Theorem 1 (Squeeze Lemma, Exercise 8.5)**.** *Suppose three sequences* (a_n)*,* (b_n)*,* (s_n) *satisfy that there is* $N_0 \in \mathbb{N}$ *such that for all* $n > N_0$*,* $a_n \le s_n \le b_n$*. If* (a_n) *and* (b_n) *both converge to the same number* s*, then* (s_n) *also converges to* s*.*

Discussion We want to arrive at the inequality $|s_n - s| < \varepsilon$, which is equivalent to $s - \varepsilon < s_n < s + \varepsilon$. In order to have $s_n < s + \varepsilon$, we want to use $s_n \le b_n$ and $b_n < s + \varepsilon$. In order to have $s_n > s - \varepsilon$, we want to use $s_n \ge a_n$ and $a_n > s - \varepsilon$. The inequalities $s_n \le b_n$ and $s_n \ge a_n$ are satisfied if $n > N_0$. The inequalities $b_n < s + \varepsilon$ and $a_n > s - \varepsilon$ can be obtained for big n using the convergence of (a_n) and (b_n).

Proof. Let $\varepsilon > 0$. Since (a_n) converges to s, there is $N_1 \in \mathbb{N}$ such that for $n > N_1$, $|a_n - s| < \varepsilon$, which implies that $a_n > s - \varepsilon$. Since (b_n) converges to s, there is $N_2 \in \mathbb{N}$ such that for $n > N_2$, $|b_n - s| < \varepsilon$, which implies that $b_n < s + \varepsilon$. Let $N = \max\{N_0, N_1, N_2\}$. If $n > N$, then $a_n \le s_n \le b_n$, $a_n > s - \varepsilon$, and $b_n < s + \varepsilon$, which together imply that $s - \varepsilon < s_n < s + \varepsilon$. So we get $|s_n - s| < \varepsilon$ for $n > N$. □

For $x \in [0, \infty)$ and $r = p/q \in \mathbb{Q}$ with $p, q \in \mathbb{N}$, we define $x^r = (x^{1/q})^p$. We have $(x^{r_1})^{r_2} = x^{r_1 r_2}$, and for $x, y \ge 0$, $x < y$ if and only if $x^r < y^r$.

Theorem 2 (Theorem 9.7)**.** *We have the following basic examples.*

(a) $\lim_{n\to\infty} (\frac{1}{n})^r = 0$ *for any* $r \in \mathbb{Q}$ *and* $r > 0$*.*

(b) $\lim_{n\to\infty} a^n = 0$ *if* $|a| < 1$*.*

(c) $\lim_{n\to\infty} n^{1/n} = 1$*.*

(d) $\lim_{n\to\infty} a^{1/n} = 1$ *if* $a > 0$*.*

Proof. (a) Let $\varepsilon > 0$. Then $\varepsilon^{1/r} > 0$. Since $1/n \to 0$, there is $N \in \mathbb{N}$ such that for $n > N$, $|1/n - 0| < \varepsilon^{1/r}$, i.e., $1/n < \varepsilon^{1/r}$, which then implies that $|(1/n)^r - 0| = (1/n)^r < (\varepsilon^{1/r})^r = \varepsilon$.

(b) If $a = 0$, then obviously $0^n \to 0$. Suppose $a \ne 0$. Then $|a| = \frac{1}{1+b}$ for some $b > 0$. We use the binomial theorem

$$(1+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Especially, $\binom{n}{0} = 1$ and $\binom{n}{1} = n$. Also, $\binom{n}{k} \ge 0$ for all $0 \le k \le n$. So $(1+b)^n \ge 1 + nb > nb$. Thus, $|a^n| = |a|^n < \frac{1}{nb}$. Since $1/n \to 0$, we get $\frac{1}{nb} \to 0$. By the corollary of the squeeze lemma, we get $a^n \to 0$.

(c) Let $s_n = n^{1/n} - 1$. Then $s_n \ge 0$ and $n = (1 + s_n)^n$. We use the binomial theorem again and the fact that $\binom{n}{2} = \frac{1}{2}n(n-1)$. So if $n \ge 2$, then

$$n = (1 + s_n)^n \ge 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 > \frac{1}{2}n(n-1)s_n^2.$$

So $s_n^2 < \frac{2}{n-1}$ for $n \geq 2$. Since $0 \leq s_n^2 < \frac{2}{n-1}$ for $n \geq 2$ and $\frac{2}{n-1} \to 0$, by squeeze lemma, $s_n^2 \to 0$. So $s_n = \sqrt{s_n^2} \to 0$. Thus, $n^{1/n} = 1 + s_n \to 1$.

(d) If $a \geq 1$, then $1 \leq a^{1/n} \leq n^{1/n}$ for $n \geq a$. By (iii) and squeeze lemma, we get $a^{1/n} \to 1$. If $0 < a \leq 1$, then $1/a \geq 1$, and so $(1/a)^{1/n} \to 1$. Thus, $a^{1/n} = 1/(1/a)^{1/n} \to 1/1 = 1$. □

Infinite Limits

Definition 1. Let (s_n) be a sequence of real numbers. We write $\lim_{n\to\infty} s_n = +\infty$ or $s_n \to +\infty$ if

$$\forall M > 0, \quad \exists N \in \mathbb{N}, \quad \text{such that } n > N \text{ implies } s_n > M. \tag{1}$$

When this holds, we say that (s_n) *diverges* to $+\infty$. We write $\lim_{n\to\infty} s_n = -\infty$ or $s_n \to -\infty$ if

$$\forall M < 0, \quad \exists N \in \mathbb{N}, \quad \text{such that } n > N \text{ implies } s_n < M. \tag{2}$$

When this holds, we say that (s_n) *diverges* to $-\infty$.

Since for any $x \in \mathbb{R}$, there are $M_1 > 0$ and $M_2 < 0$ such that $M_2 < x < M_1$, the assumption on the M in (1) and (2) can all be replaced by $M \in \mathbb{R}$. We may write ∞ for $+\infty$.

Example 1. If $s_n = n$ for each $n \in \mathbb{N}$, then $s_n \to +\infty$. To see this is true, for any $M > 0$, by Archimedean property, there is $N \in \mathbb{N}$ such that $N > M$. Then for any $n > N$, $s_n = n > N > M$.

Recall that we say that $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ for some $s \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \text{such that } n > N \text{ implies } |s_n - s| < \varepsilon. \tag{3}$$

If any of (1,2,3) hold, we say that $\lim_{n\to\infty} s_n$ exists. But only when (3) holds, we say that (s_n) converges.

When $s_n \to s \in \mathbb{R}$, (s_n) is bounded. This is not the case if $s_n \to +\infty$ or $s_n \to -\infty$. If $s_n \to +\infty$, then (s_n) is not bounded above since for any $M \in \mathbb{R}$ there is n such that $s_n > M$. But (s_n) is bounded below. To see this, taking $M = 1$, we get an $N \in \mathbb{N}$ such that $s_n > 1$ for all $n > N$. Then $\min\{s_1, \dots, s_N, 1\}$ is a lower bound of $\{s_n : n \in \mathbb{N}\}$. Similarly, if $s_n \to -\infty$, then (s_n) is bounded above and not bounded below.

Example 2. The sequence $s_n = (-1)^n$, $n \in \mathbb{N}$, has no limit. We have seen that it is divergent. It can not diverge to $+\infty$ or $-\infty$ since it is bounded.

Remark 1. The limit property of a sequence is not affected by a change of finitely many elements. This means that, if we have two sequences (s_n) and (t_n) and a number $N_0 \in \mathbb{N}$ such that $s_n = t_n$ for $n > N_0$, then if $\lim s_n$ exists, then $\lim t_n$ also exists and equals $\lim s_n$. To see this, suppose $s_n \to s \in \mathbb{R}$. Let $\varepsilon > 0$. There is $N_s \in \mathbb{N}$ such that $|s_n - s| < \varepsilon$ if $n > N_s$. Let $N_t = \max\{N_s, N_0\}$. If $n > N_t$, then $n > N_s$ and $n > N_0$. From $n > N_0$ we know $t_n = s_n$; from $n > N$ we know $|s_n - s| < \varepsilon$, which together imply that $|t_n - s| < \varepsilon$. If $s_n \to +\infty$ or $-\infty$, the argument is similar. Thus, when we talk about $\lim s_n$, we may allow that the s_n to be not defined for finitely many n.

A related fact is that when $\lim s_n$ exists, then $\lim s_{n+1}$ also exists, and two limits are equal. The converse is also true.

Remark 2. We have another way to understand the statements (3,1,2). Consider a game played by Player A and Player B. To describe the limit $s_n \to s \in \mathbb{R}$, let the sequence (s_n) and the number s be fixed. Then the two players choose the following numbers in order:

1. Player A chooses $\varepsilon > 0$.
2. Player B chooses $N \in \mathbb{N}$.
3. Player A chooses $n \in \mathbb{N}$ with $n > N$.

The rule is: if $|s_n - s| < \varepsilon$, then Player B wins the game; otherwise, Player A wins the game.

If $s_n \to s$, then Player B has a strategy to always win the game. Otherwise Player A can always win the game.

To describe $\lim s_n = +\infty$, we modify the game such that in the first step Player A chooses $M > 0$, and the final rule is changed such that B wins the game if $s_n > M$. If $s_n \to +\infty$, then Player B has a strategy to win the game; otherwise Player A can always win the game. One may similarly describe a game for $\lim s_n = -\infty$.