_n for P_{X_n} and D_n for D_{X_n}. Let $K_n=f(\bigcup_{j=0}^{l(X_n)}(X_n(j-1),X_n(j)])$. Then $K_n\in\mathcal{H}(\alpha_1)$. Write φ_n for φ_{K_n} and Ω_n for Ω_{K_n}. Let $x_n=\varphi_n\circ f(\operatorname{Tip}(X_n))$. Then $x_n\in[c_{\alpha_1},d_{\alpha_1}]$. Let $Q_n=P_n\circ f^{-1}\circ\varphi_n^{-1}$. Then Q_n is the generalized Poisson kernel in Ω_n with the pole at x_n, normalized by $Q_n(\varphi_n(p)) = 1$. From the compactness of $\mathcal{H}(\alpha_1)$, by passing to a subsequence, we may assume that $K_n \xrightarrow{\mathcal{H}} K_0 \in \mathcal{H}(\alpha_1)$ and $x_n \to x_0 \in [c_{\alpha_1}, d_{\alpha_1}]$. Write Ω_0 for Ω_{K_0} and φ_0 for φ_{K_0}. Let Q_0 be the generalized Poisson kernel in Ω_0 with the pole at x_0, normalized by $Q_0(\varphi_0(p)) = 1$. Let $D_0 = f^{-1}(\Omega \setminus K_0)$ and $P_0 = Q_0 \circ \varphi_0 \circ f$. Then P_0 is the generalized Poisson kernel in D_0 with the pole at $f^{-1} \circ \varphi_0^{-1}(x_0)$, normalized by $P_0(z_e) = 1$. Moreover, $D_n \xrightarrow{\text{Cara}} D_0$, and $P_n \xrightarrow{\text{l.u.}} P_0$ in D_0.

We extend g_n to $\text{CE}^n g_n$ that is defined on the union of lattice squares of $\delta\mathbb{Z}^2$ at whose four vertices g_n is defined. Applying Harnack's inequality to the positive discrete harmonic function g_n, we find that $(\text{CE}^n g_n)$ is locally uniformly continuous in D_0. By the Arzela–Ascoli theorem, there is a subsequence of $(\text{CE}^n g_n)$, which converges locally uniformly to some g_0 in $D_0 \setminus \{\infty\}$. We may assume that the subsequence is $(\text{CE}^n g_n)$ itself. By applying Harnack's inequality to the discrete partial derivatives of g_n, we may assume that the continuation of all discrete partial derivatives of g_n also converge to the corresponding partial derivatives of g_0. Then we conclude that g_0 is a positive harmonic function in $D_0 \setminus \{\infty\}$.

We may find a sequence of crosscuts (γ^k) in D_0 such that $(H(\gamma^k))$ is a nesting neighborhood basis of the prime end $f^{-1} \circ \varphi_0^{-1}(x_0)$ in D_0, which is the pole of P_0. Fix $k \in \mathbb{N}$, for each $n \in \mathbb{N}$, we find a crosscut γ_n^k in D_n that bounds a neighborhood $H(\gamma_n^k)$ of $\text{Tip}(X_n)$, such that γ_n^k converges to γ^k in some sense as $n \to \infty$. For each $k \geq 2$, we may construct some "hook" in the area of D_0 between γ^{k-1} and γ^{k+1} that holds the boundary of D_0 and disconnects γ^{k+1} from γ^{k-1}. We use these hooks to prove that the values of g_n outside $H(\gamma^{k+1})$ are uniformly bounded, and $g_n(w) \to 0$ as $n \to \infty$ and $w \to \partial D_n$ in $V(D^{\delta_n}) \cap (D_n \setminus H(\gamma_n^{k+1}))$ in the spherical metric. Thus $g_0(z) \to 0$ as $z \to \widehat{\mathbb{C}} \setminus D_0$ in $D_0 \setminus H(\gamma^{k+1})$ in the spherical metric. Since $(H(\gamma^k))$ is a neighborhood basis of $f^{-1} \circ \varphi_0^{-1}(x_0)$ in D_0, so if $\infty \notin D$, then g_0 must be a generalized Poisson kernel in D_0 with the pole at $f^{-1} \circ \varphi_0^{-1}(x_0)$. Since $g_0(z_e) = \lim_n \text{CE}^n g_n(w_e^\delta) = \lim g_n(w_e^\delta) = 1 = P_0(z_e)$, so $g_0 \equiv P_0$ in D_0. The sequence (w_n) has a subsequence (w_{n_k}) that converges to some $w_0 \in D$ or tends to $\widehat{\mathbb{C}} \setminus D$ in the spherical metric. In both cases, we can get a contradiction.

If $\infty \in D$, we only need to prove that g_0 is also harmonic at ∞. From Lemma 6.6, we have

$$\sum_{w \in \text{Set}(X_n) \cup (V(D^{\delta_n}) \cap \partial D)} \Delta_{D^{\delta_n}} g_n(w) = 0.$$

Choose a Jordan curve σ in D composed of line segments parallel to the x or y axis, such that ∂D is enclosed by σ. Let $U(\sigma)$ denote the intersection of D with the domain bounded by J. Let G_n be a subgraph of D^{δ_n} spanned by edges in D^{δ_n} that is incident to at least one vertex in $U(\sigma)$. Let $A = \mathrm{Set}(X_n) \cup (V(D^{\delta_n}) \cap \partial D)$, and let B be the set of vertices of G in $D \setminus U(\sigma)$. Then from Lemma 6.6, we have

$$\sum_{(w,w')\in\mathcal{P}_\sigma^n} (g_n(w) - g_n(w')) = -\sum_{w\in\mathrm{Set}(X_n)\cup(V(D^{\delta_n})\cap\partial D)} \Delta_{D^{\delta_n}} g_n(w) = 0,$$

where $\mathcal{P}_\sigma^n = \{(w,w') : w \in V(D^{\delta_n}) \cap U(\sigma), w' \in V_I(D^{\delta_n}) \setminus U(\sigma), w \sim w'\}$. Since the discrete partial derivative of g_n converges to the continuous partial derivative of g_0, so as $n \to \infty$,

$$\sum_{(w,w')\in\mathcal{P}_\sigma^n} (g_n(w) - g_n(w')) \to \int_\sigma \partial_{\mathbf{n}} g_0(z)\, ds(z).$$

Thus $\int_\sigma \partial_{\mathbf{n}} g_0(z)\, ds(z) = 0$, so g_0 is harmonic at ∞.

The reader may see Section 5 in [20] for the detailed proof of a similar proposition. □

Let the LERW curve q_δ on $[-1, \chi_\delta]$ be defined as in Section 4.2. For $-1 \le t \le \chi_\delta$, let $v_\delta(t) = \mathrm{hcap}(f \circ q_\delta((0,t]))/2$. Let $T_\delta = v_\delta(\chi_\delta)$ and $u_\delta = v_\delta^{-1}$. Let $\beta_\delta(t) = f(q_\delta(u_\delta(t)))$, $0 < t \le T_\delta$. Since $f(0_+) = 0$, so β_δ extends continuously to $[0, T_\delta]$ such that $\beta_\delta(0) = 0$. From Proposition 3.2, there is some $\xi_\delta \in C([0,T_\delta])$ such that $\beta_\delta((0,t]) = K_t^{\xi_\delta}$ for $0 \le t \le T_\delta$. For $n \in \mathbb{Z}_{\ge 0}$, let $\mathcal{F}_n$ be the σ-algebra generated by $\{n \le \chi_\delta\}$ and $q_\delta(j)$, $0 \le j \le n$. Let n_∞ be the first n such that $(q_\delta(n-1), q_\delta(n)]$ intersects ρ_0. Then n_∞ is an $\mathcal{F}_n$-stopping time and $\bigcup_{k=0}^{n_\infty}(q_\delta(k-1), q_\delta(k_j)]$ is contained in $H(\rho_1)$ because $\delta < \mathrm{dist}(\rho_0, \rho_1)$. Let $T_{\alpha_0}^\delta = v_\delta(n_\infty)$. So $K^{\xi_\delta}_{T^\delta_{\alpha_0}} \subset H(\alpha_1)$. Then $T_{\alpha_0}^\delta \le \mathrm{hcap}(H(\alpha_1))/2$, so $T_{\alpha_0}^\delta = O(1)$.

Fix any $n \in \mathbb{Z}_{[-1, n_\infty - 1]}$. Then $(q_\delta(n), q_\delta(n+1)]$ can be disconnected from ρ_1 by an annulus $A = \{\delta < |z - q_\delta(n)| < d_0\}$. Let Γ be the set of all crosscuts γ in $D \setminus \bigcup_{k=0}^n [q_\delta(k-1), q_\delta(k)]$ that is contained in A, and disconnects $(q_\delta(n), q_\delta(n+1)]$ from ρ_1 in $D \setminus \bigcup_{k=0}^n [q_\delta(k-1), q_\delta(k)]$. Then the extremal length of Γ is at most $2\pi/\ln(d_0/\delta)$. If $\gamma \in \Gamma$, then $\varphi^{\xi_\delta}_{v_\delta(n)} \circ f(\gamma)$ is a crosscut in $\mathbb{H}$, which disconnects $\varphi^{\xi_\delta}_{v_\delta(n)} \circ f((q_\delta(n), q_\delta(n+1)]) = \varphi^{\xi_\delta}_{v_\delta(n)}(K^{\xi_\delta}_{v_\delta(n+1)} \setminus K^{\xi_\delta}_{v_\delta(n)})$ from $\varphi^{\xi_\delta}_{v_\delta(n)}(\alpha_1)$ in $\mathbb{H}$. Since $K^{\xi_\delta}_{v_\delta(n)} \subset H(\alpha_0)$, and α_0 is strictly enclosed by α_1, so from the compactness of $\mathcal{H}(\alpha_0)$, the area of $H(\varphi^{\xi_\delta}_{v_\delta(n)}(\alpha_1))$ is bounded from above by a uniform constant $C_0 > 0$. By the conformal invariance, the extremal length of $f(\Gamma)$ is at most $2\pi/\ln(d_0/\delta)$. So there is $\gamma \in f(\Gamma)$ whose length is smaller than $l(\delta) := 2(C_0\pi/\ln(d_0/\delta))^{1/2}$. Then $l(\delta) = o_\delta(1)$. Since $\varphi^{\xi_\delta}_{v_\delta(n)}(K^{\xi_\delta}_{v_\delta(n+1)} \setminus K^{\xi_\delta}_{v_\delta(n)})$ is enclosed by γ, so its diameter is not bigger than $l(\delta)$. Thus there is $x_0 \in \mathbb{R}$ such that $\varphi^{\xi_\delta}_{v_\delta(n)}(K^{\xi_\delta}_{v_\delta(n+1)} \setminus K^{\xi_\delta}_{v_\delta(n)}) \subset \{z \in \mathbb{H} : |z - x_0| \le l(\delta)\}$. Thus $v_\delta(n+1) - v_\delta(n) \le \mathrm{hcap}(\{z \in \mathbb{H} : |z - x_0| \le l(\delta)\})/2 =$

$l(\delta)^2/2$ and $\xi_\delta(t) \in [x_0 - 2l(\delta), x_0 + 2l(\delta)]$ for any $t \in [v_\delta(n), v_\delta(n+1)]$, which implies that $|\xi_\delta(s) - \xi_\delta(t)| \le 4l(\delta)$ for any $s,t \in [v_\delta(n), v_\delta(n+1)]$.

Now fix a small $d > 0$. Define a nondecreasing sequence $(n_j)_{j\ge 0}$ inductively. Let $n_0 = 0$. Let n_{j+1} be the first $n \ge n_j$ such that $n = n_\infty$, or $v_\delta(n) - v_\delta(n_j) \ge d^2$, or $|\xi_\delta(n) - \xi_\delta(n_j)| \ge d$, whichever comes first. Then n_j's are stopping times w.r.t. $\{\mathcal{F}_n\}$, and are all bounded by n_∞. From the result of the last paragraph, we may let $\delta > 0$ be smaller than some positive uniform constant depending on d, such that $v_\delta(n_{j+1}) - v_\delta(n_j) \le 2d^2$ and $|\xi_\delta(v_\delta(s)) - \xi_\delta(v_\delta(n_j))| \le 2d$ for any $s \in [n_j, n_{j+1}]$, $0 \le j < \infty$. Let $\mathcal{F}'_j = \mathcal{F}_{n_j}$, $0 \le j < \infty$. For $0 \le n \le n_\infty$, let q_δ^n be the subpath of q_δ up to time n; then $q_\delta^n \in L^\delta$. Let (g_n) be the (g_n) in Proposition 2.1 for the LERW q_δ. Then $g_n = g_{q_\delta^n}$, where $g_{q_\delta^n}$ is as in Proposition 6.1. For simplicity, we write P_n for $P_{q_\delta^n}$.

From Proposition 2.1, for any $w \in V(D^\delta) \cap F_D$, $(g_{n_j}(w))_{j\ge 0}$ is a martingale w.r.t. $\{\mathcal{F}'_j\}$, so $\mathbf{E}[g_{n_{j+1}}(w)|\mathcal{F}'_j] = g_{n_j}(w)$ for any $j \in \mathbb{Z}_{\ge 0}$. From Proposition 6.1, we have $\mathbf{E}[P_{n_{j+1}}(w)|\mathcal{F}'_j] = P_{n_j}(w) + o_\delta(1)$. From Harnack's inequality, the absolute values of the gradients of P_{n_j} on F_D are bounded by a positive uniform constant. Since for any $z \in F_D$, there is $w \in V(D^\delta) \cap F_D$ with $|z - w| \le o_\delta(1)$, so for any $z \in F_D$, $\mathbf{E}[P_{n_{j+1}}(z)|\mathcal{F}'_j] = P_{n_j}(z) + o_\delta(1)$. Note that $P_n \circ f^{-1} = P^{\xi_\delta}(v_\delta(n), \xi_\delta(v_\delta(n)), \varphi^{\xi_\delta}_{v_\delta(n)}(\cdot))$. So for any $z \in F_\Omega = f(F_D)$,

$$\begin{aligned}(6.1)\qquad &\mathbf{E}[P^{\xi_\delta}(v_\delta(n_{j+1}), \xi_\delta(v_\delta(n_{j+1})), \varphi^{\xi_\delta}_{v_\delta(n_{j+1})}(z))|\mathcal{F}'_j] \\ &\quad = P^{\xi_\delta}(v_\delta(n_j), \xi_\delta(v_\delta(n_j)), \varphi^{\xi_\delta}_{v_\delta(n_j)}(z)) + o_\delta(1).\end{aligned}$$

PROPOSITION 6.2. *There are a uniform constant $d_2 > 0$, and a uniform constant $\delta(d) > 0$ depending on d, such that if $d < d_2$ and $\delta < \delta(d)$, then for all $j \in \mathbb{Z}_{\ge 0}$,*

$$\mathbf{E}\Big[(\xi_\delta(v_\delta(n_{j+1})) - \xi_\delta(v_\delta(n_j))) - \int_{v_\delta(n_j)}^{v_\delta(n_{j+1})} X_t^{\xi_\delta}\, dt\Big|\mathcal{F}'_j\Big] = O(d^3);$$

$$\mathbf{E}[(\xi_\delta(v_\delta(n_{j+1})) - \xi_\delta(v_\delta(n_j)))^2 - 2(v_\delta(n_{j+1}) - v_\delta(n_j))|\mathcal{F}'_j] = O(d^3).$$

PROOF. Note that $K^\xi_{T^\delta_{\alpha_0}} \subset H(\alpha_1)$. Let $d_1 > 0$ be the uniform constant given by Lemma 6.5 with $\alpha = \alpha_1$. Let $d_2 = (d_1/2)^{1/2}$. Suppose $d < d_2$. Fix $j \in \mathbb{Z}_{\ge 0}$. Let $a = v_\delta(n_j)$, $b = v_\delta(n_{j+1})$. Then $0 \le b - a \le 2d^2 \le 2d_2^2 = d_1$, and $|\xi_\delta(s) - \xi_\delta(t)| \le 4d$ for any $s,t \in [a,b]$. Fix $z \in F_\Omega$. From Lemma 6.5, we have

$$\begin{aligned}&P^{\xi_\delta}(b, \xi_\delta(b), \varphi_b^{\xi_\delta}(z)) - P^{\xi_\delta}(a, \xi_\delta(a), \varphi_a^{\xi_\delta}(z)) \\ &\quad = \partial_2 P^{\xi_\delta}(a, \xi_\delta(a), \varphi_a^{\xi_\delta}(z))((\xi_\delta(b) - \xi_\delta(a)) - (b-a)X_a^{\xi_\delta}) \\ &\qquad + \tfrac{1}{2}\partial_2^2 P^{\xi_\delta}(a, \xi_\delta(a), \varphi_a^{\xi_\delta}(z))((\xi_\delta(b) - \xi_\delta(a))^2 - 2(b-a)) + O(d^3).\end{aligned}$$

Take the conditional expectation of this equality with respect to $\mathcal{F}'_j$. From (6.1), we have

$$\partial_2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z))\mathbf{E}[(\xi_\delta(b)-\xi_\delta(a))-(b-a)X_a^{\xi_\delta}|\mathcal{F}'_j]$$
$$+\tfrac{1}{2}\partial_2^2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z))\mathbf{E}[(\xi_\delta(b)-\xi_\delta(a))^2-2(b-a)|\mathcal{F}'_j]$$
$$=O(d^3)+o_\delta(1).$$

Since $o_\delta(1)\to 0$ uniformly as $\delta\to 0$, so there is a positive uniform function $\delta(d)$ depending only on d such that if $\delta<\delta(d)$, then $|o_\delta(1)|\le d^3$. From Lemma 6.3, we have $X_t^{\xi_\delta}-X_a^{\xi_\delta}=O(d)$ for any $t\in[a,b]$. Thus for $\delta<\delta(d)$,

$$\partial_2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z))\mathbf{E}\left[(\xi_\delta(b)-\xi_\delta(a))-\int_a^b X_t^{\xi_\delta}\,dt\Big|\mathcal{F}'_j\right]$$
$$+\tfrac{1}{2}\partial_2^2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z))\mathbf{E}[(\xi_\delta(b)-\xi_\delta(a))^2-2(b-a)|\mathcal{F}'_j]=O(d^3).$$

Note that this is true for any $z\in F_\Omega$. We may choose $z_1\ne z_2\in F_\Omega$ and solve the linear equations to get the estimates of the two conditional expectations. We already know that $\partial_2^j P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z))=O(1)$ for $j=1,2$. So the proof will be completed if we prove that there is a uniform positive constant C_0 such that there are $z_1,z_2\in F_\Omega$ that satisfy

$$|\partial_2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z_1))\cdot\partial_2^2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z_2))$$
$$-\partial_2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z_2))\cdot\partial_2^2 P^{\xi_\delta}(a,\xi_\delta(a),\varphi_a^{\xi_\delta}(z_1))|\ge C_0.$$

This follows from the compactness of $\mathcal{H}(\alpha_1)$, and the fact that for every $K\in\mathcal{H}(\alpha_1)$ and $x\in[c_{\alpha_1},d_{\alpha_1}]$, there are $z_1,z_2\in F_\Omega$ such that

$$\begin{aligned}&\partial_2 P(K,x,\varphi_K(z_1))\partial_2^2 P(K,x,\varphi_K(z_2))\\&\qquad-\partial_2 P(K,x,\varphi_K(z_2))\partial_2^2 P(K,x,\varphi_K(z_1))\ne 0.\end{aligned}\tag{6.2}$$

Here, if (6.2) does not hold for some $K\in\mathcal{H}(\alpha_1)$ and $x\in[c_{\alpha_1},d_{\alpha_1}]$, then there is $C=C(K,x,F_\Omega)$ such that $\partial_2^2P(K,x,z)=C\partial_2P(K,x,z)$ for $z\in\varphi_K(F_\Omega)$. Since $\varphi_K(F_\Omega)$ contains an interior point, and $\partial_2^jP(K,x,\cdot)$, $j=1,2$, are harmonic in Ω_K, so $\partial_2^2P(K,x,z)=C\partial_2P(K,x,z)$ for $z\in\Omega_K$, which cannot be true because x is a pole of $\partial_2^jP(K,x,\cdot)$ of order $j+1$ for $j=1,2$. □

Let $\eta_\delta(t)=\xi_\delta(t)-2\int_0^t X_s^{\xi_\delta}\,ds$, $0\le t\le T_{\alpha_0}^\delta=v_\delta(n_\infty)$. From Lemma 6.3, we have $\int_{v_\delta(n_j)}^{v_\delta(n_{j+1})}X_s^{\xi_\delta}\,ds=O(d^2)$ for $0\le t\le T_{\alpha_0}^\delta$. Thus

$$\mathbf{E}[(\eta_\delta(v_\delta(n_{j+1}))-\eta_\delta(v_\delta(n_j)))|\mathcal{F}'_j]=O(d^3);$$
$$\mathbf{E}[(\eta_\delta(v_\delta(n_{j+1}))-\eta_\delta(v_\delta(n_j)))^2-2(v_\delta(n_{j+1})-v_\delta(n_j))|\mathcal{F}'_j]=O(d^3).$$

The following theorem can be deduced by using the Skorokhod embedding theorem. It is very similar to Theorem 3.7 in [10], so we omit the proof.

THEOREM 6.1. *For every $\varepsilon > 0$, there is a uniform constant $\delta_0 > 0$ depending on ε such that if $\delta < \delta_0$, then there is a coupling of the processes $\eta_\delta(t)$ and a Brownian motion $B(t)$ such that*

$$\mathbf{P}[\sup\{|\eta_\delta(t) - \sqrt{2}B(t)| : t \in [0, T^\delta_{\alpha_0}]\} < \varepsilon] > 1 - \varepsilon.$$

Note that for $t \in [0, T^\delta_{\alpha_0}]$, $\xi_\delta(t)$ solves the equation

$$\xi_\delta(t) = \eta_\delta(t) + 2\int_0^t X_s^{\xi_\delta}\, ds. \tag{6.3}$$

Suppose $B(t)$ is a Brownian motion, and $\xi_0(t)$, $0 \le t < T_0$, is the maximal solution to

$$\xi_0(t) = \sqrt{2}B(t) + 2\int_0^t X_s^{\xi_0}\, ds. \tag{6.4}$$

Then there is a.s. a simple curve β_0 such that $\beta_0(0) = 0$, $\beta_0(t) \in \mathbb{H}$ for $0 < t < T_0$, and $K_t^{\xi_0} = \beta_0((0,t])$ for $0 \le t < T$, and there is a continuous increasing function u_0 such that $\gamma_0(t) := f^{-1}(\beta_0(u_0^{-1}(t)))$, $0 \le t < S_0 = u_0(T_0)$, is an $\mathrm{LERW}(D; 0_+ \to z_e)$ trace.

If α is a crosscut in $\mathbb{H}$, and β defined on $[0, T)$ is a curve in $\overline{\mathbb{H}}$, let $T_\alpha(\beta)$ be the first t such that $\beta(t) \in \alpha$, if such t exists; otherwise let $T_\alpha(\beta) = T$. Since $\beta_\delta([0, T^\delta_{\alpha_0}])$ intersects α_0, so $T_{\alpha_0}(\beta_\delta) \le T^\delta_{\alpha_0}$.

THEOREM 6.2. *Suppose α is a crosscut in $\mathbb{H}$ that strictly encloses 0, and $H(\alpha) \subset \Omega \setminus \{p\}$. If $\infty \in D$, we also assume that $f(\infty) \notin H(\alpha)$. For every $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there is a coupling of the processes $\xi_\delta(t)$ and $\xi_0(t)$ such that*

$$\mathbf{P}[\sup\{|\xi_\delta(t) - \xi_0(t)| : t \in [0, T_\alpha(\beta_\delta) \vee T_\alpha(\beta_0)]\} < \varepsilon] > 1 - \varepsilon. \tag{6.5}$$

If ξ_δ or ξ_0 is not defined on $[0, T_\alpha(\beta_\delta) \vee T_\alpha(\beta_0)]$, we set the value of sup *to be $+\infty$.*

PROOF. Let ρ_j and $\alpha_j = f(\rho_j)$ be as in the beginning of this subsection such that α is strictly enclosed by α_0. From Lemma 5.5, there is $\delta_1 > 0$ such that if $K_a^\zeta \subset H(\alpha)$ and $\|\zeta - \eta\|_a < \delta_1$, then K_a^η is strictly enclosed by α_0. Since $K^{\xi_\delta}_{T^\delta_{\alpha_0}}$ intersects α_0, so if ξ_δ and ξ_0 are coupled, then on the event that $|\xi_\delta(t) - \xi_0(t)| < \delta_1$ for $0 \le t \le T^\delta_{\alpha_0}$, we have $\beta_0((0, T^\delta_{\alpha_0}]) = K^{\xi_0}_{T^\delta_{\alpha_0}} \not\subset H(\alpha)$, which implies that $T_\alpha(\beta_\delta) \vee T_\alpha(\beta_0) \le T^\delta_{\alpha_0}$. We may assume $\varepsilon < \delta_1$. Then we suffice to prove this theorem with (6.5) replaced by

$$\mathbf{P}[\sup\{|\xi_\delta(t) - \xi_0(t)| : t \in [0, T^\delta_{\alpha_0}]\} < \varepsilon] > 1 - \varepsilon. \tag{6.6}$$

Since $K^{\xi_\delta}_{T^\delta_{\alpha_0}} \subset H(\alpha_1)$, so from Lemmas 5.5 and 5.8, there are $\delta_2, C_1 > 0$ such that for any $t \in [0, T^\delta_{\alpha_0}]$, if $\|\xi_0 - \xi_\delta\|_t < \delta_2$, then $K^{\xi_0}_t \subset H(\alpha_2)$, and

$$|X^{\xi_0}_t - X^{\xi_\delta}_t| \le C_1 \|\xi_0 - \xi_\delta\|_t. \tag{6.7}$$

Let $C_2 = e^{C_1 h_1}/(2C_1)$, where $h_1 = \mathrm{hcap}(H(\alpha_1))$. From Theorem 6.1, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there is a coupling of η_δ with $\sqrt{2}B$ such that the probability that $|\eta_\delta(t) - \sqrt{2}B(t)| < (\varepsilon \wedge \delta_2)/C_2$ for $t \in [0, T^\delta_{\alpha_0}]$ is greater than $1 - \varepsilon$. Let $\mathcal{E}^\delta$ denote this event. Assume $\mathcal{E}^\delta$ occurs.

Now $\xi_0(0) = 0 = \xi_\delta(0)$. Let $[0, b)$ be maximal subinterval of $[0, T^\delta_{\alpha_0}) \cap [0, T_0)$, on which $|\xi_0(t) - \xi_\delta(t)| < \varepsilon \wedge \delta_2$. Then from (6.3), (6.4) and (6.7), we have

$$\|\xi_0 - \xi_\delta\|_t \le \|\eta_\delta - \sqrt{2}B\|_{T^\delta_{\alpha_0}} + 2C_1 \int_0^t \|\xi_0 - \xi_\delta\|_s \, ds,$$

for any $t \in [0, b]$. Solving this inequality, since $b \le T^\delta_{\alpha_0} \le h_1/2$ and $\mathcal{E}^\delta$ occurs, so

$$\|\xi_0 - \xi_\delta\|_b \le (e^{2C_1 b} - 1)/(2C_1) \|\eta_\delta - \sqrt{2}B\|_{T^\delta_{\alpha_0}} \le C_2 \|\eta_\delta - \sqrt{2}B\|_{T^\delta_{\alpha_0}} < \varepsilon \wedge \delta_2.$$

Thus $K^{\xi_0}_t \subset H(\alpha_2)$ for $0 \le t < b$. From Theorem 3.1(ii), we have $b < T_0$. Since $\|\xi_0 - \xi_\delta\|_b < \varepsilon \wedge \delta_2$, so $b = T^\delta_{\alpha_0}$. Thus $\xi_0(t)$ is defined on $[0, T^\delta_{\alpha_0}]$, and $|\xi_\delta(t) - \xi_0(t)| < \varepsilon$ for $t \in [0, T^\delta_{\alpha_0}]$ if $\mathcal{E}^\delta$ occurs. So we have (6.6). $\square$

7. Convergence of the curves.

7.1. *Local convergence.* We use the notation in Section 4.2. First we introduce a well-known lemma about random walks on $\delta\mathbb{Z}^2$.

LEMMA 7.1. *Suppose $w \in \delta\mathbb{Z}^2$ and $K \subset \mathbb{C}$ is a connected set that satisfies* $\mathrm{diam}(K) \ge R$ [*resp.* $\mathrm{diam}^\#(K) \ge R$]. *Then the probability that a random walk on $\delta\mathbb{Z}^2$ started from w will exit $\mathbf{B}(w; R)$ [resp. $\mathbf{B}^\#(w; R)$] before using an edge of $\delta\mathbb{Z}^2$ that intersects K is at most $C_0((\delta + \mathrm{dist}(w, K))/R)^{C_1}$ [resp. $C_0((\delta + \mathrm{dist}^\#(w, K))/R)^{C_1}$] for some absolute constants $C_0, C_1 > 0$.*

For $w \in V(D^\delta)$, let X_w be a random walk on D^δ started from w, stopped when it hits $V_\partial(D^\delta) \cup \{w^\delta_e\}$. Let Y_w be X_w conditioned to hit w^δ_e. Then $q_\delta = \mathrm{LE}(Y_\delta)$. Lemma 7.1 will be applied because if $w \in D$, X_w is not different from a random walk on $\delta\mathbb{Z}^2$ started from w stopped when it uses an edge that intersects ∂D or hits w^δ_e.

DEFINITION 7.1. Let $z \in \mathbb{C}$, $r, \varepsilon > 0$. A (z, r, ε)-quasi-loop in a path ω is a pair $a, b \in \omega$ such that $a, b \in \mathbf{B}(z; r)$, $|a - b| \le \varepsilon$, and the subarc of ω with endpoints a and b is not contained in $\mathbf{B}(z; 2r)$. Let $\mathcal{L}_\delta(z, r, \varepsilon)$ denote the event that q_δ has a (z, r, ε)-quasi-loop.

LEMMA 7.2. *Suppose $r>0$ and $\mathbf{B}(z_0;5r)\subset D$. Then $\mathbf{P}[\mathcal{L}_\delta(z_0,r,\varepsilon)]\to 0$, as $\varepsilon\to 0$, uniformly in δ.*

PROOF. We will use the idea in the proof of Lemma 3.4 in [16]. However, that proof does not apply here immediately, because we are dealing with the loop-erasure of a *conditional* random walk, and Wilson's algorithm does not apply to a *conditional* UST.

We will argue on the reversal path. Let X^r_w be a random walk on D^δ started from w, stopped when it hits ∂D. Let Y^r_w be X^r_w conditioned to hit the boundary vertex $\langle\delta,0\rangle$. Let $q^r_\delta=\mathrm{LE}(Y^r_{w^\delta_e})$. Then q^r_δ has the same distribution as the reversal of q_δ. Let $\mathcal{L}^r_\delta(z_0,r,\varepsilon)$ denote the event that q^r_δ has a (z_0,r,ε)-quasi-loop. Then $\mathbf{P}[\mathcal{L}^r_\delta(z_0,r,\varepsilon)]=\mathbf{P}[\mathcal{L}_\delta(z_0,r,\varepsilon)]$. It suffices to show that $\lim_{\varepsilon\to 0}\mathbf{P}[\mathcal{L}^r_\delta(z_0,r,\varepsilon)]=0$, uniformly in $\delta\in(0,\delta_1]$ for some absolute constant $\delta_1>0$ because if $\delta>\delta_1$, then $\mathcal{L}^r_\delta(z_0,r,\varepsilon)$ does not happen when $\varepsilon<\delta_1$.

Let $\mathbf{B}_k=\mathbf{B}(z_0;kr)$, $k=1,2,3,4,5$. Let $t_0=0$ and $j=0$. If t_j is defined, then define s_{j+1} to be the first time $s>t_j$ such that $Y^r_{w^\delta_e}(s)\in\mathbf{B}_1$, if such s exists; otherwise, let $M=j$ and stop here. If s_{j+1} is defined, then define t_{j+1} to be the first time $t>s_{j+1}$ such that $Y^r_{w^\delta_e}(t)\notin\mathbf{B}_2$. Let $j=j+1$ and iterate the definition. Then we get a sequence $s_1<t_1<\cdots<s_M<t_M$. Such M is a random number. Finally, for each $s\ge 0$, let $(Y^r_{w^\delta_e})^s$ be the subpath of $Y^r_{w^\delta_e}$ up to time s.

For $j\in\mathbb{N}$, let $\mathcal{Y}_j$ be the event that $j\le M$ and $\mathrm{LE}((Y^r_{w^\delta_e})^{t_j})$ has a (z_0,r,ε)-quasi-loop. Then $\mathcal{Y}_1$ is empty, and it is clear that for any $m\in\mathbb{N}$,

$$\mathcal{L}^r_\delta(z_0,r,\varepsilon)\subset\bigcup_{j=1}^{\infty}\mathcal{Y}_j\subset\{M\ge m+1\}\cup\bigcup_{j=1}^{m}\mathcal{Y}_j. \tag{7.1}$$

We first estimate $\mathbf{P}[M\ge j+1|(Y^r_{w^\delta_e})^{t_j}]$. For $w\in V(D^\delta)$, let $Q(w)$ or $Q^\delta(w)$ be the probability that X^r_w leaves D through $[\delta,0]$; let $Q_1(w)$ or $Q^\delta_1(w)$ be the probability that X^r_w avoids $\mathbf{B}_1$ and leaves D through $[\delta,0]$. Then the probability that Y^r_w does not hit $\mathbf{B}_1$ is equal to $Q_1(w)/Q(w)$. From the Markov property of Y, we have

$$\mathbf{P}[M\ge j+1|(Y^r_{w^\delta_e})^{t_j}]=1-Q_1(Y^r_{w^\delta_e}(t_j))/Q(Y^r_{w^\delta_e}(t_j)).$$

Let $F=\{2r\le|z-z_0|\le 3r\}$. Then F is a compact subset of $D\setminus\overline{\mathbf{B}_1}$, and if $\delta<r$, then $Y^r_{w^\delta_e}(t_j)\in F$. We claim that there are absolute constants $\delta_0\in(0,r)$ and $C_2>0$ such that $Q_1(w)/Q(w)\ge C_2$ for any $w\in V(D^\delta)\cap F$, if $\delta<\delta_0$. If the claim is not true, then we can find $\delta_n\to 0$, $w_n\in V(D^{\delta_n})\cap F$, and $w_n\to w_0\in F$, such that $Q^{\delta_n}_1(w_n)/Q^{\delta_n}(w_n)\to 0$. Let $I^{\delta_n}=Q^{\delta_n}(\cdot)/Q^{\delta_n}(w_n)$ and $J^{\delta_n}=(Q^{\delta_n}(\cdot)-Q^{\delta_n}_1(\cdot))/Q^{\delta_n}(w_n)$. Let P be the generalized Poisson kernels in D with the pole at 0_+, normalized by $P(w_0)=1$. Then I^{δ_n} converges to P locally uniformly in D. Since J^{δ_n} vanishes on the boundary vertices of D^δ including 0, agrees with I^{δ_n} on the vertices in $\mathbf{B}_1$, and is discrete harmonic in $D\setminus\overline{\mathbf{B}_1}$, so J_n^δ converges to a continuous function H locally uniformly in $\overline{D}\setminus\mathbf{B}_1$, where H vanishes on ∂D, agrees with P on $\partial\mathbf{B}_1$, and is harmonic in $D\setminus\overline{\mathbf{B}_1}$. Then $H\le P$ in $D\setminus\overline{\mathbf{B}_1}$. From $J^{\delta_n}(w_n)\to 1$, we have $H(w_0)=1=P(w_0)$. From the maximum principle of harmonic functions, we have $P(w)-H(w)=0$ for any $w\in D\setminus\overline{\mathbf{B}_1}$, which is impossible. So the claim is justified. Suppose $\delta<\delta_0$. Then $\mathbf{P}[M\ge j+1|(Y^r_{w_e^\delta})^{t_j}]\le 1-C_2$. By induction, we find that

$$\mathbf{P}[M\ge m+1]\le(1-C_2)^m. \tag{7.2}$$

We now estimate $\mathbf{P}[\mathcal{Y}_{j+1}|\neg\mathcal{Y}_j,(Y^r_{w_e^\delta})^{t_j}]$. Let $\mathcal{Q}_j$ be the set of components of intersection of $\mathbf{B}_2$ with $\mathrm{LE}((Y^r_{w_e^\delta})^{s_{j+1}})$ that do not contain $Y^r_{w_e^\delta}(s_{j+1})$. Observe that if $\mathcal{Y}_j$ does not occur, then for $\mathcal{Y}_{j+1}$ to occur, there must be a $K\in\mathcal{Q}_j$ such that $Y^r_{w_e^\delta}$ comes at some time $t\in[s_{j+1},t_{j+1}]$ within distance ε of $K\cap\mathbf{B}_1$ but $Y^r_{w_e^\delta}(t)\notin K$ for all $t\in[s_{j+1},t_{j+1}]$. But if $Y^r_{w_e^\delta}(t)$ is close to K for some $t\in[s_{j+1},t_{j+1}]$, then Lemma 7.1 can be applied, to estimate the probability that $Y^r_{w_e^\delta}(t)$ will not hit K before time t_{j+1}.

Suppose $\delta<\delta_1:=\delta_0\wedge\mathrm{dist}(0,\mathbf{B}_5)$; then $\delta\notin\mathbf{B}_5$, so Q is discrete harmonic inside $\mathbf{B}_5$, and $Q(w)>0$ for any $w\in V(D^\delta)\cap\mathbf{B}_5$. Applying Harnack's inequality to Q, we get an absolute constant $C_1\ge 1$ such that $Q(w_1)\le C_1Q(w_2)$ for any $w_1,w_2\in V(D^\delta)\cap\mathbf{B}_4$. Let T_3 be the first time that a path leaves $\mathbf{B}_3$ or hits K. Then for any $w\in V(D^\delta)\cap\mathbf{B}_3$, $X^r_w(t)$ and $Y^r_w(t)$, $t=0,1,\dots,T_3$, are contained in $\mathbf{B}_4$ because $\delta<\delta_0<r$. Note that for any path $(w_0,w_1,\dots,w_n)$ on D^δ that is contained in $\mathbf{B}_4$,

$$\mathbf{P}[Y^r_{w_0}(j)=w_j,1\le j\le n]\,/\,\mathbf{P}[X^r_{w_0}(j)=w_j,1\le j\le n]=Q(w_n)/Q(w_0)\le C_1.$$

Therefore, conditioned on $Y^r_{w_e^\delta}(s_{j+1})$, for each given $K\in\mathcal{Q}_j$, the probability that $Y^r_{w_e^\delta}([s_{j+1},t_{j+1}])$ gets to within distance ε of K but does not hit K is at most $C_3((\delta+\varepsilon)/r)^{C_4}$ for some absolute constant $C_3,C_4>0$. Note that if $\delta>\varepsilon$, then the above event cannot happen, so the probability is at most $C_3(2\varepsilon/r)^{C_4}$. Observe that $|\mathcal{Q}_j|$, the cardinality of $\mathcal{Q}_j$, is at most j. Let $C_5=C_3(2/r)^{C_4}$. Then

$$\mathbf{P}[\mathcal{Y}_{j+1}|\neg\mathcal{Y}_j]\le jC_5\varepsilon^{C_4}.$$

This gives

$$\mathbf{P}\left[\bigcup_{j=1}^m\mathcal{Y}_j\right]=\sum_{j=1}^{m-1}\mathbf{P}[\mathcal{Y}_{j+1}\cap\neg\mathcal{Y}_j]\le\sum_{j=1}^{m-1}\mathbf{P}[\mathcal{Y}_{j+1}|\neg\mathcal{Y}_j]$$
$$\le\sum_{j=1}^{m-1}jC_5\varepsilon^{C_4}\le m^2C_5\varepsilon^{C_4}.$$

Combining this with (7.1) and (7.2), we find that

$$\mathbf{P}[\mathcal{L}^r_\delta(z_0,r,\varepsilon)] \le (1-C_2)^m + m^2 C_5 \varepsilon^{C_4}.$$

Since $C_2>0$, the lemma follows by taking $m=\lfloor \varepsilon^{-C_4/3}\rfloor$, say. □

DEFINITION 7.2. Let $F\subset\mathbb{C}$, and $r,\varepsilon>0$. An (F,r,ε)-quasi-loop in a path ω is a pair $a,b\in\omega$ such that $a\in F$, $|a-b|<\varepsilon$, and the subarc of ω with endpoints a and b is not contained in $\mathbf{B}(a;r)$.

COROLLARY 7.1. *Suppose F is a compact subset of $D\setminus\{\infty\}$, and $r>0$. Then the probability that q_δ contains an (F,r,ε)-quasi-loop tends to 0 as $\varepsilon\to 0$, uniformly in δ.*

PROOF. Let $\mathcal{L}_\delta(F,r,\varepsilon)$ denote this event. We may find $r_0\in(0,r/3)$ and finitely many points $z_1,\dots,z_n\in F$, such that $\mathbf{B}(z_j;5r_0)\subset D$ for each $j\in\mathbb{Z}_{[1,n]}$, and $F\subset\bigcup_{j=1}^n\mathbf{B}(z_j;r_0/2)$. It is easy to check that if $\varepsilon<r_0/2$, then $\mathcal{L}_\delta(F,r,\varepsilon)\subset\bigcup_{j=1}^n\mathcal{L}_\delta(z_j,r_0,\varepsilon)$. The conclusion follows from Lemma 7.2. □

COROLLARY 7.2. *Suppose F is a compact subset of $\Omega\setminus\{f(\infty)\}$, and $r>0$. Then the probability that β_δ contains an (F,r,ε)-quasi-loop tends to 0 as $\varepsilon\to 0$, uniformly in δ.*

PROOF. This follows from the last corollary, and the facts that f maps D conformally onto Ω, f (resp. f^{-1}) is uniformly continuous on each compact subset of $D\setminus\{\infty\}$ (resp. $\Omega\setminus\{f(\infty)\}$), and that β_δ is a time-change of $f\circ q_\delta$. □

For a domain E and $\varepsilon>0$, let $\partial^\#_\varepsilon E:=\{z\in E:\operatorname{dist}^\#(z,\widehat{\mathbb{C}}\setminus E)<\varepsilon\}$. For any $\varepsilon>0$ there are $\varepsilon_1,\varepsilon_2>0$ such that $f(\partial^\#_{\varepsilon_1}D)\subset\partial^\#_\varepsilon\Omega$ and $f^{-1}(\partial^\#_{\varepsilon_2}\Omega)\subset\partial^\#_\varepsilon D$. In the following lemmas, let F_D (resp. F_Ω) be a compact subset of $D\setminus\{z_e,\infty\}$ [resp. $\Omega\setminus\{p,f(\infty)\}$].

LEMMA 7.3. *The probability that Y_δ or q_δ visits $\partial^\#_\varepsilon D$ after visiting F_D tends to 0 as $\varepsilon,\delta\to 0$.*

PROOF. Since q_δ is the loop-erasure of Y_δ, so we only need to consider Y_δ. By the Markov property of Y, we need to prove that the probability that Y_w visits $\partial^\#_\varepsilon D$ tends to 0 as $\varepsilon,\delta\to 0$, uniformly in $w\in F_D$. For $w\in V(D^\delta)$, let $Q(w)$ be the probability that X_w visits w_e^δ. Let $P_\varepsilon(w)$ be the probability that Y_w hits $\partial^\#_\varepsilon D$. Then $Q(w)P_\varepsilon(w)$ equals the probability that X_w first hits $\partial^\#_\varepsilon D$ and then w_e^δ, which is not bigger than $\sup\{Q(w):w\in\partial^\#_\varepsilon D\}$.

Choose $z_0\in F_D$. Let w_0^δ be the vertex of D^δ closest to z_0. As $\delta\to 0$, $Q(\cdot)/Q(w_0^\delta)$ converges to $G(D,z_e;\cdot)/G(D,z_e;z_0)$ uniformly on any subset of $\overline{D}$ bounded away from z_e. Thus $\sup\{Q(w):w\in\partial^\#_\varepsilon(D)\}/\inf\{Q(w):w\in F_D\}\to 0$ as $\delta,\varepsilon\to 0$. So $P_\varepsilon(w)\to 0$ as $\varepsilon,\delta\to 0$, uniform on $w\in F_D$. □

COROLLARY 7.3. *The probability that β_δ visits $\partial^{\#}_{\varepsilon}\Omega$ after F_Ω tends to* 0 *as* $\varepsilon,\delta\to 0$.

LEMMA 7.4. *For any $\varepsilon>0$, there are $M,\delta_0>0$ such that if $\delta<\delta_0$, then with probability greater than $1-\varepsilon$, q_δ stays in $\mathbf{B}(0;M)$ after visiting F_D.*

PROOF. This follows from Lemma 7.1 and the idea in the proof of Lemma 7.3. □

LEMMA 7.5. *Let $T^\delta_{F_\Omega}$ be the first time that β_δ visits F_Ω. For any $\varepsilon>0$, there are $\varepsilon_0,\delta_0>0$ such that for $\delta<\delta_0$, with probability greater than $1-\varepsilon$, β_δ satisfies that if $|\beta_\delta(t_1)-\beta_\delta(t_2)|<\varepsilon_0$ for some $t_1,t_2\ge T^\delta_{F_\Omega}$, then $\operatorname{diam}(\beta_\delta([t_1,t_2]))<\varepsilon$.*

PROOF. From Lemma 7.4, there are $M,\delta_1>0$ such that if $\delta<\delta_1$, then with probability greater than $1-\varepsilon/3$, q_δ stays in $\mathbf{B}(0;M)$ after visiting $f^{-1}(F_\Omega)$, so β_δ stays in $f(D\cap\mathbf{B}(0;M))$ after $T^\delta_{F_\Omega}$. Let $\mathcal{E}^\delta_1$ denote this event. From Corollary 7.3, there are $\delta_2,\varepsilon_1>0$ such that if $\delta<\delta_2$, then with probability greater than $1-\varepsilon/3$, $\beta_\delta(t)\in F:=\Omega\setminus\partial^{\#}_{\varepsilon_1}\Omega$ for $t\ge a$. Let $\mathcal{E}^\delta_2$ denote this event. Let $F_0=F\setminus f(D\cap\{|z|>M\})$. Then F_0 is a compact subset of $\Omega\setminus\{f(\infty)\}$, so from Corollary 7.2, there is $\varepsilon_0>0$ such that with probability greater than $1-\varepsilon/3$, β_δ does not contain an $(F_0,\varepsilon/3,\varepsilon_0)$-quasi-loop. Let $\mathcal{E}^\delta_3$ denote this event. Let $\delta_0=\delta_1\wedge\delta_2$ and $\mathcal{E}^\delta=\mathcal{E}^\delta_1\cap\mathcal{E}^\delta_2\cap\mathcal{E}^\delta_3$. Suppose $\delta<\delta_0$. Then $\mathbf{P}[\mathcal{E}^\delta]>1-\varepsilon$. Assume $\mathcal{E}^\delta$ occurs. Suppose $t_1,t_2\ge T^\delta_{F_\Omega}$ and $|\beta_\delta(t_1)-\beta_\delta(t_2)|<\varepsilon_0$. Since $\mathcal{E}^\delta_1$ and $\mathcal{E}^\delta_2$ occur, so $\beta_\delta(t_1)\in F_0$. Since $\mathcal{E}^\delta_3$ occurs, so β_δ does not contain an $(F_0,\varepsilon/3,\varepsilon_0)$-quasi-loop. Thus $\beta_\delta([t_1,t_2])\subset\mathbf{B}(\beta_\delta(t_1);\varepsilon/3)$, whose diameter is less than ε. □

THEOREM 7.1. *Let α be a crosscut in $\mathbb{H}$ that strictly encloses 0, such that $H(\alpha)\subset\Omega\setminus\{p,f(\infty)\}$. For every $\varepsilon>0$, there is $\delta_0>0$ depending on α and ε, such that if $\delta<\delta_0$, then there is a coupling of the processes $\beta_\delta(t)$ and $\beta_0(t)$ such that*

$$\mathbf{P}[\sup\{|\beta_\delta(t)-\beta_0(t)|:t\in[0,T_\alpha(\beta_\delta)\vee T_\alpha(\beta_0)]\}<\varepsilon]>1-\varepsilon. \tag{7.3}$$

PROOF. Let α_0 be a crosscut in $\mathbb{H}$ that strictly encloses α such that $H(\alpha_0)\subset\Omega\setminus\{p,f(\infty)\}$. Let $d_0=\operatorname{dist}(\alpha,\alpha_0)>0$. Since $\beta_0((0,T_{\alpha_0}(\beta_0)])$ intersects α_0, so if β_δ and β_0 are coupled, then on the event that $|\beta_\delta(t)-\beta_0(t)|<d_0$ for $0\le t\le T_{\alpha_0}(\beta_0)$, we have $\beta_\delta((0,T_{\alpha_0}(\beta_0)])\notin H(\alpha)$, which implies that $T_\alpha(\beta_\delta)\vee T_\alpha(\beta_0)\le T_{\alpha_0}(\beta_0)$. We may assume $\varepsilon<d_0$. Then we suffice to prove this theorem with (7.3) replaced by

$$\mathbf{P}[\sup\{|\beta_\delta(t)-\beta_0(t)|:t\in[0,T_{\alpha_0}(\beta_0)]\}<\varepsilon]>1-\varepsilon. \tag{7.4}$$

Choose a crosscut α_1 in $\mathbb{H}$ that strictly encloses α_0, such that $H(\alpha_1)\subset\Omega\setminus\{p,f(\infty)\}$. Suppose the theorem is not true; then there exist $\varepsilon_0>0$ and a sequence $\delta_n\to 0$, such that for each δ_n, there is no coupling of β_{δ_n} with β_0 such that (7.4) holds with $\delta=\delta_n$. From Theorem 6.2, and by passing to a subsequence, we may assume that for each n, there is a coupling of ξ_{δ_n} and ξ_0 such that

$$\mathbf{P}[\sup\{|\xi_{\delta_n}(t)-\xi_0(t)|:t\in[0,T_{\alpha_1}(\beta_0)]\}\ge 1/2^n]<1/2^n. \tag{7.5}$$

We may assume that all ξ_{δ_n} and ξ_0 are defined in the same probability space, and (7.5) is satisfied. By discarding a null event, we have

$$\|\xi_{\delta_n}-\xi_0(t)\|_{T_{\alpha_1}(\beta_0)}\to 0. \tag{7.6}$$

Fix any $t\in[0,T_{\alpha_1}(\beta_0)]$. Suppose F is any compact subset of $\mathbb{H}\setminus\beta_0((0,t])$. From $\|\xi_{\delta_n}-\xi_0(t)\|_t\to 0$, we see that $\varphi_t^{\xi_{\delta_n}}\to\varphi_t^{\xi_0}$ uniformly on F, and $F\subset\mathbb{H}\setminus\beta_{\delta_n}((0,t])$ for all but finitely many n. Thus $(\mathbb{H}\setminus\beta_{\delta_n}((0,t]))\cap(\mathbb{H}\setminus\beta_0((0,t]))\xrightarrow{\text{Cara}}\mathbb{H}\setminus\beta_0((0,t])$. From Lemma 5.1, $(\varphi_t^{\xi_{\delta_n}})^{-1}\xrightarrow{\text{l.u.}}(\varphi_t^{\xi_0})^{-1}$ in $\mathbb{H}=\varphi_t^{\xi_0}(\mathbb{H}\setminus\beta_0((0,t]))$. Thus we have $\mathbb{H}\setminus\beta_{\delta_n}((0,t])\xrightarrow{\text{Cara}}\mathbb{H}\setminus\beta_0((0,t])$ for any $t\in[0,T_{\alpha_1}(\beta_0)]$.

We may assume that $\mathbf{B}(0;\varepsilon_0)\cap\mathbb{H}\subset H(\alpha_0)$. Since β_0 is a continuous curve started from 0, so there is $b>0$ such that with probability greater than $1-\varepsilon_0/5$, β_0 is defined on $[0,b]$, and $\beta_0([0,b])\subset\mathbf{B}(0;\varepsilon_0/4)$. Let $\mathcal{E}_1^0$ denote this event. If $\mathcal{E}_1^0$ occurs, then $b<T_{\alpha_1}(\beta_0)$. For each $n\in\mathbb{N}$, let $\mathcal{E}_1^n$ denote the event that β_n is defined on $[0,b]$ and $\beta_n([0,b])\subset\mathbf{B}(0;\varepsilon_0/3)$. From (7.6) and Lemma 5.5, we have $\mathcal{E}_1^0\subset\liminf\mathcal{E}_1^n$. So there is $N_1\in\mathbb{N}$ such that $\mathbf{P}[\mathcal{E}_1^n]>1-\varepsilon_0/5$ if $n>N_1$.

Let $a=b/2$. If $\mathcal{E}_1^0$ occurs, then $\beta_0((0,a])\subset H(\alpha_1)\subset\Omega\setminus\{p,f(\infty)\}$. So there is a nonempty compact subset F_1 of $\Omega\setminus\{p,f(\infty)\}$ such that $\mathbf{P}[\mathcal{E}_2^0]>1-\varepsilon_0/5$, where $\mathcal{E}_2^0$ is the subevent of $\mathcal{E}_1^0$ on which $\beta_0((0,a])\cap F_1\ne\varnothing$. Choose another compact subset F_2 of $\Omega\setminus\{p,f(\infty)\}$ such that F_1 is contained in the interior of F_2. Let $\mathcal{E}_2^n$ denote the event that β_{δ_n} is defined on $[0,a]$, and $\beta_{\delta_n}((0,a])\cap F_2\ne\varnothing$. If $\mathcal{E}_2^0$ occurs, then $a\le T_{\alpha_1}(\beta_0)$, so $\mathbb{H}\setminus\beta_{\delta_n}((0,a])\xrightarrow{\text{Cara}}\mathbb{H}\setminus\beta_0((0,a])$, and so $\operatorname{dist}(z_0,\beta_{\delta_n}((0,a]))\to 0$ for any $z_0\in\beta_0((0,a])$. Thus $\mathcal{E}_2^0\subset\liminf\mathcal{E}_2^n$. So there is $N_2\in\mathbb{N}$ such that $\mathbf{P}[\mathcal{E}_2^n]>1-\varepsilon_0/5$ if $n>N_2$. Note that if $\mathcal{E}_2^n$ occurs, then $a\ge T_{F_2}^{\delta_n}$, where $T_{F_2}^{\delta_n}$ is the first time that β_{δ_n} visits F_2.

From Theorem 6.2 and Lemma 7.5, there are $\varepsilon_1\in(0,\varepsilon_0)$ and $N_3\in\mathbb{N}$ such that if $n\ge N_3$, then with probability at least $1-\varepsilon_0/5$, ξ_{δ_n} is defined on $[0,T_{\alpha_1}(\beta_0)]$, and if $|\beta_{\delta_n}(t_2)-\beta_{\delta_n}(t_1)|<\varepsilon_1$ for some $t_1,t_2\ge T_{F_2}^{\delta_n}$, then $\operatorname{diam}(\beta_{\delta_n}([t_1,t_2]))<\varepsilon_0/3$. Let $\mathcal{E}_3^n$ denote this event.

Since β_0 is continuous on $[a,T_{\alpha_1}(\beta_0)]$, $\operatorname{dist}(\beta_0([a,T_{\alpha_1}(\beta_0)]),\mathbb{R})>0$ and $T_{\alpha_0}(\beta_0)<T_{\alpha_1}(\beta_0)$, so there is $\Delta,h>0$ such that with probability at least $1-\varepsilon_0/5$, the followings hold: $T_{\alpha_1}(\beta_0)-T_{\alpha_0}(\beta_0)>\Delta$, $\operatorname{Im}\beta_0(t)\ge h$ for any

$t \in [a, T_{\alpha_1}(\beta_0)]$, and if $t_1, t_2 \in [a, T_{\alpha_1}(\beta_0)]$ and $|t_1 - t_2| \le \Delta$, then $|\beta_0(t_1) - \beta_0(t_2)| < \varepsilon_1/3$. Let $\mathcal{E}_4$ denote this event.

Let $A = \text{hcap}(H(\alpha_1))/2$. Then $T_{\alpha_1}(\beta_0) \le A$. Choose $N \in \mathbb{N}$ such that $A/N < (\Delta \wedge b)/2$, and define $t_k = a + (T_{\alpha_1}(\beta_0) - a)k/N$, $k = 0, 1, \dots, N$. Then $t_0 = a$, $t_N = T_{\alpha_1}(\beta_0)$ and $t_1 \le b$, $t_{N-1} \ge T_{\alpha_0}(\beta_0)$. Fix $k \in \mathbb{Z}_{[1,N]}$. Since $\beta_0(t_k) \in \mathbb{H} \setminus \beta_0((0, t_{k-1}])$ and $\mathbb{H} \setminus \beta_{\delta_n}((0, t_{k-1}]) \xrightarrow{\text{Cara}} \mathbb{H} \setminus \beta_0((0, t_{k-1}])$, so there is $M_k^1 \in \mathbb{N}$ such that $\beta_0(t_k) \notin \beta_{\delta_n}((0, t_{k-1}])$ when $n > M_k^1$. Since $\beta_0(t_k)$ is a boundary point of $\mathbb{H} \setminus \beta_0((0, t_k])$ and $\mathbb{H} \setminus \beta_{\delta_n}((0, t_k]) \xrightarrow{\text{Cara}} \mathbb{H} \setminus \beta_0((0, t_k])$, so there is $M_k^2 \in \mathbb{N}$ such that when $n > M_k^2$, there is $z_n \in \partial(\mathbb{H} \setminus \beta_{\delta_n}((0, t_k]))$ with $|z_n - \beta_0(t_k)| < (\varepsilon_1/3) \wedge h$. If event $\mathcal{E}_4$ occurs, and $n > M_k^1 \vee M_k^2$, then $z_n \notin \mathbb{R}$ and $z_n \notin \beta((0, t_{k-1}])$, which implies that $z_n = \beta_{\delta_n}(s_k)$ for some $s_k \in (t_{k-1}, t_k]$. Thus if $\mathcal{E}_4$ occurs and $n > M := \bigvee_{k=1}^N (M_k^1 \vee M_k^2)$, then we have $s_k \in (t_{k-1}, t_k]$, $k = 1, 2, \dots, N$, such that $|\beta_{\delta_n}(s_k) - \beta_0(t_k)| < \varepsilon_1/3$.

Let $L = \bigvee_{j=1}^3 N_j \vee M$ and $\mathcal{E}^n = \bigcap_{j=1}^3 \mathcal{E}_j^n \cap \mathcal{E}_1^0 \cap \mathcal{E}_4$. Suppose $n > L$. Then $\mathbf{P}[\mathcal{E}^n] > 1 - \varepsilon_0$. Assume $\mathcal{E}^n$ occurs. Fix $t \in [0, T_{\alpha_0}(\beta_0)]$. If $t \le b$, then $\beta_{\delta_n}(t)$, $\beta_0(t) \in \mathbf{B}(0; \varepsilon_0/3)$ because $\mathcal{E}_1^0$ and $\mathcal{E}_1^n$ both occur and $n > N_1$, so $|\beta_{\delta_n}(t) - \beta_0(t)| < \varepsilon_0$. Now suppose $t \ge b$. Then $t \in [b, T_{\alpha_0}(\beta_0)] \subset [t_1, t_{N-1}] \subset [s_1, s_N]$. Thus $t \in [s_k, s_{k+1}]$ for some $k \in \mathbb{Z}_{[1,N-1]}$. Since $n > M$, $t_k, t_{k+1} \in [a, T_{\alpha_1}(\beta_0)]$, $|t_k - t_{k+1}| < \Delta$, and $\mathcal{E}_4$ occurs, so

$$
\begin{aligned}
|\beta_{\delta_n}(s_k) - \beta_{\delta_n}(s_{k+1})| &\le |\beta_{\delta_n}(s_k) - \beta_0(t_k)| + |\beta_0(t_k) - \beta_0(t_{k+1})| \\
&\quad + |\beta_0(t_{k+1}) - \beta_{\delta_n}(s_{k+1})| \\
&< \varepsilon_1/3 + \varepsilon_1/3 + \varepsilon_1/3 = \varepsilon_1.
\end{aligned}
$$

Since $n > N_2$ and $\mathcal{E}_2^n$ occurs, so $s_k, s_{k+1} \ge a \ge T_{F_2}^{\delta_n}$. Since $n > N_3$, $\mathcal{E}_3^n$ occurs, and $t \in [s_k, s_{k+1}]$, so $|\beta_{\delta_n}(t) - \beta_{\delta_n}(s_k)| < \varepsilon_0/3$. Since $t \in [s_k, s_{k+1}] \subset [t_{k-1}, t_{k+1}]$, so $|t - t_k| < \Delta$. Since $\mathcal{E}_4$ occurs, so $|\beta_0(t) - \beta_0(t_k)| < \varepsilon_1/3$. Thus

$$
\begin{aligned}
|\beta_{\delta_n}(t) - \beta_0(t)| &\le |\beta_{\delta_n}(t) - \beta_{\delta_n}(s_k)| + |\beta_{\delta_n}(s_k) - \beta_0(t_k)| + |\beta_0(t_k) - \beta_0(t)| \\
&\le \varepsilon_0/3 + \varepsilon_1/3 + \varepsilon_1/3 < \varepsilon_0/3 + \varepsilon_0/3 + \varepsilon_0/3 = \varepsilon_0.
\end{aligned}
$$

Thus with probability greater than $1 - \varepsilon_0$, $|\beta_{\delta_n}(t) - \beta_0(t)| < \varepsilon_0$ for $0 \le t \le T_{\alpha_0}(\beta_0)$, which contradicts the choice of (δ_n). $\square$

7.2. *Global convergence.* We restrict β_δ to $[0, T_\delta)$. Then $\lim_{t \to T_\delta} \beta_\delta(t) = f(w_e^\delta)$. Recall that β_0 is defined on $[0, T_0)$, where $[0, T_0)$ is the maximal interval on which the solution to (6.4) exists. Let $\mathcal{B}$ denote the set of continuous curves $\beta : [0, T(\beta)) \to \Omega \cup \mathbb{R}$, for some $T(\beta) \in (0, \infty]$, with $\beta(0) = 0$ and $\beta(t) \in \Omega$ for $t \in (0, T(\beta))$. So T is a function taking values in $(0, \infty]$ on $\mathcal{B}$ that describes the length of lifetime. Then β_0 and β_δ are $\mathcal{B}$-valued random variables, and $T(\beta_\delta) = T_\delta$, $T(\beta_0) = T_0$.

Let $\mathcal{A}$ denote the set of crosscuts α in $\mathbb{H}$ that strictly enclose 0, and such that $H(\alpha) \subset \Omega \setminus \{p, f(\infty)\}$. For $\alpha_1, \alpha_2 \in A$, we write $\alpha_1 \prec \alpha_2$ or $\alpha_2 \succ \alpha_1$ if α_1

is strictly enclosed by α_2. For any $\beta\in\mathcal{B}$ and $\alpha\in\mathcal{A}$, let $T_\alpha(\beta)$ be the biggest $T\in(0,T(\beta)]$ such that $\beta(t)\notin\alpha$ for $0\le t<T$. It is clear that $T_{\alpha_1}\le T_{\alpha_2}$ if $\alpha_1\prec\alpha_2$. Define $T^+_\alpha=\bigwedge_{\alpha'\succ\alpha}T_{\alpha'}$. If β does not leave $H(\alpha)$ immediately after hitting α, then $T_\alpha(\beta)<T^+_\alpha(\beta)$.

Suppose $\alpha\in\mathcal{A}$. For $\beta_1,\beta_2\in\mathcal{B}$, let $\Delta(\beta_1,\beta)$ be 0 if $\beta_1=\beta_2$ and 1 otherwise, where $\beta_1=\beta_2$ means that $T(\beta_1)=T(\beta_2)$ and $\beta_1(t)=\beta_2(t)$ for $0\le t<T(\beta_1)$, and define

$$d^\vee_\alpha(\beta_1,\beta_2)=\Delta(\beta_1,\beta_2)\wedge\sup\{|\beta_1(t)-\beta_2(t)|:t\in[0,T_\alpha(\beta_1)\vee T_\alpha(\beta_2)]\},$$

where the value of the sup is set to be ∞ if either $\beta_1(t)$ or $\beta_2(t)$ is not defined at some t in the interval of the formula. Then $0\le d^\vee_\alpha\le 1$. Now define

$$d_\alpha(\beta_1,\beta_2)=\inf\left\{\sum_{k=1}^n d^\vee_\alpha(\gamma_{k-1},\gamma_k):\gamma_0=\beta_1,\gamma_n=\beta_2,\gamma_k\in\mathcal{B},k\in\mathbb{Z}_{[1,n-1]},n\in\mathbb{N}\right\}.$$

Then d_α is a pseudo-metric on $\mathcal{B}$, and $d_\alpha\le d^\vee_\alpha$. For $\alpha\in\mathcal{A}$, $\beta_1\in\mathcal{B}$ and $r>0$, let $\mathbf{B}_\alpha(\beta_1;r)=\{\beta\in\mathcal{B}:d_\alpha(\beta,\beta_1)<r\}$. Let $\mathcal{T}_\alpha$ denote the topology generated by d_α. It is clear that if $\alpha_1\prec\alpha_2$, then $d^\vee_{\alpha_1}\le d^\vee_{\alpha_2}$, so $d_{\alpha_1}\le d_{\alpha_2}$, from which follows that $\mathcal{T}_{\alpha_1}\subset\mathcal{T}_{\alpha_2}$. Let $\mathcal{T}^+_\alpha=\bigcap_{\alpha'\succ\alpha}\mathcal{T}_{\alpha'}$.

Lemma 7.6. *Suppose* $\alpha_1\prec\alpha_2\in\mathcal{A}$ *and* $d_0=1\wedge\operatorname{dist}(\alpha_1,\alpha_2)>0$. *Suppose* $\beta_1,\beta_2\in\mathcal{B}$, *and* $d_{\alpha_2}(\beta_1,\beta_2)<d_0$. *Then* $d^\vee_{\alpha_1}(\beta_1,\beta_2)\le d_{\alpha_2}(\beta_1,\beta_2)$.

Proof. Choose $d_1\in(d_{\alpha_2}(\beta_1,\beta_2),d_0)$. Then there are $\gamma_0,\gamma_1,\dots,\gamma_n\in\mathcal{B}$ such that $\gamma_0=\beta_1$, $\gamma_n=\beta_2$ and $\sum_{j=1}^n d^\vee_{\alpha_2}(\gamma_{j-1},\gamma_j)<d_1$. For each $j\in\mathbb{Z}_{[1,n]}$, since $d^\vee(\gamma_{j-1},\gamma_j)<d_1<1$, so

$$d^\vee_{\alpha_2}(\gamma_{j-1},\gamma_j)=\sup\{|\gamma_{j-1}(t)-\gamma_j(t)|:0\le t\le T_{\alpha_2}(\gamma_{j-1})\vee T_{\alpha_2}(\gamma_j)\}.$$

Let $t_0=T_{\alpha_1}(\beta_1)\vee T_{\alpha_1}(\beta_2)$. Assume, for example, that $t_0=T_{\alpha_1}(\beta_1)=T_{\alpha_1}(\gamma_0)$.

We claim that $t_0\le T_{\alpha_2}(\gamma_j)$ for any $0\le j\le n$. Since $t_0=T_{\alpha_1}(\gamma_0)<T_{\alpha_2}(\gamma_0)$, if the claim is not true, then there is $k\in\mathbb{Z}_{[1,n]}$ such that $t_0>T_{\alpha_2}(\gamma_k)$ and $t_0\le T_{\alpha_2}(\gamma_j)$ for $0\le j\le k-1$. Let $t_1=T_{\alpha_2}(\gamma_k)$. So $t_1\in[0,T_{\alpha_2}(\gamma_j)]$, $0\le j\le k$. Then we have

$$d_0>d_1>\sum_{j=1}^k d^\vee_{\alpha_2}(\gamma_{j-1},\gamma_j)\ge\sum_{j=1}^k 1\wedge|\gamma_{j-1}(t_1)-\gamma_j(t_1)|\ge 1\wedge|\gamma_0(t_1)-\gamma_k(t_1)|.$$

Since t_0 is the first t such that $\beta_1(t)\in\alpha_1$, and $t_1<t_0$, so $\gamma_0(t_1)=\beta_1(t_1)$ is enclosed by α_1. Since $\gamma_k(t_1)\in\alpha_2$ and $\alpha_1\prec\alpha_2$, so $|\gamma_0(t_1)-\gamma_k(t_1)|\ge\operatorname{dist}(\alpha_1,\alpha_2)$. This implies that $1\wedge|\gamma_0(t_1)-\gamma_k(t_1)|\ge d_0$, which is a contradiction. So the claim is justified.

Thus for any $t\in[0,t_0]$, we have $t\in[0,T_{\alpha_2}(\gamma_j)]$ for any $0\le j\le n$, so

$$|\beta_1(t)-\beta_2(t)|\le\sum_{j=1}^n|\gamma_{j-1}(t)-\gamma_j(t)|\le\sum_{j=1}^n d^{\vee}_{\alpha_2}(\gamma_{j-1},\gamma_j)<d_1.$$

Since this is true for any $t\in[0,t_0]=[0,T_{\alpha_1}(\beta_1)\vee T_{\alpha_1}(\beta_2)]$ and $d_1\in(d_{\alpha_2}(\beta_1,\beta_2),d_0)$, so $d^{\vee}_{\alpha_1}(\beta_1,\beta_2)\le d_{\alpha_2}(\beta_1,\beta_2)$. $\square$

LEMMA 7.7. *$\{T^+_{\alpha_1}<T_{\alpha_2}\}\in\mathcal{T}^+_{\alpha_1}$ for any $\alpha_1,\alpha_2\in\mathcal{A}$.*

PROOF. Fix any $\alpha_1''\in\mathcal{A}$ such that $\alpha_1''\succ\alpha_1$. There is $\alpha_1'\in\mathcal{A}$ with $\alpha_1''\succ\alpha_1'\succ\alpha_1$. Suppose $\beta_1\in\{T^+_{\alpha_1}<T_{\alpha_2}\}$. Then there is $a>0$ such that $a<T_{\alpha_1'}(\beta_1)\wedge T_{\alpha_2}(\beta_1)$ and $\beta_1(a)\notin H(\alpha_1)$. Let $d_0=1\wedge\operatorname{dist}(\beta_1(a),H(\alpha_1))\wedge\operatorname{dist}(\alpha_1',\alpha_1'')\wedge\operatorname{dist}(\beta_1([0,a]),\alpha_2)>0$. Suppose $\beta_2\in\mathbf{B}_{\alpha_1''}(\beta_1;d_0)$. From Lemma 7.6, $d^{\vee}_{\alpha_1'}(\beta_2,\beta_1)<d_0$. Since $d_0\le 1$, so $|\beta_2(t)-\beta_1(t)|<d_0$ for $0\le t\le T_{\alpha_1'}(\beta_1)$. Since $a<T_{\alpha_1'}(\beta_1)$, so $|\beta_2(t)-\beta_1(t)|<d_0$ for any $t\in[0,a]$. Since $\beta_1([0,a])$ is strictly enclosed by α_2, and $d_0\le\operatorname{dist}(\beta_1([0,a]),\alpha_2)$, so $\beta_2([0,a])$ is also strictly enclosed by α_2, which implies that $a<T_{\alpha_2}(\beta_2)$. Since $|\beta_2(a)-\beta_1(a)|<d_0$ and $d_0\le\operatorname{dist}(\beta_1(a),H(\alpha_1))$, so $\beta_2(a)\notin H(\alpha_1)$, which implies that $T^+_{\alpha_1}(\beta_2)<a$. Thus $T^+_{\alpha_1}(\beta_2)<T_{\alpha_2}(\beta_2)$, that is, $\beta_2\in\{T^+_{\alpha_1}<T_{\alpha_2}\}$. So $\mathbf{B}_{\alpha_1''}(\beta_1;d_0)\subset\{T^+_{\alpha_1}<T_{\alpha_2}\}$. Thus $\{T^+_{\alpha_1}<T_{\alpha_2}\}\in\mathcal{T}_{\alpha_1''}$. Since $\alpha_1''\succ\alpha_1$ is chosen arbitrarily, so $\{T^+_{\alpha_1}<T_{\alpha_2}\}\in\mathcal{T}^+_{\alpha_1}$. $\square$

LEMMA 7.8. *Suppose $\alpha_1,\alpha_2\in\mathcal{A}$ and $B\in\mathcal{T}^+_{\alpha_1}$. Then $B\cap\{T^+_{\alpha_1}<T_{\alpha_2}\}\in\mathcal{T}_{\alpha_2}$.*

PROOF. Fix $\beta_1\in B\cap\{T^+_{\alpha_1}<T_{\alpha_2}\}$. Then there is $a>0$ such that $a<T_{\alpha_2}(\beta_1)$ and $\beta_1(a)\notin H(\alpha_1)$. We may choose $\alpha_1'\succ\alpha$ and $\alpha_2'\prec\alpha_2$ such that $\beta_1(a)\notin H(\alpha_1')$ and $\beta_1([0,a])$ is strictly enclosed by α_2'. Since $B\in\mathcal{T}^+_{\alpha_1}\subset\mathcal{T}_{\alpha_1'}$, so there is $d_0>0$ such that $\mathbf{B}_{\alpha_1'}(\beta_1;d_0)\subset B$. Let $d_1=1\wedge d_0\wedge\operatorname{dist}(\beta_1(a),H(\alpha_1'))\wedge\operatorname{dist}(\alpha_2',\alpha_2)$. Suppose $\beta_2\in\mathbf{B}_{\alpha_2}(\beta_1;d_1)$. From Lemma 7.6, $d^{\vee}_{\alpha_2'}(\beta_2,\beta_1)<d_1$. Since $d_1\le 1$, so $|\beta_2(t)-\beta_1(t)|<d_1$ for $0\le t\le T_{\alpha_2'}(\beta_1)$. Since $a<T_{\alpha_2'}(\beta_1)$, so $|\beta_2(t)-\beta_1(t)|<d_1$ for $0\le t\le a$. Since $d_1\le\operatorname{dist}(\beta_1(a),H(\alpha_1'))$, so $\beta_2(a)\notin H(\alpha_1')$. Thus $T_{\alpha_1'}(\beta_2)\vee T_{\alpha_1'}(\beta_1)<a$. So we have

$$d_{\alpha_1'}(\beta_2,\beta_1)\le d^{\vee}_{\alpha_1'}(\beta_2,\beta_1)\le\sup\{|\beta_2(t)-\beta_1(t)|:0\le t\le a\}<d_1\le d_0.$$

Thus $\beta_2\in\mathbf{B}_{\alpha_1'}(\beta_1;d_0)\subset B$. Since $\beta_1([0,a])$ is strictly enclosed by α_2', $\alpha_2'\prec\alpha_2$, and $|\beta_2(t)-\beta_1(t)|<d_1\le\operatorname{dist}(\alpha_2',\alpha_2)$ for $0\le t\le a$, so $\beta_2([0,a])$ is strictly enclosed by α_2. Thus $T^+_{\alpha_1}(\beta_2)\le T_{\alpha_1'}(\beta_2)<a<T_{\alpha_2}(\beta_2)$, that is, $\beta_2\in\{T^+_{\alpha_1}<T_{\alpha_2}\}$. So $\mathbf{B}_{\alpha_2}(\beta_1;d_1)\subset B\cap\{T^+_{\alpha_1}<T_{\alpha_2}\}$. Thus $B\cap\{T^+_{\alpha_1}<T_{\alpha_2}\}\in\mathcal{T}_{\alpha_2}$. $\square$

COROLLARY 7.4. *$\{T^+_{\alpha_1}<T_{\alpha_2}\}\in\mathcal{T}_{\alpha_2}$ for any $\alpha_1,\alpha_2\in\mathcal{A}$.*

Let μ_δ and μ_0 be the distribution of β_δ and β_0, respectively. From Theorem 7.1, for any $\alpha\in\mathcal{A}$, $\mu_\delta\to\mu_0$ weakly w.r.t. d_α, as $\delta\to 0$. Suppose A is a nonempty finite subset of $\mathcal{A}$. Let $d_A=\bigvee_{\alpha\in A} d_\alpha$ and $\mathcal{T}_A$ be the topology generated by d_A. So $\mathcal{T}_A=\bigvee_{\alpha\in A}\mathcal{T}_A$. For $\beta_1\in\mathcal{B}$ and $r>0$, let $\mathbf{B}_A(\beta_1;r):=\{\beta\in\mathcal{B}: d_A(\beta,\beta_1)<r\}=\bigcap_{\alpha\in A}\mathbf{B}(\beta_1;r)$. Let $\mathcal{B}_A^+:=\{\bigvee_{\alpha\in A}T_\alpha^+<T\}$, that is, the set of $\beta\in\mathcal{B}$ that are not contained in $\bigcup_{\alpha\in A}H(\alpha)$.

Lemma 7.9. *$(\mathcal{B}_A^+,d_A)$ is separable.*

Proof. For $r\in\mathbb{Q}_{>0}$, let $\mathcal{C}_r$ denote the set of continuous curves $\gamma:[0,r]\to\Omega\cup\mathbb{R}$ with $\gamma(0)=0$ and $\gamma(t)\in\Omega$ for $t\in(0,r]$. Then $\mathcal{C}_r$ is a subset of $C([0,r],\mathbb{C})$. Let d_r be the restriction of $\|\cdot\|_r$ to $\mathcal{C}_r$, that is, $d_r(\gamma_1,\gamma_2)=\sup\{|\gamma_1(t)-\gamma_2(t)|:0\le t\le r\}$. Then $(\mathcal{C}_r,d_r)$ is a subspace of $(C([0,r],\mathbb{C}),\|\cdot\|_r)$, so is separable. Let $\{\gamma_{r,n}:n\in\mathbb{N}\}$ be dense in $(\mathcal{C}_r,d_r)$. For each $r\in\mathbb{Q}_{>0}$ and $n\in\mathbb{N}$, we choose $\beta_{r,n}\in\mathcal{B}$ such that $T(\beta_{r,n})>r$ and $\beta_{r,n}(t)=\gamma_{r,n}(t)$ for $0\le t\le r$. Then $\{\beta_{r,n}:r\in\mathbb{Q}_{>0},n\in\mathbb{N}\}$ is countable.

Suppose $\beta_1\in\mathcal{B}_A^+$, and $d_0>0$. There is $r_0\in\mathbb{Q}_{>0}$ such that $\bigvee_{\alpha\in A}T_\alpha^+(\beta_1)<r_0<T(\beta_1)$. For each $\alpha\in A$, there is $t_\alpha\in(0,r_0)$ such that $\beta_1(t_\alpha)\notin H(\alpha)$. Let $d_1=\bigwedge_{\alpha\in A}\operatorname{dist}(\beta_1(t_\alpha),H(\alpha))\wedge d_0>0$. From the denseness of $\{\gamma_{r_0,n}:n\in\mathbb{N}\}$ in $(\mathcal{C}_{r_0},d_{r_0})$, we have $n_0\in\mathbb{N}$ such that $|\beta_{r_0,n_0}(t)-\beta_1(t)|=|\gamma_{r_0,n_0}(t)-\beta_1(t)|<d_1$ for $0\le t\le r_0$. Fix $\alpha\in A$. Since $|\beta_{r_0,n_0}(t_\alpha)-\beta_1(t_\alpha)|<d_1\le\operatorname{dist}(\beta_1(t_\alpha),H(\alpha))$, so $\beta_{r_0,n_0}(t_\alpha)\notin H(\alpha)$. Thus $T_\alpha(\beta_{r_0,n_0})<r_0<T(\beta_{r_0,n_0})$. Since this is true for any $\alpha\in A$, so $\beta_{r_0,n_0}\in\mathcal{B}_A^+$. Since

$$d_\alpha(\beta_{r_0,n_0},\beta_1)\le d_\alpha^\vee(\beta_{r_0,n_0},\beta_1)\le\sup\{|\beta_{r_0,n_0}(t)-\beta_1(t)|:0\le t\le r_0\}<d_1\le d_0$$

for any $\alpha\in A$, so $d_A(\beta_{r_0,n_0},\beta_1)<d_0$. Thus $\{\beta_{r,n}\}\cap\mathcal{B}_A^+$ is dense in $(\mathcal{B}_A^+,d_A)$. □

Theorem 7.2. *$\mu_\delta\to\mu_0$ weakly w.r.t. d_A, as $\delta\to 0$.*

Proof. Suppose $A=\{\alpha_1,\dots,\alpha_n\}$. The case $n=1$ follows from Theorem 7.1. Now suppose $n\ge 2$. We suffice to show that for any $G\in\mathcal{T}_A$, $\liminf_{\delta\to 0}\mu_\delta(G)\ge\mu_0(G)$.

We may find polygonal paths $\alpha_j^0\in\mathcal{A}$, $1\le j\le n$, such that $\alpha_j^0\succ\alpha_j$ for each j, and such that for $j\ne k$, any line segment on α_j^0 is not parallel to any line segment on α_k^0. Fix $j\in\mathbb{Z}_{[1,n]}$. List the vertices on $\overline{\alpha_j^0}$ in the counterclockwise order as $z_0^0,z_1^0,\dots,z_m^0$. We may find $z_0^1>0>z_m^1$, and $z_k^1\in\Omega$, $1\le k\le m-1$, and let $\alpha_j^1=\bigcup_{k=1}^{m-1}(z_{k-1}^1,z_k^1]\cup(z_{m-1}^1,z_m^1)$, such that $\mathcal{A}\ni\alpha_j^1\succ\alpha_j^0$, $[z_{k-1}^1,z_k^1]$ is parallel to $[z_{k-1}^0,z_k^0]$ for $1\le k\le m$, and $[z_l^0,z_l^1]\cap[z_k^0,z_k^1]=\varnothing$ for $1\le l<k\le m$. For $r\in[0,1]$, let $z_k(r)=z_k^0+r(z_k^1-z_k^0)$, $0\le k\le m$, and let $\alpha_j(r)=\bigcup_{k=1}^{m-1}(z_{k-1}(r),z_k(r)]\cup(z_{m-1}(r),z_m(r))$. Then $\alpha_j(r)\in\mathcal{A}$ for all $r\in[0,1]$, and $\alpha_j(s)\prec\alpha_j(r)$ if $0\le s<r\le 1$. And for any $s\in[0,1)$, if $\alpha_j(s)\prec\alpha\in\mathcal{A}$, then

there is $r\in(s,1)$ such that $\alpha_j(r)\prec\alpha$. Thus for any $\beta\in\mathcal{B}$, we have that $r\mapsto T_{\alpha_j(r)}(\beta)$ is increasing on $[0,1]$, and for any $s\in[0,1)$, $T^+_{\alpha_j(s)}=\lim_{r\downarrow s}T_{\alpha_j(r)}$, so there are at most countably many $r\in[0,1]$ such that $T^+_{\alpha_j(r)}(\beta)>T_{\alpha_j(r)}(\beta)$. So there is $r_j\in(0,1)$ such that $\mu_0(\{T^+_{\alpha_j(r_j)}>T_{\alpha_j(r_j)}\})=0$. For $j=1,\dots,k$, let $\alpha_j^2=\alpha_j(r_j)$, then $\alpha_j\prec\alpha_j^2$, and $\mu_0(\{T^+_{\alpha_j^2}>T_{\alpha_j^2}\})=0$.

Suppose $j\ne k\in\mathbb{Z}_{[1,n]}$. Since any line segment on α_j^2 is not parallel to any line segment on α_k^2, so $S_{j,k}:=\alpha_j^2\cap\alpha_k^2$ is a finite set. If for some $j\ne k$ and $\beta\in\mathcal{B}$, $T_{\alpha_j^2}(\beta)=T_{\alpha_k^2}(\beta)<T(\beta)$, then β must pass through $S_{j,k}$. From Theorem 3.1(ii), we have $T_{\alpha_j^2}(\beta_0),T_{\alpha_k^2}(\beta_0)<T(\beta_0)$. Thus $\{T_{\alpha_j^2}(\beta_0)=T_{\alpha_k^2}(\beta_0)\}\subset\{\beta_0$ passes through $S_{j,k}\}$. From the property of chordal SLE_2, for any $z_0\in\Omega$, the probability that β_0 passes through z_0 is 0, which implies $\mathbf{P}[\beta_0$ passes through $S_{j,k}]=0$, so $\mu_0(\{T_{\alpha_j^2}=T_{\alpha_k^2}\})=0$.

For $j\in\mathbb{Z}_{[1,n]}$, let $I_j=\mathbb{Z}_{[1,n]}\setminus\{j\}$ and $B_j=\{\bigvee_{k\in I_j}T^+_{\alpha_k^2}<T_{\alpha_j^2}\}=\bigcap_{k\in I_j}\{T^+_{\alpha_k^2}<T_{\alpha_j^2}\}$, which belongs to $\mathcal{T}_{\alpha_j^2}$ from Corollary 7.4. Then $B_1,\dots,B_n$ are mutually disjoint. Let $N=\mathcal{B}\setminus\bigcup_{j=1}^m B_j$. Then

$$N\subset\bigcup_{1\le j\le n}\{T^+_{\alpha_j^2}>T_{\alpha_j^2}\}\cup\bigcup_{1\le j<k\le n}\{T_{\alpha_j^2}=T_{\alpha_k^2}\}.$$

Thus $\mu_0(N)=0$. Fix $j\in\mathbb{Z}_{[1,n]}$. If $B\in\mathcal{T}_{\alpha_j}$, then $B\in\mathcal{T}_{\alpha_j^2}$, so $B\cap B_j\in\mathcal{T}_{\alpha_j^2}$. If $B\in\mathcal{T}_{\alpha_k}$ for some $k\in I_j$, then $B\in\mathcal{T}^+_{\alpha_k^2}$. From Lemma 7.8, we have $B\cap\{T^+_{\alpha_k^2}<T_{\alpha_j^2}\}\in\mathcal{T}_{\alpha_j^2}$. Thus $B\cap B_j=B\cap\{T^+_{\alpha_k^2}<T_{\alpha_j^2}\}\cap B_j\in\mathcal{T}_{\alpha_j^2}$. Let $\mathcal{T}_j$ denote the collection of sets $B\subset\mathcal{B}$ such that $B\cap B_j\in\mathcal{T}_{\alpha_j^2}$. Then $\mathcal{T}_j$ is a topology. We have proved that $\mathcal{T}_{\alpha_k}\subset\mathcal{T}_j$ for any $k\in\mathbb{Z}_{[1,n]}$. Thus $\mathcal{T}_A=\bigvee_{k=1}^n\mathcal{T}_{\alpha_j}\subset\mathcal{T}_j$.

Suppose $G\in\mathcal{T}_A$. Let $G_j=G\cap B_j$, $1\le j\le n$. For each $j\in\mathbb{Z}_{[1,n]}$, since $G\in\mathcal{T}_A\subset\mathcal{T}_j$, so $G_j=G\cap B_j\in\mathcal{T}_{\alpha_j^2}$. Since $\mu_\delta\to\mu_0$ w.r.t. $d_{\alpha_j^2}$, so $\liminf_{\delta\downarrow 0}\mu_\delta(G_j)\ge\mu_0(G_j)$. Since G is the disjoint union of $G\cap N$ and G_j, $1\le j\le n$, and $\mu_0(G\cap N)=0$, so

$$\liminf_{\delta\to 0}\mu_\delta(G)\ge\sum_{j=1}^n\liminf_{\delta\to 0}\mu_\delta(G_j)\ge\sum_{j=1}^n\mu_0(G_j)=\mu_0(G).$$

Since this is true for any $G\in\mathcal{T}_A$, so we have $\mu_\delta\to\mu_0$ weakly w.r.t. d_A, as $\delta\to 0$. □

We may find a sequence $\{\breve\alpha_n:n\in\mathbb{N}\}$ in $\mathcal{A}$ such that for any $\alpha\in\mathcal{A}$, there is $n\in\mathbb{N}$ such that $\breve\alpha_n\succ\alpha$. For $n\in\mathbb{N}$, let $T_n=\bigvee_{j=1}^n T_{\breve\alpha_j}$. Then for any $\beta\in\mathcal{B}$, $\bigvee_{n=1}^\infty T_n(\beta)=\bigvee_{\alpha\in\mathcal{A}}T_\alpha(\beta)$. If β_0 does not visit $f(\infty)$, then $\bigvee_{n=1}^\infty T_n(\beta_0)=T(\beta_0)=T_0$. From the property of chordal SLE_2, β_0 does not visit $f(\infty)$ a.s., so $\bigvee_{n=1}^\infty T_n(\beta_0)=T_0$ a.s.

THEOREM 7.3. *For any $n \in \mathbb{N}$ and $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there is a coupling of β_δ and β_0 such that with probability greater than $1-\varepsilon$, $|\beta_\delta(t) - \beta_0(t)| < \varepsilon$ for $t \in [0, T_n(\beta_0)]$.*

PROOF. For each $1 \le j \le n$, choose $\alpha_j \succ \breve{\alpha}_j$. Let $A = \{\alpha_1, \dots, \alpha_n\}$. From Theorem 3.1(ii), we have $\beta_0 \in \mathcal{B}_A^+$. As $\delta \to 0$, $w_e^\delta \to z_e$, so $f(w_e^\delta) \to p \notin \bigcup_{\alpha \in A} H(\alpha)$. There is $\delta_1 > 0$, such that if $\delta < \delta_1$, then $f(w_e^\delta) \notin \bigcup_{\alpha \in A} H(\alpha)$, so $\beta_\delta \in \mathcal{B}_A^+$. Thus μ_0 and μ_δ are supported by $\mathcal{B}_A^+$ when $\delta < \delta_1$. From Theorem 7.2, $\mu_\delta \to \mu_0$ weakly as $\delta \to 0$, w.r.t. d_A. From Lemma 7.9, $(\mathcal{B}_A^+, d_A)$ is separable. So from the coupling theorem in [3], there is $\delta_0 \in (0, \delta_1)$ such that if $\delta < \delta_0$, there is a coupling of β_δ and β_0 such that

$$\mathbf{P}\left[d_A(\beta_\delta, \beta_0) < \bigwedge_{j=1}^n \operatorname{dist}(\alpha_j, \breve{\alpha}_j) \wedge 1 \wedge \varepsilon \right] > 1 - \varepsilon. \tag{7.7}$$

Assume $d_A(\beta_\delta, \beta_0) < \bigwedge_{j=1}^n \operatorname{dist}(\alpha_j, \breve{\alpha}_j) \wedge 1 \wedge \varepsilon$. Then for each $j \in \{1, \dots, n\}$, we have $d_{\alpha_j}(\beta_\delta, \beta_0) < \operatorname{dist}(\alpha_j, \breve{\alpha}_j) \wedge 1 \wedge \varepsilon$, which implies $d_{\breve{\alpha}_j}^\vee(\beta_\delta, \beta_0) < 1 \wedge \varepsilon$ from Lemma 7.6, so $|\beta_\delta(t) - \beta_0(t)| < 1 \wedge \varepsilon$ for $0 \le t \le T_{\breve{\alpha}_j}(\beta_\delta) \vee T_{\breve{\alpha}_j}(\beta_0)$. Since $T_n(\beta_0) = \bigvee_{j=1}^n T_{\breve{\alpha}_j}(\beta_0)$, so $|\beta_\delta(t) - \beta_0(t)| < \varepsilon$ for $t \in [0, T_n(\beta_0)]$. □

THEOREM 7.4. (i) *For any $\alpha \in \mathcal{A}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there is a coupling of β_δ and β_0 such that with probability greater than $1-\varepsilon$, $|f^{-1}(\beta_\delta(t)) - f^{-1}(\beta_0(t))| < \varepsilon$ for $t \in [T_\alpha(\beta_0), T_n(\beta_0)]$.*

(ii) *Suppose 0_+ is degenerate. Then for any $n \in \mathbb{N}$ and $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then there is a coupling of β_δ and β_0 such that with probability greater than $1-\varepsilon$, $|f^{-1}(\beta_\delta(t)) - f^{-1}(\beta_0(t))| < \varepsilon$ for $t \in (0, T_n(\beta_0)]$.*

PROOF. (i) Since $\beta_0([T_\alpha(\beta_0), T_n(\beta_0)])$ is a compact subset of $\Omega \setminus \{f(\infty)\}$, on which f^{-1} is continuous in Euclidean metric, so there is $\varepsilon_0 > 0$ such that $\mathbf{P}[\mathcal{E}_1] > 1 - \varepsilon/2$, where $\mathcal{E}_1$ is the event that $|f^{-1}(z_2) - f^{-1}(z_1)| < \varepsilon$ for any $z_1 \in \beta_0([T_\alpha(\beta_0), T_n(\beta_0)])$ and $z_2 \in \Omega$ with $|z_2 - z_1| < \varepsilon_0$. From Theorem 7.3 there is $\delta_0 > 0$ such that if $\delta < \delta_0$, then β_δ and β_0 can be coupled such that with probability greater than $1 - \varepsilon/2$, $|\beta_\delta(t) - \beta_0(t)| < \varepsilon_0$ for $t \in [0, T_n(\beta_0)]$. Let $\mathcal{E}_2^\delta$ denote this event. Let $\mathcal{E}^\delta = \mathcal{E}_1 \cap \mathcal{E}_2^\delta$. Suppose $\delta < \delta_0$. Then $\mathbf{P}[\mathcal{E}^\delta] > 1 - \varepsilon$. Assume $\mathcal{E}^\delta$ occurs. Then for $t \in [T_\alpha(\beta_0), T_n(\beta_0)]$, $|\beta_\delta(t) - \beta_0(t)| < \varepsilon_0$, so $|f^{-1}(\beta_\delta(t)) - f^{-1}(\beta_0(t))| < \varepsilon$.

(ii) Suppose 0_+ is degenerate. From [13], f^{-1} extends continuously to $\Omega \cup \{0\}$. Since $\beta_0([0, T_n(\beta_0)])$ is a compact subset of $(\Omega \setminus \{f(\infty)\}) \cup \{0\}$, so the above argument still works here. □

Let $\bar{\gamma}_0 = f^{-1} \circ \beta_0$ and $\bar{\gamma}_\delta = f^{-1} \circ \beta_\delta$. Then $\bar{\gamma}_0$ is a time-change of γ_0, and $\bar{\gamma}_\delta$ is a time-change of q_δ.

THEOREM 7.5. *$\lim_{t\to S_0}\gamma_0(t)=\lim_{t\to T_0}\bar\gamma_0(t)=z_e$ almost surely.*

PROOF. Let L be the set of spherical subsequential limits of $\bar\gamma_0(t)$ as $t\to T_0$. We first claim that $L\cap\partial^\# D=\varnothing$ a.s. If the claim is not true, then there is $\varepsilon_0>0$ such that $\mathbf{P}[L\cap\partial^\# D\ne\varnothing]>\varepsilon_0$. Since $\bar\gamma_0([T_1(\beta_0),T_2(\beta_0)])\subset D\setminus\{z_e,\infty\}$, so for every $\varepsilon>0$ there is a compact subset F_1 of $D\setminus\{z_e,\infty\}$ such that $\mathbf{P}[\mathcal{E}_0]>1-\varepsilon_0/3$, where $\mathcal{E}_0$ is the event that $\bar\gamma_0([T_1(\beta_0),T_2(\beta_0)])$ intersects F_1. Let F_2 be a compact subset of $D\setminus\{z_e,\infty\}$ such that F_1 is contained in the interior of F_2. Let $d_0=\operatorname{dist}(F_1,\partial F_2)>0$. From Lemma 7.3, there are $\varepsilon_1,\delta_1>0$ such that if $\delta<\delta_1$, then the probability that $\bar\gamma_\delta$ visits $\partial^\#_{\varepsilon_1}D$ after F_2 is smaller than $\varepsilon_0/3$. Since $\mathbf{P}[\bar\gamma_0([T_2(\beta_0),T_0))\cap\partial^\#_{\varepsilon_1/2}D\ne\varnothing]>\varepsilon_0$ and $T_0=\bigvee_{n=1}^\infty T_n(\beta_0)$ a.s., so there is $n_0\in\mathbb{N}$ such that $\mathbf{P}[\mathcal{E}_1]>\varepsilon_0$, where $\mathcal{E}_1$ is the event that $\bar\gamma_0([T_2(\beta_0),T_{n_0}(\beta_0)])\cap\partial^\#_{\varepsilon_1/2}D\ne\varnothing$. Note that $T_1=T_{\breve\alpha_1}$. From Theorem 7.4(i), there are $\delta_2<\delta_1$ and a coupling of $\bar\gamma_{\delta_2}$ with $\bar\gamma_0$ such that with probability greater than $1-\varepsilon_0/3$, $|\bar\gamma_{\delta_2}(t)-\bar\gamma_0(t)|<(\varepsilon_1/4)\wedge d_0$ for $t\in[T_1(\beta_0),T_{n_0}(\beta_0)]$. Let $\mathcal{E}_2$ denote this event. Since $\delta_2<\delta_1$, so the probability that $\bar\gamma_{\delta_2}$ does not visit $\partial^\#_{\varepsilon_1}D$ after F_2 is greater than $1-\varepsilon_0/3$. Let $\mathcal{E}_3$ denote this event. Let $\mathcal{E}=\bigcap_{j=0}^3\mathcal{E}_j$. Then $\mathbf{P}[\mathcal{E}]>0$. So $\mathcal{E}$ is nonempty. Assume $\mathcal{E}$ occurs. Since $\mathcal{E}_0$ occurs, so there is $t_0\in[T_1(\beta_0),T_2(\beta_0)]$ such that $\bar\gamma_0(t_0)\in F_1$. Since $\mathcal{E}_2$ occurs, so $|\bar\gamma_{\delta_2}(t_0)-\bar\gamma_0(t_0)|<d_0$, which implies that $\bar\gamma_{\delta_2}(t_0)\in F_2$. Since $\mathcal{E}_1$ occurs, there is $t_1\in[T_2(\beta_0),T_{n_0}(\beta_0)]$ such that $\bar\gamma_0(t_1)\in\partial^\#_{\varepsilon_1/2}D$. Since $\mathcal{E}_2$ occurs, so $\operatorname{dist}^\#(\bar\gamma_{\delta_2}(t_1),\bar\gamma_0(t_1))\le 2\operatorname{dist}(\bar\gamma_{\delta_2}(t_1)-\bar\gamma_0(t_1))<\varepsilon_1/2$, which implies that $\bar\gamma_{\delta_2}(t_1)\in\partial^\#_{\varepsilon_1}D$. Since $t_0\le T_2(\beta_0)\le t_1$, so $\bar\gamma_{\delta_2}$ visits $\partial^\#_{\varepsilon_1}D$ after F_2, which means that $\mathcal{E}_3$ cannot occur. So we get a contradiction. Thus $L\cap\partial^\# D=\varnothing$ a.s.

Second, we claim that $\operatorname{diam}^\#(L)=0$ a.s. If the claim is not true, then from the last paragraph we have $\mathbf{P}[\operatorname{diam}^\#(L)>0,L\subset D]>0$. Then there are $z_0\in D\setminus\{\infty\}$ and $r_0,\varepsilon_0>0$ such that $\overline{\mathbf{B}(z_0;4r_0)}\subset D$ and the probability that $L\cap\mathbf{B}(z_0;r_0/2)\ne\varnothing$ and $L\setminus\mathbf{B}(z_0;4r_0)\ne\varnothing$ is greater than ε_0. Let $\mathcal{E}_0$ denote this event. From Corollary 7.1, there is $\varepsilon_1>0$ such that with probability greater than $1-\varepsilon_0/2$, $\bar\gamma_\delta$ does not contain a $(\overline{\mathbf{B}(z_0;r_0)},r_0,\varepsilon_1)$-quasi-loop. For $n\in\mathbb{N}$, let $\mathcal{E}_0^n$ denote the event that there are $t_1<t_0<t_2<T_n(\beta_0)$ with $\bar\gamma_0(t_1),\bar\gamma_0(t_2)\in\mathbf{B}(z_0;r_0/2)$, $|\bar\gamma_0(t_1)-\bar\gamma_0(t_2)|<\varepsilon_1/3$, and $\bar\gamma_0(t_0)\notin\mathbf{B}(z_0;3r_0)$. If $\mathcal{E}_0$ occurs, then since $T_0=\bigvee_{n=1}^\infty T_n(\beta_0)$ a.s., and $\beta_0(t)$ has subsequential limits, as $t\to T_0$, inside $\mathbf{B}(z_0;r_0/2)$ and outside $\mathbf{B}(z_0;4r_0)$, so some $\mathcal{E}_0^n$, $n\in\mathbb{N}$, must occur. Thus $\mathcal{E}_0\subset\bigcup_{n=1}^\infty\mathcal{E}_0^n$. Since $\mathbf{P}[\mathcal{E}_0]>\varepsilon_0$, and $(\mathcal{E}_n^0)$ is increasing, so there is $n_0\in\mathbb{N}$ such that $\mathbf{P}[\mathcal{E}_0^{n_0}]>\varepsilon_0$. Choose $\alpha\in\mathcal{A}$ such that $f^{-1}(H(\alpha))\cap\overline{\mathbf{B}(z_0;4r_0)}=\varnothing$. From Theorem 7.4, there are $\delta_0>0$ and a coupling of $\bar\gamma_{\delta_0}$ and $\bar\gamma_0$ such that with probability greater than $1-\varepsilon_0/2$, $|\bar\gamma_{\delta_0}(t)-\bar\gamma_0(t)|<(r_0/2)\wedge(\varepsilon_1/3)$ for $t\in[T_\alpha(\beta_0),T_{n_0}(\beta_0)]$. Let $\mathcal{E}_1$ denote this event. Let $\mathcal{E}_2$ denote the event that $\bar\gamma_{\delta_0}$ does not contain a $(\overline{\mathbf{B}(z_0;r_0)},r_0,\varepsilon_1)$-quasi-loop. Then $\mathbf{P}[\mathcal{E}_2]>1-\varepsilon_0/2$ from the choice of ε_1. Let $\mathcal{E}=\mathcal{E}_0^{n_0}\cap$

$\mathcal{E}_1 \cap \mathcal{E}_2$. Then $\mathbf{P}[\mathcal{E}] > 0$. So $\mathcal{E}$ is nonempty. Assume $\mathcal{E}$ occurs. Since $\mathcal{E}_0^{n_0}$ occurs, so there are $t_1 < t_0 < t_2 < T_{n_0}(\beta_0)$ with $\bar\gamma_0(t_1), \bar\gamma_0(t_2) \in \mathbf{B}(z_0; r_0/2)$, $|\bar\gamma_0(t_1) - \bar\gamma_0(t_2)| < \varepsilon_1/3$, and $\bar\gamma_0(t_0) \notin \mathbf{B}(z_0; 3r_0)$. For $j = 1, 2$, since $\bar\gamma_0(t_j) \in \mathbf{B}(z_0; r_0/2)$, so $\beta_0(t_j) \not\subset H(\alpha)$, which implies that $t_j \ge T_\alpha(\beta_0)$. Since $\mathcal{E}_1$ occurs, so $|\bar\gamma_{\delta_0}(t_j) - \bar\gamma_0(t_j)| < (r_0/2) \wedge (\varepsilon_1/3)$, $j = 1, 2$, and $|\bar\gamma_{\delta_0}(t_0) - \bar\gamma_0(t_0)| < r_0$, which implies that $\bar\gamma_{\delta_0}(t_1) \in \mathbf{B}(z_0; r_0)$, $|\bar\gamma_{\delta_0}(t_1) - \bar\gamma_{\delta_0}(t_2)| < \varepsilon_1$ and $\bar\gamma_{\delta_0}(t_0) \notin \mathbf{B}(z_0; 2r_0)$, so $|\bar\gamma_{\delta_0}(t_0) - \bar\gamma_{\delta_0}(t_1)| \ge r_0$. So we find a $(\overline{\mathbf{B}(z_0; r_0)}, r_0, \varepsilon_1)$-quasi-loop on $\bar\gamma_{\delta_0}$, which contradicts $\mathcal{E}_2$. So $\mathbf{P}[\mathrm{diam}^{\#}(L) > 0] = 0$.

Thus almost surely L is a single point in D, which means that $\lim_{t\to T_0} \bar\gamma_0(t)$ exists in the spherical metric and lies in D. Now we claim that $\lim_{t\to T_0} \bar\gamma_0(t) \notin \bar\gamma_0([0, T_0))$ a.s. If the claim is not true, then there exist $z_0 \in D$ and $r_0 > 0$ such that with a positive probability, we have $\lim_{t\to T_0} \bar\gamma_0(t) \in \bar\gamma_0([0, T_0)) \cap \mathbf{B}(z_0; r_0/2)$ and $\bar\gamma_0([0, T_0))) \not\subset \mathbf{B}(z_0; 4r_0)$, so we can use an argument that is similar to the last paragraph to find a contradiction. Note that almost surely $\bar\gamma_0$ does not visit ∞. Thus almost surely we may extend $\bar\gamma_0$ to be a simple continuous curve defined on $[0, T_0]$ such that $\bar\gamma_0(T_0) \in D \setminus \{\infty\}$. If $\mathbf{P}[\bar\gamma_0(T_0) \ne z_e] > 0$, then there is $n_0 \in \mathbb{N}$ such that the probability that $\bar\gamma_0([0, T_0])$ is enclosed by $f^{-1}(\breve\alpha_{n_0})$ is positive, which contradicts Theorem 3.1(ii). Thus $\mathbf{P}[\bar\gamma_0(T_0) = z_e] = 1$. Since γ_0 is a time-change of $\bar\gamma_0$, so $\lim_{t\to S_0} \gamma_0(t) = \lim_{t\to T_0} \bar\gamma_0(t) = z_e$ a.s. □

Proof of Theorem 4.2. (i) Choose $r > 0$ such that $\mathbf{B} := \overline{\mathbf{B}(z_e; r)} \subset D$. From Corollary 7.1, there is $\varepsilon_0 \in (0, \varepsilon)$ such that the probability that $\bar\gamma_\delta$ does not contain a $(\mathbf{B}, \varepsilon/6, \varepsilon_0)$-quasi-loop is greater than $1 - \varepsilon/3$. Let $\mathcal{E}_0^\delta$ denote this event. There is δ_1 such that if $\delta < \delta_1$, then $|w_e^\delta - z_e| < r \wedge (\varepsilon_0/3)$. From Theorem 7.5, we have $\lim_{t\to T_0} \bar\gamma_0(t) = z_e$ a.s. Since $T_0 = \bigvee_{n=1}^\infty T_n(\beta_0)$ a.s., so there is $n_0 \in \mathbb{N}$ such that with probability greater than $1 - \varepsilon/3$, the diameter of $\bar\gamma_0([T_{n_0}(\beta_0), T_0))$ is less than $\varepsilon_0/3$. Let $\mathcal{E}_1$ denote this event. Choose $\alpha \in \mathcal{A}$ such that $f^{-1}(H(\alpha)) \subset U$. Then $T_\alpha(\beta_0) \le T_U(\bar\gamma_0)$. From Theorem 7.4(i), there is $\delta_0 < \delta_1$ such that if $\delta < \delta_0$, then there is a coupling of $\bar\gamma_\delta$ and $\bar\gamma_0$ such that with probability greater than $1 - \varepsilon/3$, $|\bar\gamma_{\delta_2}(t) - \bar\gamma_0(t)| < \varepsilon_0/3$ for $t \in [T_U(\bar\gamma_0), T_{n_0}(\beta_0)]$. Let $\mathcal{E}_2$ denote this event. Let $\mathcal{E}^\delta = \mathcal{E}_0^\delta \cap \mathcal{E}_1 \cap \mathcal{E}_2$. Suppose $\delta < \delta_0$. Then $\mathbf{P}[\mathcal{E}^\delta] > 1 - \varepsilon$. Assume $\mathcal{E}^\delta$ occurs. Let $T_e = T_{n_0}(\beta_0)$. Then $|\bar\gamma_\delta(t) - \bar\gamma_0(t)| < \varepsilon_0/3 < \varepsilon/3$ for $T_U(\bar\gamma_0) \le t \le T_e$. And $|\bar\gamma_\delta(T_e) - w_e^\delta| \le |\bar\gamma_\delta(T_e) - \bar\gamma_0(T_e)| + |\bar\gamma_0(T_e) - z_e| + |z_e - w_e^\delta| < \varepsilon_0$. Since $\bar\gamma_\delta(T_\delta) = w_e^\delta \in \mathbf{B}$ and $\bar\gamma_\delta$ does not contain a $(\mathbf{B}, \varepsilon/6, \varepsilon_0)$-quasi-loop, so the diameter of $\bar\gamma_\delta([T_e, T_\delta))$ is less than $\varepsilon/3$. Choose $\mathring{u}$ that maps $[T_U(\bar\gamma_0), T_\delta)$ onto $[T_U(\bar\gamma_0), T_0)$ such that $\mathring{u}(t) = t$ for $T_U(\bar\gamma_0) \le t \le T_e$; then $|\bar\gamma_\delta(\mathring{u}^{-1}(t)) - \bar\gamma_0(t)| < \varepsilon$ for $T_U(\bar\gamma_0) \le t < T_0$. Since $\bar\gamma_\delta$ and $\bar\gamma_0$ are time-changes of q_δ and γ_0, respectively, so the proof of (i) is finished.

(ii) If 0_+ is degenerate, then we use Theorem 7.4(ii) in the above proof. □

8. Other kinds of targets.

8.1. *When the target is a prime end.* Now we consider the case that the target is a prime end. We use the notation and boundary conditions given in Section 4.2 for the discrete LERW aimed at a prime end w_e. Suppose f maps D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(0_+) = 0$.

We will go through the propositions in Sections 6 and 7, and explain how they can be modified to prove Theorem 4.2 in this case. We only consider D^δ for $\delta \in \mathcal{M}$, so the words "$\delta < *$" should be replaced by "$\delta \in \mathcal{M}$ and $\delta < *$," and the words "$\delta \to 0$" should be replaced by "$\delta \to 0$ along $\mathcal{M}$."

Let X_t^ξ and $P^\xi(t, x, \cdot)$ be notation in the case that the target is a prime end defined in Sections 3.4 and 4.1. Then all lemmas in Section 6.1 still hold. For Proposition 6.1, redefine P_X to be the generalized Poisson kernel in D_X with the pole at $\mathrm{Tip}(X)$, normalized by $\partial_{\mathbf{n}} P_X(w_e) = 1$; let h_X be defined on $V(D^\delta)$ that satisfies $h_X \equiv 0$ on $V_\partial(D^\delta) \cup \mathrm{Set}(X) \setminus \{\mathrm{Tip}(X)\}$, $\Delta_{D^\delta} h_X \equiv 0$ on $V_I(D^\delta) \setminus \mathrm{Set}(X)$, and $\Delta_{D^\delta} h_X(w_e) = 1$. Proposition 6.1 should be restated as Proposition 8.1 below, which together with Proposition 2.1 implies Proposition 6.2, and then all theorems in Section 6.2.

PROPOSITION 8.1. *For any $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta \in \mathcal{M}$ and $\delta < \delta_0$, then for any $X \in L^\delta$, and any $w \in V(D^\delta) \cap (D \setminus H(\rho_2))$, we have $|\delta \cdot h_X(w) - P_X(w)| < \varepsilon$.*

PROOF. Fix $z_0 \in D \setminus H(\rho_2)$ and let w_0^δ be a vertex on D^δ that is closest to z_0. For $\delta \in \mathcal{M}$ and $X \in L^\delta$, let $g_X^0(w) = h_X(w)/h_X(w_0^\delta)$. Then from Proposition 6.1, g_X^0 converges to the generalized Poisson kernel P_X^0 in D_X with the pole at $\mathrm{Tip}(X)$, normalized by $P_X^0(z_0) = 1$, uniformly on $D \setminus H(\rho_3)$ for any crosscut ρ_3 in D such that $H(\rho_1) \subset H(\rho_3)$ and $\overline{\rho_1} \cap \overline{\rho_3} = \varnothing$. Since ∂D is flat near w_e, and g_X^0 vanishes on ∂D near w_e, so g_X^0 can be naturally extended to be a discrete harmonic function on $\delta\mathbb{Z}^2 \cap \mathbf{B}(w_e; r_0)$ for some $r_0 > 0$. We may also extend P_X^0 to be a harmonic function defined in $\mathbf{B}(w_e; r_0)$ by the Schwarz reflection principle. Then we can prove that the discrete partial derivatives of g_X^0 approximate the corresponding partial derivatives of P_X^0 locally uniformly in $\mathbf{B}(w_e; r_0)$. Especially, we have $(g_X^0(w_e^\delta) - g_X^0(w_e))/\delta \to \partial_{\mathbf{n}} P_X^0(w_e)$ as $\delta \to 0$, because w_e^δ is the unique adjacent vertex of w_e in D^δ. Note that $\Delta_{D^\delta} g_X^0(w_e) = g_X^0(w_e^\delta) - g_X^0(w_e)$. From the definition of g_X^0, we have $\Delta_{D^\delta} h_X(w_e)/(\delta \cdot h_X(w_0^\delta)) \to \partial_{\mathbf{n}} P_X^0(w_e)$ as $\delta \to 0$. Since $\Delta_{D^\delta} h_X(w_e) = 1$, so $1/(\delta \cdot h_X(w_0^\delta)) \to \partial_{\mathbf{n}} P_X^0(w_e)$ as $\delta \to 0$. Thus $\delta \cdot h_X(w) = g_X(w) \cdot \delta \cdot h_X(w_0^\delta)$ converges to $P_X^0(w)/\partial_{\mathbf{n}} P_X^0(w_e) = P_X(w)$ uniformly on $D \setminus H(\rho_2)$. $\square$

In Section 7, redefine X_w to be a random walk on D^δ started from w, stopped when it hits $V_\partial(D^\delta)$, and Y_w to be that X_w conditioned to hit $V_\partial(D^\delta)$ at w_e. Then q_δ is the loop-erasure of Y_δ. Lemma 7.2 still holds.

For the proof, we argue on Y_w instead of the reversal path. Then Corollary 7.1 and Corollary 7.2 immediately follow. Let F_D (resp. F_Ω) be a compact subset of $D\setminus\{\infty\}$ [resp. $\Omega\setminus\{f(\infty)\}$]. Lemma 7.4 still holds. Lemma 7.3, Corollary 7.3 and Lemma 7.5 should be restated as Lemma 8.1, Corollary 8.1 and Lemma 8.2, respectively, whose proofs are similar. Then we have Theorem 7.1.

LEMMA 8.1. *Suppose U_e is a neighborhood of w_e in D. Then the probability that Y_δ or q_δ visits $(D\setminus U_e)\cap\partial^\#_\varepsilon D$ after visiting F_D tends to 0 as $\varepsilon\to 0$ and $\delta\to 0$ along $\mathcal{M}$.*

COROLLARY 8.1. *Suppose U_e is a neighborhood of $f(w_e)$ in Ω. Then the probability that β_δ visits $(\Omega\setminus U_e)\cap\partial^\#_\varepsilon\Omega$ after visiting F_Ω tends to 0 as $\varepsilon\to 0$ and $\delta\to 0$ along $\mathcal{M}$.*

LEMMA 8.2. *Suppose U_e is a neighborhood of $f(w_e)$ in Ω. Let $T^\delta_{F_\Omega}$ (resp. T^δ_e) be the first time β_δ hits F_Ω (resp. $\overline{U_e}$). For any $\varepsilon>0$, there are $\varepsilon_0,\delta_0>0$ such that for $\delta<\delta_0$, with probability greater than $1-\varepsilon$, β_δ satisfies that if $|\beta_\delta(t_1)-\beta_\delta(t_2)|<\varepsilon_0$ for some $t_1,t_2\in[T^\delta_{F_\Omega},T^\delta_e]$, then $\operatorname{diam}(\beta_\delta([t_1,t_2]))<\varepsilon$.*

In Section 7.2, keep $\mathcal{B}$ unchanged, but redefine $\mathcal{A}$ to be the set of crosscuts α in $\mathbb{H}$ such that α strictly encloses 0, $H(\alpha)\subset\Omega\setminus\{f(\infty)\}$, and $H(\alpha)$ is bounded away from $f(w_e)$. Then Theorems 7.2, 7.3 and 7.4 still hold. Using Lemma 8.3 below, we can prove Theorem 7.5 with z_e replaced by w_e, and finally Theorem 4.2.

LEMMA 8.3. *For $r>0$, the probability that q_δ visits $D\setminus\mathbf{B}(w_e;r)$ after $D\cap\mathbf{B}(w_e;\varepsilon)$ tends to 0 as $\varepsilon\to 0$ and $\delta\to 0$ along $\mathcal{M}$.*

PROOF. Let Y^r_w be that X_w conditioned to leave D through $[\delta,0]$. Let $q^r_\delta=\mathrm{LE}(Y^r_{w^\delta_e})$. Then q^r_δ has the same distribution as the reversal of q_δ. Let P_Y be the probability that $Y^r_{w^\delta_e}$ visits $D\cap\mathbf{B}(w_e;\varepsilon)$ after $D\setminus\mathbf{B}(w_e;r)$. We suffice to prove that P_Y tends to 0 as $\varepsilon\to 0$ and $\delta\to 0$ along $\mathcal{M}$.

We may assume that $\varepsilon<r<r_e/2$, where $r_e>0$ satisfies $\mathbf{B}(w_e;r_e)\cap D=(w_e+a\mathbb{H})\cap\mathbf{B}(w_e;r_e)$ for some $a\in\{\pm1,\pm i\}$. Let $Q(w)$ be the probability that X_w leaves D through $[\delta,0]$. Let P_X be the probability that $X_{w^\delta_e}$ visits $D\cap\mathbf{B}(w_e;\varepsilon)$ after $D\setminus\mathbf{B}(w_e;r)$, and leaves D through $[\delta,0]$. Then $P_Y=P_X/Q(w^\delta_e)$. Let $Q_r(w)$ be the probability that X_w reaches $D\setminus\mathbf{B}(w_e;r)$. Then $P_X\le Q_r(w^\delta_e)\sup\{Q(w):w\in\mathbf{B}(w_e;\varepsilon)\cap D\}$. Choose $z_0\in D$ and $r_0>0$ such that $B:=\overline{\mathbf{B}(z_0;r_0)}\subset D$. Let $Q_B(w)$ be the probability that X_w visits B before ∂D. Then $Q(w^\delta_e)\ge Q_B(w^\delta_e)\inf\{Q(w):w\in B\}$. Thus

$$P_Y\le\frac{Q_r(w^\delta_e)}{Q_B(w^\delta_e)}\cdot\frac{\sup\{Q(w):w\in\mathbf{B}(w_e;\varepsilon)\cap D\}}{\inf\{Q(w):w\in B\}}. \tag{8.1}$$

Let w_0^δ be a vertex of D^δ closest to z_0. As $\delta \to 0$ along $\mathcal{M}$, $Q(\cdot)/Q(w_0^\delta)$ converges to the generalized Poisson kernel P in D with the pole at 0_+, normalized by $P(z_0)=1$, uniformly on any subset of $\overline{D}$ that is bounded away from 0_+. Thus

$$\sup\{Q(w): w \in \mathbf{B}(w_e;\varepsilon) \cap D\}/\inf\{Q(w): w \in B\} \to 0 \tag{8.2}$$

as $\varepsilon \to 0$ and $\delta \to 0$ along $\mathcal{M}$.

As $\delta \to 0$ along $\mathcal{M}$, Q_B converges to $H(D \setminus B, \partial B; \cdot)$ in $D \setminus B$. Let $U = D \cap \mathbf{B}(w_e; r)$ and $\rho = \{|z - w_e| = r\} \cap D$. As $\delta \to 0$ along $\mathcal{M}$, Q_r converges to $H(U, \rho; \cdot)$ in U. Since ∂D is flat near w_e, so Q_r and Q_B extend to be a discrete harmonic function on $\delta\mathbb{Z}^2 \cap (D \cup \mathbf{B}(w_e; r))$. So the discrete partial derivatives of Q_B and Q_r converge to the continuous partial derivatives of $H(D \setminus B, \partial B; \cdot)$ and $H(U, \rho; \cdot)$, respectively, in $D \cup \mathbf{B}(w_e; r)$. Thus $Q_B(w_e^\delta)/\delta \to \partial_{\mathbf{n}} H(D \setminus B, \partial B; w_e)$ and $Q_r(w_e^\delta)/\delta \to \partial_{\mathbf{n}} H(U, \rho; w_e)$ as $\delta \to 0$ along $\mathcal{M}$. So we have

$$Q_r(w_e^\delta)/Q_B(w_e^\delta) \to \partial_{\mathbf{n}} H(U, \rho; w_e)/\partial_{\mathbf{n}} H(D \setminus B, \partial B; w_e) \tag{8.3}$$

as $\delta \to 0$ along $\mathcal{M}$. The conclusion follows from (8.1), (8.2) and (8.3). $\square$

In the proof of Lemma 8.3, we consider the LERW curve q_δ^r, which has the same distribution as the reversal of q_δ. If ∂D is flat near 0, then we have the convergence of q_δ^r to a continuous LERW$(D; w_e \to 0_+)$ trace. From the conformal invariance of continuous LERW, we have the reversibility of continuous LERW.

COROLLARY 8.2. *Suppose $w_1 \ne w_2$ are two prime ends of D. For $j = 1, 2$, suppose $\gamma_j(t)$, $0 < t < S_j$, is an* LERW$(D; w_j \to w_{3-j})$ *trace. Then there is a random continuous decreasing function u_r that maps $(0, S_1)$ onto $(0, S_2)$ such that $(\gamma_1 \circ u_r^{-1}(t), 0 < t < S_2)$ has the same distribution as $(\gamma_2(t), 0 < t < S_2)$.*

8.2. *When the target is a side arc.* Now we consider the case that the target is a side arc. We use the notation and boundary conditions given in Section 4.2 for the discrete LERW aimed at a side arc I_e. Let f map D conformally onto an almost $\mathbb{H}$ domain Ω such that $f(0_+) = 0$.

We will modify the propositions in Sections 6 and 7 to prove Theorem 4.2 in this case. Recall that if I_e is not a whole side, then we only consider D^δ for $\delta \in \mathcal{M}$, so the words "$\delta < *$" should be replaced by "$\delta \in \mathcal{M}$ and $\delta < *$," and the words "$\delta \to 0$" should be replaced by "$\delta \to 0$ along $\mathcal{M}$." If I_e is a whole side, we may consider D^δ for any small δ. For consistency, let $\mathcal{M} = (0, \infty)$ in this case.

Let X_t^ξ and $P^\xi(t, x, \cdot)$ be notation in the case that the target is a side arc defined in Sections 3.4 and 4.1. Then all lemmas in Section 6.1 still hold. For

Proposition 6.1, redefine P_X to be the generalized Poisson kernel in D_X with the pole at $\mathrm{Tip}(X)$, normalized by $\int_{I_e} \partial_{\mathbf{n}} P_X(z)\,ds(z) = 1$; let h_X be defined on $V(D^\delta)$ that satisfies $h_X \equiv 0$ on $V_\partial(D^\delta) \cup \mathrm{Set}(X) \setminus \{\mathrm{Tip}(X)\}$, $\Delta_{D^\delta} h_X \equiv 0$ on $V_I(D^\delta) \setminus \mathrm{Set}(X)$, and $\sum_{w \in I_e^\delta} \Delta_{D^\delta} h_X(w) = 1$. Then Proposition 6.1 should be restated as Proposition 8.2, which together with Proposition 2.1 implies Proposition 6.2, and then all theorems in Section 6.2.

PROPOSITION 8.2. *For any $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta \in \mathcal{M}$ and $\delta < \delta_0$, then for any $X \in L^\delta$, and any $w \in V(D^\delta) \cap (D \setminus H(\rho_2))$, we have $|h_X(w) - P_X(w)| < \varepsilon$.*

PROOF. Let z_0, w_0^δ, h_X^0 and P_X^0 be as in the proof of Proposition 8.1. Then we have the convergence of h_X^0 to P_X^0. Now we suffice to prove that $\sum_{w \in I_e^\delta} \Delta_{D^\delta} h_X^0(w) \to \int_{I_e} \partial_{\mathbf{n}} P_X^0(z)\,ds(z)$ as $\delta \to 0$ along $\mathcal{M}$.

We first consider the case that I_e is a whole side. Then we may choose a polygonal Jordan curve σ in D that disconnects I_e from other sides of D, such that σ is disjoint from ρ_2, and every line segment on σ is parallel to either the x or y axis. Let $U(\sigma)$ denote the doubly connected domain bounded by I_e and σ. Since P_X^0 is bounded and harmonic in $U(\sigma)$, so we have

$$\int_{I_e} \partial_{\mathbf{n}} P_X^0(z)\,ds(z) = -\int_\sigma \partial_{\mathbf{n}} P_X^0(z)\,ds(z), \tag{8.4}$$

where $\mathbf{n}$ is the inward unit normal vector on the boundary of $U(\sigma)$.

Suppose δ is smaller than the Euclidean distance from σ to ρ_2 and any side of D. Let G be the subgraph of D^δ spanned by the set of edges in D^δ that is incident to at least one vertex in $U(\sigma)$. Let A be the set of vertices of G on I_e, and let B be the set of vertices of G in $D \setminus U(\sigma)$. From Lemma 6.6, we have

$$\sum_{w \in I_e^\delta} \Delta_{D^\delta} h_X^0(w) = -\sum_{(w,w') \in \mathcal{P}_\sigma} (h_X^0(w) - h_X^0(w')), \tag{8.5}$$

where $\mathcal{P}_\sigma = \{(w, w') : w \in V(D^\delta) \cap U(\sigma), w' \in V_I(D^\delta) \setminus U(\sigma), w \sim w'\}$.

Since the discrete partial derivatives of h_X^0 converge to the corresponding partial derivatives of P_X^0 uniformly on σ, so as $\delta \to 0$, we have

$$\sum_{(w,w') \in \mathcal{P}_\sigma} (h_X^0(w) - h_X^0(w')) \to \int_\sigma \partial_{\mathbf{n}} P_X^0(z)\,ds(z).$$

This together with (8.4) and (8.5) finishes the proof of the first case.

The second case is that I_e is not a whole side. We assume that ∂D is flat near the two ends z_e^1 and z_e^2 of I. We may choose a polygonal cross-cut σ in D composed of line segments parallel to x or y axis, such that its two ends approach to z_e^1 and z_e^2, respectively, and σ disconnects I_e from $\partial D\setminus\overline{I_e}$. Since P_X^0 is bounded and harmonic in $H(\sigma)$, so $\int_{I_e}\partial_{\mathbf{n}}P_X^0(z)\,ds(z)=-\int_\sigma\partial_{\mathbf{n}}P_X^0(z)\,ds(z)$, where $\mathbf{n}$ is the inward unit normal vector on the boundary of $H(\sigma)$. An argument similar to the last paragraph gives

$$\sum_{w\in I_e^\delta}\Delta_{D^\delta}h_X^0(w)=-\sum_{(w,w')\in\mathcal{P}_\sigma}(h_X^0(w)-h_X^0(w')),$$

where $\mathcal{P}_\sigma=\{(w,w'):w\in V(D^\delta)\cap H(\sigma),w'\in V_I(D^\delta)\setminus H(\sigma),w\sim w'\}$. So we suffice to show that

$$\sum_{(w,w')\in\mathcal{P}_\sigma}(h_X^0(w)-h_X^0(w'))\to\int_\sigma\partial_{\mathbf{n}}P_X^0(z)\,ds(z) \tag{8.6}$$

as $\delta\to 0$ along $\mathcal{M}$. To prove this, we use the flat boundary conditions at w_e^1 and w_e^2 to extend h_X^0 and P_X^0 harmonically across ∂D near w_e^1 and w_e^2. Since $\overline{\sigma}$ is compact in the extended domain: D unions two balls centered at w_e^1 and w_e^2, respectively, so we get the uniform convergence of the discrete partial derivatives of h_X^0 to the corresponding partial derivatives of P_X^0 on σ. Then we are done. $\square$

In Section 7, redefine X_w to be a random walk on D^δ started from w, stopped when it hits $V_\partial(D^\delta)$, and Y_w to be that X_w conditioned to hit $V_\partial(D^\delta)$ at I_e^δ. Then q_δ is the loop-erasure of Y_δ. Lemma 7.2 still holds, and Corollaries 7.1 and 7.2 immediately follow from this lemma. Let F_D (resp. F_Ω) be a compact subset of $D\setminus\{\infty\}$ [resp. $\Omega\setminus\{f(\infty)\}$]. Lemma 7.4 still holds, and Lemma 8.1, Corollary 8.1, and Lemma 8.2 hold with w_e replaced by I_e. Using this, we can obtain Theorem 7.1.

In Section 7.2, keep $\mathcal{B}$ unchanged, but redefine $\mathcal{A}$ to be the set of crosscuts α in $\mathbb{H}$ that strictly encloses 0, such that $H(\alpha)\subset\Omega\setminus\{f(\infty)\}$ and $H(\alpha)$ is bounded away from $f(I_e)$. Then we have Theorems 7.2, 7.3 and 7.4. Let $\bar\gamma_\delta=f^{-1}\circ\beta_\delta$ and $\bar\gamma_0=f^{-1}\circ\beta_0$. Using Lemma 8.4 and Theorem 8.1 below, we can prove Theorem 4.2 in this case.

LEMMA 8.4. *Let $T_{F_D}^\delta$ be the first time that $\bar\gamma_\delta$ visits F_D. For $a>0$, let $\partial_a D=\{z\in D:\operatorname{dist}(z,\partial D)<a\}$. For any $\varepsilon\in(0,1)$, there are $\varepsilon_0,\delta_0>0$ such that if $\delta\in\mathcal{M}$ and $\delta<\delta_0$, then with probability greater than $1-\varepsilon$, if $\bar\gamma_\delta(t_0)\in\partial_{\varepsilon_0}D$ for some $t_0\ge T_{F_D}^\delta$, then $\bar\gamma_\delta(t)\in\mathbf{B}(\bar\gamma_\delta(t_0),\varepsilon)$ for $t\ge t_0$.*

PROOF. Since $\bar\gamma_\delta$ is a time-change of q_δ, which is the loop-erasure of Y_δ, so we suffice to prove this lemma with Y_δ replacing $\bar\gamma_\delta$. We first consider the case that I_e is not a whole side. Choose $r>0$ such that $D\cap\mathbf{B}(w_e^j,3r)=(w_e^j+a^j\mathbb{H})\cap\mathbf{B}(w_e^j,3r)$ for $j=1,2$, where $a^1,a^2\in\{\pm1,\pm i\}$, and $\mathbf{B}(w_e^1;3r)$ is disjoint from $\mathbf{B}(w_e^2;3r)$. Let $B^j=\overline{\mathbf{B}(w_e^j;r)}$ and $\sigma^j=D\cap\partial B^j$, $j=1,2$.

Let $Q(w)$ be the probability that X_w hits ∂D at I_e^δ. Let $Q_r(w)$ be the probability that X_w visits $B_1 \cup B_2$ before leaving D. Then Q and Q_r converge to $H(D, I_e; \cdot)$ and $H(D \setminus (B^1 \cup B^2), \sigma^1 \cup \sigma^2; \cdot)$, respectively, uniformly on F_D. Thus $Q_r(w)/Q(w) \to 0$ as $r \to 0$ and $\delta \to 0$ along $\mathcal{M}$, uniformly in $w \in F_D$. Note that $Q_r(w)/Q(w)$ is the probability that Y_w visits $B^1 \cup B^2$. From the Markov property of Y, the probability that Y_δ visits $B^1 \cup B^2$ after F_D tends to 0 as $r \to 0$ and $\delta \to 0$ along $\mathcal{M}$. So we may choose $r, \delta_e > 0$ such that $\mathbf{P}[\mathcal{E}_e^\delta] < \varepsilon/3$ if $\delta \in \mathcal{M}$ and $\delta < \delta_e$, where $\mathcal{E}_e^\delta$ is the event that Y_δ visits $B^1 \cup B^2$ after F_D.

For $j = 1, 2$, every point on $[w_e^j - 2a^j r, w_e^j + 2a^j r]$ corresponds to a prime end of D. Since w_e^1 and w_e^2 are end points of I_e, so $I_e \cap [w_e^j - 2a^j r, w_e^j + 2a^j r] = [w_e^j, w_e^j + 2c^j a^j r]$ for some $c^j \in \{\pm 1\}$, $j = 1, 2$. For $j = 1, 2$, let $z^j = w_e^j - c^j a^j r$; then z^j is the end point of σ^j that does not lie on I_e. For $j = 1, 2$, choose $\theta_1^j \ne \theta_2^j \in \sigma^j$ such that θ_1^j is closer to z^j than θ_2^j, and let ρ_k^j denote the open arc on σ^j bounded by z^j and θ_k^j, $k = 1, 2$. We may find two closed simple curves ρ_1^0 and ρ_2^0 in D such that for $k = 1, 2$, θ_k^1 and θ_k^2 are end points of ρ_k^0, $\rho_k^0 \cap \sigma^j = \{\theta_k^j\}$, $j = 1, 2$; $\rho_1^0 \cap \rho_2^0 = \varnothing$; and $\rho_1 := \rho_1^0 \cup \rho_1^1 \cup \rho_1^2$ disconnects I_e from any side of D that does not contain I_e, and so ρ_1 is a crosscut in D, and $H(\rho_1)$ is a neighborhood of I_e. Let $\rho_2 = \rho_2^0 \cup \rho_2^1 \cup \rho_2^2$. Then ρ_2 is also a crosscut in D, and $H(\rho_2) \subset H(\rho_1)$.

For $j = 1, 2$, let $\rho_3^j = \sigma^j \setminus \rho_2^j$. Let $\rho_3 = \rho_2^0 \cup \rho_3^1 \cup \rho_3^2$ and $\rho_{1.5} = \rho_1^0 \cup (w_e^1, \theta_1^1] \cup (w_e^2, \theta_1^2]$. Then ρ_3 and $\rho_{1.5}$ are also crosscuts in D, $H(\rho_3) \subset H(\rho_{1.5})$, and $d_1 := \operatorname{dist}(\rho_3, \rho_{1.5}) > 0$. From Lemma 7.1, there are $\delta_1, \varepsilon_1 > 0$ such that if $\delta \in \mathcal{M}$, $\delta < \delta_1$, and $w \in \partial_{\varepsilon_1} D$, then the probability that X_w leaves $\mathbf{B}(w; (d_1/2) \wedge (\varepsilon/3))$ is less than $\varepsilon/6$. For $w \in H(\rho_3)$, if X_w hits $V_\partial(D^\delta) \setminus I_e^\delta$, then X_w must intersect both ρ_3 and $\rho_{1.5}$, so X_w must leave $\mathbf{B}(w; d_1/2)$ before it hits ∂D. Thus if $\delta \in \mathcal{M}$, $\delta < \delta_1$ and $w \in H(\rho_3) \cap \partial_{\varepsilon_1} D$, then $Q(w) \ge 1 - \varepsilon/6 \ge 1/2$. Since Y_w is X_w conditioned to hit I_e^δ, so the probability that Y_w leaves $\mathbf{B}(w; \varepsilon/3)$ before it hits ∂D is at most 2 times the probability that X_w leaves $\mathbf{B}(w; \varepsilon/3)$ before it hits ∂D, and so is less than $\varepsilon/3$ when $\delta < \delta_1$. From the Markov property of Y_w, if $\delta \in \mathcal{M}$ and $\delta < \delta_1$, then with probability greater than $1 - \varepsilon/3$, Y_δ satisfies that if $Y_\delta(t_1) \in H(\rho_3) \cap \partial_{\varepsilon_1} D$, then $Y_\delta(t) \in \mathbf{B}(Y_\delta(t_1); \varepsilon)$ for $t \ge t_1$. Let $\mathcal{E}_1^\delta$ denote this event.

Let $U_e = H(\rho_2) \setminus \rho_2$. Then U_e is a neighborhood of I_e in D. From Lemma 8.1, there are $\delta_2, \varepsilon_2 > 0$ such that if $\delta \in \mathcal{M}$ and $\delta < \delta_2$, then with probability greater than $1 - \varepsilon/3$, Y_δ does not visit $\partial_{\varepsilon_2} D \setminus U_e$ after $T_{F_D}^\delta$. Let $\mathcal{E}_2^\delta$ denote this event.

Let $\delta_0 = \delta_e \wedge \delta_1 \wedge \delta_2$, $\varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2$, and $\mathcal{E}^\delta = \mathcal{E}_1^\delta \cap \mathcal{E}_2^\delta \setminus \mathcal{E}_e^\delta$. Suppose $\delta \in \mathcal{M}$ and $\delta < \delta_0$. Then $\mathbf{P}[\mathcal{E}^\delta] > 1 - \varepsilon$. Assume $\mathcal{E}^\delta$ occurs. Suppose $Y_\delta(t_0) \in \partial_{\varepsilon_0} D$ for some $t_0 \ge T_{F_D}^\delta$. Since $\delta < \delta_2$ and $\mathcal{E}_2^\delta$ occurs, so $Y_\delta(t_0) \in U_e$. Since $\delta < \delta_e$ and $\mathcal{E}_e^\delta$ does not occur, so $Y_\delta(t_0) \in H(\rho_2) \setminus (B^1 \cup B^2) \subset H(\rho_3)$. Since $\delta < \delta_1$ and $\mathcal{E}_1^\delta$ occurs, and $Y_\delta(t_0) \in H(\rho_3) \cap \partial_{\varepsilon_3} D$, so $Y_\delta(t) \in \mathbf{B}(Y_\delta^\delta(t_0); \varepsilon)$ for $t \ge t_0$.

The case that I_e is a whole side is easier. We may choose a Jordan curve ρ in D that disconnects I_e from other sides of D. Let U_e denote the domain bounded by I_e and ρ. From the argument used in the first part of the proof, we have $\delta_1, \varepsilon_1 > 0$ such that if $\delta \in \mathcal{M}$ and $\delta < \delta_1$, then with probability greater than $1-\varepsilon/3$, Y_δ satisfies that if $Y_\delta(t_1) \in U_e$, then $Y_\delta(t) \in \mathbf{B}(Y_\delta(t_1); \varepsilon)$ for $t \ge t_1$. Let $\mathcal{E}_1^\delta$ denote this event. From Lemma 8.1, there are $\delta_2, \varepsilon_2 > 0$ such that if $\delta \in \mathcal{M}$ and $\delta < \delta_2$, then with probability greater than $1-\varepsilon/3$, Y_δ does not visit $\partial_{\varepsilon_2} D \setminus U_e$ after $T_{F_D}^\delta$. Let $\mathcal{E}_2^\delta$ denote this event. Let $\delta_0 = \delta_1 \wedge \delta_2$, $\varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2$ and $\mathcal{E}^\delta = \mathcal{E}_1^\delta \cap \mathcal{E}_2^\delta$. Assume $\delta \in \mathcal{M}$ and $\delta < \delta_0$, then $\mathbf{P}[\mathcal{E}^\delta] > 1-\varepsilon$. If $\mathcal{E}^\delta$ occurs and $Y_\delta(t_0) \in \partial_{\varepsilon_0} D$ for some $t_0 \ge T_{F_D}^\delta$, then $Y_\delta(t) \in \mathbf{B}(Y_\delta(t_0), \varepsilon)$ for $t \ge t_0$. $\square$

THEOREM 8.1. *Almost surely* $\lim_{t\to S_0} \gamma_0(t) = \lim_{t \to T_0} \bar{\gamma}_0(t)$ *exists and lies on* ∂D.

PROOF. Let L be the set of subsequential limits of $\bar{\gamma}_0(t)$ as $t \to T_0$, in the spherical metric. From Lemma 7.4, Theorem 7.4, and the idea in the first paragraph of the proof of Theorem 7.5, we have $\infty \notin L$ a.s. So L is the set of subsequential limits of $\bar{\gamma}_0(t)$ as $t \to T_0$, in the Euclidean metric. From Theorem 3.1(ii), we have $L \cap \partial D \neq \varnothing$ a.s. From Theorem 7.4, Lemma 8.4, and the idea in the second paragraph of the proof of Theorem 7.5, we have $\operatorname{diam}(L) = 0$ a.s. So we are done. $\square$

From the property of discrete LERW and the conformal invariance of continuous LERW, we then have the following corollary.

COROLLARY 8.3. *Suppose* $\gamma(t)$, $0 \le t < S$, *is an* $\mathrm{LERW}(D; w_0 \to I_e)$ *trace; then almost surely* $\widehat{\lim}_{t\to S}\gamma(t)$, *the limit of* $\gamma(t)$ *in* $\widehat{D}$, *as* $t \to S$, *exists and lies on* I_e, *and the distribution of* $\widehat{\lim}_{t\to S}\gamma(t)$ *is the same as the distribution of the limit point in* $\widehat{D}$ *of the Brownian excursion in* D *started from* w_0 *conditioned to hit* I_e. *And if* J_e *is a subarc of* I_e, *then after a time-change,* $\gamma(t)$ *conditioned on the event that* $\widehat{\lim}_{t\to S}\gamma(t) \in J_e$ *has the same distribution as an* $\mathrm{LERW}(D; w_0 \to J_e)$ *trace.*

QUESTION. Can we prove Theorem 7.5, Corollary 8.2 and Corollary 8.3 directly from the definition of continuous LERW?

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