Time-reversal of Multiple-force-point $\mathrm{SLE}_\kappa(\underline{\rho})$ with All Force Points Lying on the Same Side

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June 13, 2019

Abstract

We define intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ and reversed intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ processes using Appell-Lauricella multiple hypergeometric functions, and use them to describe the time-reversal of multiple-force-point chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curves in the case that all force points are on the boundary and lie on the same side of the initial point, and κ and $\underline{\rho}=(\rho_1,\dots,\rho_m)$ satisfy that either $\kappa\in(0,4]$ and $\sum_{j=1}^k \rho_j>-2$ for all $1\le k\le m$, or $\kappa\in(4,8)$ and $\sum_{j=1}^k \rho_j\ge\frac{\kappa}{2}-2$ for all $1\le k\le m$.

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1 Introduction

The Schramm-Loewner evolution (SLE), first introduced by Oded Schramm in 1999 ([15]), is a one-parameter ($\kappa \in (0,\infty)$) family of measures on non-self-crossing curves, which has received a lot of attention over the past two decades. It has been shown that, modulo time parametrization, a number of discrete random paths on grids have SLE with different parameters as their scaling limits. We refer the reader to Lawler's textbook [3] for basic properties of SLE.

Based on the convergence of various discrete lattice models to SLE, Rohde and Schramm conjectured (cf. [13]) that, for $\kappa \in (0,8)$, chordal SLE_κ (growing in a simply connected domain from one boundary point to another) satisfies reversibility, i.e., the time reversal of a chordal SLE_κ curve is also a chordal SLE_κ curve, modulo a time reparametrization.

The conjecture was first proved for $\kappa \in (0,4]$ in [23], which constructed a commutation coupling of two chordal SLE_κ curves growing towards each other, and used the coupling to show that the two curves are time-reversal of each other. The conjecture for $\kappa \in (4,8)$ was proved by [7] using the celebrated imaginary geometry theory.

People have also worked on the reversibility property of other types of SLE. Radial SLE grows in a simply connected domain from a boundary point to an interior point, and obviously does not satisfy reversibility. However, one may consider its close relative: whole-plane SLE, which grows in the Riemann sphere from one (interior) point to another point. Conditionally on an initial segment of a whole-plane SLE_κ curve, the rest of the curve is a radial SLE_κ curve in the remaining domain. The reversibility of whole-plane SLE_κ was first proved for $\kappa \in (0,4]$ in [20], and later for $\kappa \in (4,8)$ in [6]. The work [20] also describes the time-reversal of radial SLE_κ for $\kappa \in (0,4]$ although the reversibility does not hold.

$\text{SLE}_\kappa(\underline{\rho})$ is another important type of SLE, whose growth is affected by some additional marked points, called force points, besides the target. They were introduced in [5] for the construction of restriction measures, and were later used in the series [9, 8, 7, 6] as building blocks of the imaginary geometry.

The paper [21] proves the reversibility of a single-force-point chordal $\text{SLE}_\kappa(\rho)$ in the case that $\kappa \in (0,4]$, $\rho \geq \kappa/2-2$, and that the only force point lies on the boundary, and is degenerate, i.e., lies immediately next to the initial point. A new process called intermediate $\text{SLE}_\kappa(\rho)$ was introduced there to describe the time-reversal of chordal $\text{SLE}_\kappa(\rho)$ in the case that the boundary force point is not degenerate. An intermediate $\text{SLE}_\kappa(\rho)$ is a two-force-point process defined using a hypergeometric function, and is different from the $\text{SLE}_\kappa(\underline{\rho})$ in [5]. It is also proved in [21] that intermediate $\text{SLE}_\kappa(\rho)$ satisfies reversibility for $\kappa \in (0,4]$ and $\rho \geq \kappa/2-2$.

The intermediate SLE was later called hypergeometric SLE or hSLE in [11] and [17]. The latter paper [17] extends the reversibility of intermediate $\mathrm{SLE}_\kappa(\rho)$ to $\kappa \in (0,8)$ and $\rho \geq \kappa/2-2$ in the case that both force points are not degenerate, and proved that intermediate $\mathrm{SLE}_\kappa(2)$ is the marginal law of a single curve in a multiple 2-SLE_κ configuration.

The papers [8, 7] established the reversibility of chordal $\mathrm{SLE}_\kappa(\rho_1,\rho_2)$ in the case that $\kappa \in (0,4)$ and $\rho_1,\rho_2 > -2$, or $\kappa \in (4,8)$ and $\rho_1,\rho_2 \geq \frac{\kappa}{2}-4$, and that the two boundary force points are both degenerate, one on each side. But those papers did not provide description of the time-reversal of chordal $\mathrm{SLE}_\kappa(\rho_1,\rho_2)$ in the case that any force point is not degenerate.

The current paper studies the time-reversal of multiple-force-point chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ in the case that all force points are boundary points and lie on the same side of the initial point. The first result of this form was obtained in [22] for $\kappa=4$, where it was shown that if the force points $\underline{v}=(v_1,\dots,v_m)$ are ordered such that v_j is closer to the initial point than v_k when $j<k$, and if the corresponding force values $\underline{\rho}=(\rho_1,\dots,\rho_m)$ satisfy that $\sum_{j=1}^k \rho_j \geq 0$ for all $1\leq k\leq m$, then the time-reversal of a chordal $\mathrm{SLE}_4(\underline{\rho})$ curve is a chordal $\mathrm{SLE}_4(-\underline{\rho},-\rho_\infty)$ curve, where $\rho_\infty = -\sum_{j=1}^m \rho_j$, the value $-\rho_j$ force point for the time-reversal is still v_j, $1\leq j\leq m$, and the value $-\rho_\infty$ force point for the time-reversal lies immediately next to the initial point of the reversal curve, i.e., the terminal point of the original curve.

Below are the main theorems of the paper, which extend the results of [21].

Theorem 1.1. *Let $v_1>\cdots>v_m \in (-\infty,0)\cup\{0^-\}$ or $v_1<\cdots<v_m\in(0,+\infty)\cup\{0^+\}$. Let $\sigma\in\{+,-\}$ be the sign of v_j's. Suppose κ and $\rho_1,\dots,\rho_m$ satisfy either*

(I) $\kappa\in(0,4]$ and for any $1\leq k\leq m$, $\sum_{j=1}^k \rho_j > -2$; or

(II) $\kappa\in(4,8)$ and for any $1\leq k\leq m$, $\sum_{j=1}^k\rho_j \geq \frac{\kappa}{2}-2$.

Let η be a chordal $SLE_\kappa(\rho_1,\dots,\rho_m)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $v_1,\dots,v_m$. Let $J(z)=-1/z$. Let $\rho_\infty=-\sum_{j=1}^m\rho_j$. Then the time-reversal of $J\circ\eta$ may be reparametrized by half-plane capacity and become a chordal Loewner curve, whose law is absolutely continuous w.r.t. that of a chordal $SLE_\kappa(-\rho_1,\dots,-\rho_m,-\rho_\infty)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $J(v_1),\dots,J(v_m),0^{-\sigma}$. Here we use the convention that $J(0^\pm)=\mp\infty$.

Theorem 1.2. *Let κ, $\rho_1,\dots,\rho_m,\rho_\infty$, $v_1,\dots,v_m$, σ, and J be as in Theorem 1.1. Let $v_\infty\in(v_m,+\infty)\cup\{+\infty\}$ if $\sigma=+$; and $\in\{-\infty\}\cup(-\infty,v_m)$ if $\sigma=-$. Let $v_j^r=J(v_j)$ and $\rho_j^r=-\rho_j$, $j\in\{1,\dots,m,\infty\}$. Here we use the convention that $J(0^\pm)=\mp\infty$ and $J(\pm\infty)=0^\mp$. Let η be an $iSLE_\kappa(\rho_1,\dots,\rho_m)$ curve (Definition 3.2) in $\mathbb{H}$ from 0 to ∞ with force points $v_1,\dots,v_m,v_\infty$. Let η^r be an $iSLE_\kappa^r(\rho_1,\dots,\rho_m)$ curve (Definition 3.4) in $\mathbb{H}$ from 0 to ∞ with force points $v_1^r,\dots,v_m^r,v_\infty^r$. Then up to a time-change, the law of the time-reversal of $J(\eta)$ agrees with the law of η^r, which is absolutely continuous w.r.t. that of a chordal $SLE_\kappa(\rho_1^r,\dots,\rho_m^r,\rho_\infty^r)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $v_1^r,\dots,v_m^r,v_\infty^r$.*

The $\mathrm{iSLE}_\kappa(\underline{\rho})$ and $\mathrm{iSLE}_\kappa^r(\underline{\rho})$ (shorthands for intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ and reversed intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$, respectively) processes will be defined using Appell-Lauricella multiple hypergeometric functions. When $v_\infty=\sigma\cdot\infty$, the η in Theorem 1.2 agrees with the η in Theorem 1.1. So

Theorem 1.1 is a special case of Theorem 1.2, and we have a description of the law of η^r in Theorem 1.1. Unless $\kappa = 4$, an $\text{iSLE}_\kappa^r(\underline{\rho})$ curve is not a chordal $\text{SLE}_\kappa^r(\underline{\rho}')$. So the time-reversal of a chordal $\text{SLE}_\kappa(\underline{\rho})$ may not be a chordal $\text{SLE}_\kappa(\underline{\rho}')$ curve.

The proof of Theorem 1.2 in the case that $\kappa \in (0,4]$ uses the stochastic coupling technique introduced in [23, 22]. We will construct a commutation coupling of an $\text{iSLE}_\kappa(\underline{\rho})$ curve with an $\text{iSLE}_\kappa^r(\underline{\rho})$ curve, and use the commutation relation to prove that the two curves are time-reversal of each other. The proof in the case that $\kappa \in (4,8)$ uses the reversibility of chordal SLE_κ established in [7]. We will show that when none of the force points is degenerate, the laws of both η and η^r are absolutely continuous w.r.t. that of a chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞, and the Radon-Nikodym derivatives are related by the map J. We will then extend the result to the case that some force points are degenerate using commutation couplings.

The definitions of $\text{iSLE}_\kappa(\underline{\rho})$ and $\text{iSLE}_\kappa^r(\underline{\rho})$ are valid for all $\kappa \in (0,8)$ and $\underline{\rho} = (\rho_1, \dots, \rho_m)$ satisfying that $\sum_{j=1}^k \rho_j > \max\{-2, \frac{\kappa}{2} - 4\}$ for $1 \le k \le m$. We believe that both theorems should hold if Condition (II) is weakened to $\kappa \in (4,8)$ and $\sum_{j=1}^k \rho_j > \frac{\kappa}{2} - 4$ for $1 \le k \le m$. Actually, we believe that the theorems should also hold if there are force points on both sides. The expected extension of Theorem 1.1 is the following conjecture.

Conjecture 1.3. *Suppose $v_1^- > \cdots > v_{m_-}^- \in (-\infty, 0) \cup \{0^-\}$ and $v_1^+ < \cdots < v_{m_+}^+ \in (0, +\infty) \cup \{0^+\}$. Let κ and ρ_j^σ, $1 \le j \le m_\sigma$, $\sigma \in \{+,-\}$, satisfy that $\kappa \in (0,8)$ and $\sum_{j=1}^k \rho_j^\sigma > \max\{-2, \frac{\kappa}{2} - 4\}$ for all $1 \le k \le m_\sigma$ and $\sigma \in \{+,-\}$. Let $\underline{\rho}^\sigma = (\rho_1^\sigma, \dots, \rho_{m_\sigma}^\sigma)$ and $\underline{v}^\sigma = (v_1^\sigma, \dots, v_{m_\sigma}^\sigma)$, $\sigma \in \{+,-\}$. Let η be a chordal $SLE_\kappa(\underline{\rho}^+, \underline{\rho}^-)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $(\underline{v}^+, \underline{v}^-)$. Let $\rho_\infty^\sigma = -\sum_{j=1}^{m_\sigma} \rho_j^\sigma$, $\sigma \in \{+,-\}$. Then the time-reversal of $J(\eta)$ may be reparametrized by half-plane capacity and become a chordal Loewner curve, whose law is absolutely continuous w.r.t. that of a chordal $SLE_\kappa(-\underline{\rho}^+, -\rho_\infty^+, -\underline{\rho}^-, -\rho_\infty^-)$ curve with force points $J(v_1^+), \dots, J(v_{m_+}^+), 0^-, J(v_1^-), \dots, J(v_{m_-}^-), 0^+$.*

The conjecture is known to be true (cf. [22, Theorem 5.5]) in the case that $\kappa = 4$ and $\rho_j^\pm$ satisfy that $\sum_{j=1}^k \rho_j^\pm \ge 0$ for all $1 \le k \le m_\pm$. In that case, the time-reversal is exactly a chordal $\text{SLE}_4(-\underline{\rho}^+, -\rho_\infty^+, -\underline{\rho}^-, -\rho_\infty^-)$ curve. For other κ, we have not found the correct definitions of two-sided $\text{iSLE}_\kappa(\underline{\rho}^+, \underline{\rho}^-)$ and $\text{iSLE}_\kappa^r(\underline{\rho}^+, \underline{\rho}^-)$ curves to make the extension of Theorem 1.2 holds, even in the simplest case that $m_+ = m_- = 1$.

Below is the outline of the rest of the paper. In the next section, we recall $\mathbb{H}$-hulls, chordal Loewner equation, chordal $\text{SLE}_\kappa(\underline{\rho})$, and multiple hypergeometric functions. In Section 3, we define $\text{iSLE}_\kappa(\underline{\rho})$ and $\text{iSLE}_\kappa^r(\underline{\rho})$ curves, and study some basic properties. In Section 4, we construct a commutation coupling of an $\text{iSLE}_\kappa(\underline{\rho})$ curve with an $\text{iSLE}_\kappa^r(\underline{\rho})$ curve. We prove the main theorems in the last section.

2 Preliminary

We first fix some notation. We write $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ for $x, y \in \mathbb{R}$. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $z_0 \in \mathbb{C}$ and $S \subset \mathbb{C}$, let

$\mathrm{rad}_{z_0}(S) = \sup\{|z-z_0| : z \in S \cup \{z_0\}\}$. Let $D \subsetneq \mathbb{C}$ be a simply connected domain. The conformal radius of D w.r.t. any $z_0 \in D$ is defined by $\mathrm{crad}_{z_0}(D) = 1/|g_{z_0}'(0)|$ if g_{z_0} maps D conformally onto $\mathbb{D}$ such that $g_{z_0}(z_0) = 0$. We write $\mathrm{crad}^{(4)}_{z_0}(D)$ for $\mathrm{crad}_{z_0}(D)/4$. Then for $x > y \in \mathbb{R}$, $\mathrm{crad}^{(4)}_x(\mathbb{C} \setminus (-\infty, y]) = |x-y|$. By Koebe's 1/4 theorem, we have $\mathrm{dist}(z_0, D^c)/4 \le \mathrm{crad}^{(4)}_{z_0}(D) \le \mathrm{dist}(z_0, D^c)$. The boundary Poisson kernel w.r.t. $z \ne w \in \partial D$ at which ∂D is smooth is defined by $H_D(z,w) = \frac{|g'(z)||g'(w)|}{(g(z)-g(w))^2}$, where g maps D conformally onto $\mathbb{H}$ such that $g(z), g(w) \ne \infty$. The value does not depend on the choice of g.

2.1 $\mathbb{H}$-hulls

A relatively closed subset K of $\mathbb{H}$ is called an $\mathbb{H}$-hull if K is bounded and $\mathbb{H} \setminus K$ is a simply connected domain. If S is a bounded subset of $\overline{\mathbb{H}}$ such that $\overline{S} \cup \mathbb{R}$ is connected, then the unbounded connected component of $\mathbb{H} \setminus \overline{S}$ is a simply connected domain, whose complement in $\mathbb{H}$ is an $\mathbb{H}$-hull, which is called the $\mathbb{H}$-hull generated by S, and denoted by $\mathrm{Hull}(S)$. For an $\mathbb{H}$-hull K, there is a unique conformal map g_K from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ such that $g_K(z) = z + \frac{c}{z} + O(\frac{1}{z^2})$ as $z \to \infty$ for some $c \ge 0$. The constant c, denoted by $\mathrm{hcap}(K)$, is called the $\mathbb{H}$-capacity of K, which is zero iff $K = \emptyset$. We write $\mathrm{hcap}_2(K)$ for $\mathrm{hcap}(K)/2$.

If $K_1 \subset K_2$ are two $\mathbb{H}$-hulls, then the quotient hull K_2/K_1 is defined as $g_{K_1}(K_2 \setminus K_1)$, which is also an $\mathbb{H}$-hull, and we have $g_{K_2} = g_{K_2/K_1} \circ g_{K_1}$ and $\mathrm{hcap}(K_2) = \mathrm{hcap}(K_2/K_1) + \mathrm{hcap}(K_1)$. From $\mathrm{hcap} \ge 0$ we see that $\mathrm{hcap}(K_1), \mathrm{hcap}(K_2/K_1) \le \mathrm{hcap}(K_2)$. If $K_1 \subset K_2 \subset K_3$ are $\mathbb{H}$-hulls, then $K_2/K_1 \subset K_3/K_1$ and $(K_3/K_1)/(K_2/K_1) = K_3/K_2$.

Let K be a non-empty $\mathbb{H}$-hull. Let $K^{\mathrm{doub}} = \overline{K} \cup \{\overline{z} : z \in K\}$, where $\overline{K}$ is the closure of K, and $\overline{z}$ is the complex conjugate of z. By Schwarz reflection principle, there is a compact set $S_K \subset \mathbb{R}$ such that g_K extends to a conformal map from $\mathbb{C} \setminus K^{\mathrm{doub}}$ onto $\mathbb{C} \setminus S_K$. Let $a_K = \min(\overline{K} \cap \mathbb{R})$, $b_K = \max(\overline{K} \cap \mathbb{R})$, $c_K = \min S_K$, $d_K = \max S_K$. Then g_K maps $\mathbb{C} \setminus (K^{\mathrm{doub}} \cup [a_K, b_K])$ conformally onto $\mathbb{C} \setminus [c_K, d_K]$. Below is an important example.

Example 2.1. For $x_0 \in \mathbb{R}$ and $r > 0$, $H := \{z \in \mathbb{H} : |z - x_0| \le r\}$ is an $\mathbb{H}$-hull with $g_H(z) = z + \frac{r^2}{z - x_0}$, $\mathrm{hcap}(H) = r^2$, $a_H = x_0 - r$, $b_H = x_0 + r$, $H^{\mathrm{doub}} = \{z \in \mathbb{C} : |z - x_0| \le r\}$, $c_H = x_0 - 2r$, $d_H = x_0 + 2r$.

The next proposition combines Lemmas 5.2 and 5.3 of [24].

Proposition 2.2. *If $L \subset K$ are two non-empty $\mathbb{H}$-hulls, then $[a_L, b_L] \subset [a_K, b_K] \subset [c_K, d_K]$, $[c_L, d_L] \subset [c_K, d_K]$, and $[c_{K/L}, d_{K/L}] \subset [c_K, d_K]$.*

Proposition 2.3. *Let $x_0 \in \mathbb{R}$ and $r > 0$. If K is an $\mathbb{H}$-hull with $\mathrm{rad}_{x_0}(K) \le r$, then $\mathrm{hcap}(K) \le r^2$, $\mathrm{rad}_{x_0}(S_K) \le 2r$, and $|g_K(z) - z| \le 3r$ for any $z \in \mathbb{C} \setminus K^{\mathrm{doub}}$.*

Proof. We have $K \subset H := \{z \in \mathbb{H} : |z - x_0| \le r\}$. So $\mathrm{hcap}(K) \le \mathrm{hcap}(H) = r^2$. From Proposition 2.2, $S_K \subset [c_K, d_K] \subset [c_H, d_H] = [x_0 - 2r, x_0 + 2r]$. So $\mathrm{rad}_{x_0}(S_K) \le 2r$. Since $g_K(z) - z$ is analytic on $\mathbb{C} \setminus K^{\mathrm{doub}}$ and tends to 0 as $z \to \infty$, by the maximum modulus principle,

$$\sup_{z\in\mathbb{C}\setminus K^{\rm doub}}|g_K(z)-z|\le \limsup_{\mathbb{C}\setminus K^{\rm doub}\ni z\to K^{\rm doub}}|(g_K(z)-x_0)-(z-x_0)|\le 2r+r=3r,$$

where the second inequality follows from the facts that $z\to K^{\rm doub}$ implies that $g_K(z)\to S_K$, $\mathrm{rad}_{x_0}(S_K)\le 2r$, and $\mathrm{rad}_{x_0}(K^{\rm doub})\le r$. □

Let $f_K=g_K^{-1}$. By [14, Lemma C.1], there is a measure μ_K supported on S_K with $|\mu_K|=\mathrm{hcap}(K)$ such that for any $w\in\mathbb{C}\setminus S_K$,

$$f_K(w)-w=\int_{S_K}\frac{-1}{w-y}\,d\mu_K(y). \tag{2.1}$$

Differentiating the equality about w, we get

$$f_K'(w)-1=\int_{S_K}\frac{1}{(w-y)^2}\,d\mu_K(y). \tag{2.2}$$

From this formula we see that $f_K'\ge 1$ on $\mathbb{R}\setminus S_K$, and is decreasing on (d_K,∞) and increasing on $(-\infty,c_K)$. So $g_K'\in(0,1]$ on $\mathbb{R}\setminus\overline{K}$, is increasing on (b_K,∞) and decreasing on $(-\infty,a_K)$. Moreover, we have the following proposition.

Proposition 2.4. *Let K be an $\mathbb{H}$-hull contained in $\{|z|\le R\}$. If $|z|\ge 7R$, then $|\log\frac{|g_K(z)|}{|z|}|\le 1.5\frac{R^2}{|z|^2}$, $|\log|g_K'(z)||\le 2.25\frac{R^2}{|z|^2}$, and $|Sg_K(z)|\le 35\frac{R^2}{|z|^4}$, where Sg_K is the Schwarzian derivative of g_K, i.e., $Sg_K=\frac{g_K'''}{g_K'}-\frac{3}{2}(\frac{g_K''}{g_K'})^2$.*

Proof. Suppose $|w|\ge 6R$. Since μ_K is supported by $S_K\subset[c_K,d_K]\subset[-2R,2R]$, and $|\mu_K|=\mathrm{hcap}(K)\le R^2$, by (2.1),

$$|f_K(w)-w|=\Big|\int_{-2R}^{2R}\frac{-1}{w-y}d\mu_K(y)\Big|\le\frac{R^2}{|w|-2R}\le\frac{R^2}{6R-2R}=\frac{R}{4}, \tag{2.3}$$

Thus, $f_K(\{|w|=6R\})$ is a Jordan curve contained in $\{|w|<6.25R\}$. Since f_K maps $\{|w|>6R\}$ onto the exterior of $f_K(\{|w|=6R\})$, which contains $\{|w|>6.25R\}$, we see that $g_K=f_K^{-1}$ maps $\{|z|>6.25R\}$ into $\{|z|>6R\}$.

Suppose now $|z|\ge 7R$. Then $|g_K(z)|\ge 6R$, and by (2.3), $|g_K(z)-z|\le\frac{R}{4}$. So $|g_K(z)|-2R\ge|z|-\frac{9}{4}R\ge\frac{19}{28}|z|$. By (2.3) again, $|g_K(z)-z|\le\frac{R^2}{|g_K(z)|-2R}\le\frac{28R^2}{19|z|}$, which implies that $|\frac{|g_K(z)|}{|z|}-1|\le\frac{28R^2}{19|z|^2}\le\frac{1}{33}$. Since $|\log x|\le 1.016|x-1|$ if $|x-1|<\frac{1}{33}$, we get $|\log\frac{|g_K(z)|}{|z|}|\le 1.016\frac{28R^2}{19|z|^2}\le\frac{3R^2}{2|z|^2}$. From (2.2) we get $|f_K'(g_K(z))-1|\le(\frac{R}{|g_K(z)|-2R})^2\le(\frac{28R}{19|z|})^2\le(\frac{4}{19})^2$. Since $f_K'(g_K(z))=1/g_K'(z)$, and $|\log x|\le 1.03|x-1|$ if $|x-1|<(\frac{4}{19})^2$, we get $|\log|g_K'(z)||\le 1.03(\frac{28R}{19|z|})^2\le 2.25\frac{R^2}{|z|^2}$.

Differentiating (2.2) further w.r.t. z twice and then replacing w by $g_K(z)$ and using $|g_K(z)|-2R \ge \frac{19}{28}|z|$ and $|z|\ge 7R$, we get

$$|f_K''(g_K(z))| \le \frac{2\cdot 28^3 R^2}{19^3|z|^3} \le \frac{8\cdot 28^2 R}{19^3|z|^2}, \quad |f_K'''(g_K(z))| \le \frac{6\cdot 28^4 R^2}{19^4|z|^4}.$$

Using that $|1/f_K'(g_K(z))| = |g_K'(z)| \le e^{2.25\frac{R^2}{|z|^2}} \le e^{\frac{2.25}{49}} \le 1.05$ and the chain rule for Schwarzian derivative, we get

$$|Sg_K(z)| = |Sf_K(g_K(z))|\cdot|g_K'(z)|^2 \le 1.5\cdot 1.05^4|f_K''(g_K(z))|^2 + 1.05^3|f_K'''(g_K(z))|.$$

Combining the above two displayed formulas, we get $|Sg_K(z)| \le 35\frac{R^2}{|z|^4}$. □

The following proposition is essentially Lemma 2.8 in [4].

Proposition 2.5. *Let ϕ be a conformal map, which maps a real open interval containing x_0 into $\mathbb{R}$, and satisfies $\phi'(x_0)>0$. Then*

$$\lim_{H\to x_0}\frac{\mathrm{hcap}(\phi(H))}{\mathrm{hcap}(H)} = |\phi'(x_0)|^2,$$

where $H\to x_0$ means that $\mathrm{rad}_{x_0}(H)\to 0$ *with H being a nonempty $\mathbb{H}$-hull.*

Definition 2.6. For $w\in\mathbb{R}$, let $\mathbb{R}_w = (\mathbb{R}\setminus\{w\})\cup\{w^+,w^-\}$. Let K be an $\mathbb{H}$-hull. Let the interval $[a_K^w,b_K^w]$ be the convex hull generated by w and $\overline{K}\cap\mathbb{R}$. Then g_K maps $\mathbb{C}\setminus(K^{\mathrm{doub}}\cup[a_K^w,b_K^w])$ conformally onto $\mathbb{C}\setminus[c_K^w,d_K^w]$ for some interval $[c_K^w,d_K^w]$. We define g_K^w from $\mathbb{R}_w\cup\{+\infty,-\infty\}$ onto $[-\infty,c_K^w]\cup[d_K^w,+\infty]$ such that $g_K^w(\pm\infty)=\pm\infty$; if $x\in\mathbb{R}\setminus[a_K^w,b_K^w]$, $g_K^w(x)=g_K(x)$; if $x\in[a_K^w,w)\cup\{w^-\}$, $g_K^w(x)=c_K^w$; and if $x\in(w,b_K^w]\cup\{w^+\}$, $g_K^w(x)=d_K^w$.

Remark 2.7. The maps g_K^w will be useful in describing force point processes for $\mathrm{SLE}_\kappa(\underline{\rho})$. Note that if $K=\emptyset$, $a_K^w=b_K^w=c_K^w=d_K^w=w$; if $w\in[a_K,b_K]$, then $a_K^w=a_K$, $b_K^w=b_K$, $c_K^w=c_K$, and $d_K^w=d_K$. It is clear that g_K^w is increasing. Since $g_K^w=g_K$ and $g_K'\in(0,1]$ on $(-\infty,a_K^w)\cup(b_K^w,\infty)$, and g_K^w maps $[a_K^w,w)\cup\{w^-\}$ and $(w,b_K^w]\cup\{w^+\}$ respectively to c_K^w and d_K^w, we see that g_K is a contraction on $(-\infty,w)\cup\{w^-\}$ and $(w,\infty)\cup\{w^+\}$.

2.2 Chordal Loewner equation

Let $\widehat{w}\in C([0,T),\mathbb{R})$ for some $T\in(0,\infty]$. The chordal Loewner equation driven by $\widehat{w}$ is

$$\partial_t g_t(z) = \frac{2}{g_t(z)-\widehat{w}(t)}, \quad 0\le t<T; \quad g_0(z)=z.$$

For every $z\in\mathbb{C}$, let τ_z be the first time that the solution $t\mapsto g_t(z)$ blows up; if such time does not exist, then set $\tau_z=\infty$. For $t\in[0,T)$, let $K_t=\{z\in\mathbb{H}:\tau_z\le t\}$. It turns out that for each $t\ge 0$, K_t is an $\mathbb{H}$-hull with $\mathrm{hcap}(K_t)=2t$, $K_t^{\mathrm{doub}}=\{z\in\mathbb{C}:\tau_z\le t\}$, and g_t agrees with

g_{K_t}. We call g_t and K_t the chordal Loewner maps and hulls, respectively, driven by $\widehat{w}$. Since we write $\mathrm{hcap}_2(K)$ for $\mathrm{hcap}(K)/2$, $\mathrm{hcap}_2(K_t)=t$ for all t.

If for every $t\in[0,T)$, $f_{K_t}=g_{K_t}^{-1}$ extends continuously from $\mathbb{H}$ to $\overline{\mathbb{H}}$, and $\eta(t):=f_{K_t}(\widehat{w}(t))$, $0\le t<T$, is continuous in t, then we say that η is the chordal Loewner curve driven by $\widehat{w}$. Such η may not exist in general. When it exists, we have $\eta(0)=\widehat{w}(0)\in\mathbb{R}$, and $K_t=\mathrm{Hull}(\eta([0,t]))$ for all t, and we say that K_t, $0\le t<T$, are generated by η.

Let u be a continuous and strictly increasing function on $[0,T)$ such that $u(0)=0$. Suppose that g_t and K_t, $0\le t<T$, satisfy that $g_{u^{-1}(t)}$ and $K_{u^{-1}(t)}$, $0\le t<u(T)$, are chordal Loewner maps and hulls, respectively, driven by $\widehat{w}\circ u^{-1}$. Then we say that g_t and K_t, $0\le t<T$, are chordal Loewner maps and hulls, respectively, driven by $\widehat{w}$ with speed du, and call $(K_{u^{-1}(t)})$ the normalization of (K_t). If (K_t) are generated by a curve η, i.e., $K_t=\mathrm{Hull}(\eta([0,t]))$ for all t, then η is called a chordal Loewner curve driven by $\widehat{w}$ with speed du, and $\eta\circ u^{-1}$ is called the normalization of η. If u is absolutely continuous, we also say that the speed is u'. In this case, the g_t satisfy the differential equation $\partial_t g_t(z)=\frac{2u'(t)}{g_t(z)-\widehat{w}(t)}$. The original Loewner maps and hulls then have speed 1.

The following proposition is a slight variation of Theorem 2.6 of [4].

Proposition 2.8. *The $\mathbb{H}$-hulls K_t, $0\le t<T$, are chordal Loewner hulls with some speed if and only if for any fixed $a\in[0,T)$, $\lim_{\delta\downarrow 0}\sup_{0\le t\le a}\mathrm{diam}(K_{t+\delta}/K_t)=0$. Moreover, the driving function $\widehat{w}$ satisfies that $\{\widehat{w}(t)\}=\bigcap_{\delta>0}\overline{K_{t+\delta}/K_t}$, $0\le t<T$; and the speed is du, where $u(t)=\mathrm{hcap}_2(K_t)$, $0\le t<T$.*

Proposition 2.9. *Suppose K_t, $0\le t<T$, are chordal Loewner hulls driven by $\widehat{w}$ with some speed. Then for any $t_0\in(0,T)$ and $t\in[0,t_0]$, $c_{K_{t_0}}\le\widehat{w}(t)\le d_{K_{t_0}}$.*

Proof. Let $t_0\in(0,T)$. If $0\le t<t_0$, by Propositions 2.2 and 2.8, $\widehat{w}(t)\in[a_{K_{t_0}/K_t},b_{K_{t_0}/K_t}]\subset[c_{K_{t_0}/K_t},d_{K_{t_0}/K_t}]\subset[c_{K_{t_0}},d_{K_{t_0}}]$. By the continuity of $\widehat{w}$, we also have $\widehat{w}(t_0)\in[c_{K_{t_0}},d_{K_{t_0}}]$. □

We now cite [18, Proposition 2.13] below, which is a corollary of [9, Lemma 2.5] and [8, Lemma 3.3].

Proposition 2.10. *Let K_t and $\eta(t)$, $0\le t<T$, be chordal Loewner hulls and curve driven by $\widehat{w}$ with speed q. Suppose the Lebesgue measure of $\eta([0,T))\cap\mathbb{R}$ is 0. Let $w=\widehat{w}(0)$, and $x\in\mathbb{R}_w$. Define $X(t)=g^w_{K_t}(x)$, $0\le t<T$. Then the set of t such that $X(t)\ne\widehat{w}(t)$ is zero, and X is absolutely continuous with $X'(t)=\mathbf{1}_{\{X(t)\ne\widehat{w}(t)\}}\frac{2q(t)}{X(t)-\widehat{w}(t)}$ almost everywhere on $[0,T)$.*

2.3 Chordal SLE$_\kappa$ and SLE$_\kappa(\underline{\rho})$ processes

If $\widehat{w}(t)=\sqrt{\kappa}B(t)$, $0\le t<\infty$, where $\kappa>0$ and $B(t)$ is a standard Brownian motion, then the chordal Loewner curve η driven by $\widehat{w}$ is known to exist (cf. [13]), and is called a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. It satisfies $\eta(0)=0$ and $\lim_{t\to\infty}\eta(t)=\infty$. The behavior of η depends on κ: if $\kappa\in(0,4]$, η is simple and intersects $\mathbb{R}$ only at 0; if $\kappa\ge 8$, η is space-filling, i.e., $\overline{\mathbb{H}}=\eta(\mathbb{R}_+)$; if $\kappa\in(4,8)$, η is neither simple nor space-filling. If D is a simply connected domain with two distinct marked boundary points (or more precisely, prime ends (cf. [1])) a and b, the chordal SLE_κ curve in D from a to b is defined to be the conformal image of a chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞ under a conformal map from $(\mathbb{H};0,\infty)$ onto $(D;a,b)$.

For any $\kappa>0$, chordal SLE_κ satisfies Domain Markov Property (DMP): if η is a chordal SLE_κ curve in D from a to b, and τ is a stopping time for η, then conditionally on the part of η before τ and the event that η does not end at the time τ, the part of η after τ is a chordal SLE_κ curve from $\eta(\tau)$ to b in the connected component of $D\setminus\eta([0,\tau])$ whose boundary contains b.

The $\mathrm{SLE}_\kappa(\underline{\rho})$ processes, first appeared in [5], are natural variants of SLE_κ, where one keeps track of additional marked points, often called force points, which may lie on the boundary or interior. For the generality needed here, all force points will lie on the boundary. We now review the definition and properties of $\mathrm{SLE}_\kappa(\underline{\rho})$ developed in [9].

Let $\kappa>0$, $n\in\mathbb{N}$, $\rho_1,\dots,\rho_n\in\mathbb{R}$, $w\in\mathbb{R}$, $v_1,\dots,v_n\in\mathbb{R}_w\cup\{+\infty,-\infty\}$. Recall that $\mathbb{R}_w=(\mathbb{R}\setminus\{w\})\cup\{w^+,w^-\}$. We require that for $\sigma\in\{+,-\}$, $\sum_{j:v_j=w^\sigma}\rho_j>-2$. The chordal $\mathrm{SLE}_\kappa(\rho_1,\dots,\rho_n)$ process in $\mathbb{H}$ started from w with force points $v_1,\dots,v_n$ is the chordal Loewner process driven by the function $\widehat{w}(t)$, $0\le t<T$, which drives chordal Loewner maps g_t and hulls K_t, and satisfies the following system of SDE:

$$d\widehat{w}(t)=\sqrt{\kappa}dB(t)+\sum_{j=1}^n \mathbf{1}_{\{\widehat{w}(t)\ne\widehat{v}_j(t)\}}\frac{\rho_j}{\widehat{w}(t)-\widehat{v}_j(t)}\,dt,\quad \widehat{w}(0)=w, \tag{2.4}$$

where B is a standard Brownian motion, and $\widehat{v}_j(t)=g^w_{K_t}(v_j)$, $1\le j\le n$. Here we used Definition 2.6. The SDE should be understood as an integral equation, i.e., $\widehat{w}(t)-w-\sqrt{\kappa}B(t)$ equals the Lebesgue integral of the summation from 0 to t. The solution exists uniquely up to the first time T (called a continuation threshold) that $\sum_{j:\widehat{v}_j(t)=c_{K_t}}\rho_j\le-2$ or $\sum_{j:\widehat{v}_j(t)=d_{K_t}}\rho_j\le-2$, whichever comes first. If there does not exist a continuation threshold, then the lifetime is ∞. The $\widehat{v}_j$ is called the force point function started from v_j. If $v_j=+\infty$ or $-\infty$, then $\widehat{v}_j$ is constant $+\infty$ or $-\infty$, and the term $\frac{\rho_j}{\widehat{w}(t)-\widehat{v}_j(t)}$ is constant 0, which means that the force point $+\infty$ or $-\infty$ does not play a role. If $v_j\not\in\{+\infty,-\infty\}$, then $\widehat{v}_j$ satisfies the ODE:

$$d\widehat{v}_j(t)=\mathbf{1}_{\{\widehat{w}(t)\ne\widehat{v}_j(t)\}}\frac{2}{\widehat{v}_j(t)-\widehat{w}(t)},\quad \widehat{v}_j(0)=v_j,\quad 1\le j\le n. \tag{2.5}$$

This equation should also be understood as an integral equation, which means that $\widehat{v}_j$ is absolutely continuous. If $v_j>w$, then $\widehat{v}_j\ge\widehat{w}$, and $\widehat{v}_j$ is increasing; if $v_j<w$, then $\widehat{v}_j\le\widehat{w}$, and $\widehat{v}_j$ is decreasing. Here the sets $\{\widehat{v}_j\ne\widehat{w}\}$ have Lebesgue measure zero. So we may omit the factors $\mathbf{1}_{\{\widehat{w}(t)\ne\widehat{v}_j(t)\}}$ in (2.4) and (2.5).

A chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ process generates a chordal Loewner curve η in $\overline{\mathbb{H}}$ started from w up to the continuation threshold. If no force point is swallowed by the process at any time, this fact follows from the existence of chordal SLE_κ curve and Girsanov Theorem. The existence of the curve in the general case was proved in [9]. The chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η satisfies the following DMP. If τ is a stopping time for η, then conditionally on the process before τ and the event that τ is less than the lifetime of η, $\widehat{w}(\tau+\cdot)$ and $\widehat{v}_j(\tau+\cdot)$, $1\le j\le n$ are the driving function and force point functions for a chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η^τ started from $\widehat{w}(\tau)$ with force points at $\widehat{v}_1(\tau),\dots,\widehat{v}_n(\tau)$. Moreover, $\eta(\tau+\cdot)=f_{K(\tau)}(\eta^\tau)$, where $K(\tau):=\mathrm{Hull}(\eta([0,\tau]))$. Here if for some j, $\widehat{v}_j(\tau)=\widehat{w}(\tau)$, then $\widehat{v}_j(\tau)$ as a force point for η^τ is treated as $\widehat{w}(\tau)^+$ if $v_j\ge w^+$, or $\widehat{v}(\tau)^-$ if $v_j\le w^-$.

If two force points v_j and v_k are equal, we may treat them as a single force point with force value $\rho_j+\rho_k$. By merging the force points and removing $+\infty$ and $-\infty$, we may assume that the force points $v_1,\dots,v_n$ are mutually distinct finite numbers. We now relabel them by $v^{(\sigma)}_j$, $1\le j\le n_\sigma$, $\sigma\in\{+,-\}$, such that $v^{(-)}_{n_-}<\cdots<v^{(-)}_1\le w^-<w^+\le v^{(+)}_1<\cdots<v^{(+)}_{n_+}$, where n_- or n_+ could be 0. Then $\widehat{v}^{(-)}_{n_-}\le\cdots\le\widehat{v}^{(-)}_1\le\widehat{w}\le\widehat{v}^{(+)}_1\le\cdots\le\widehat{v}^{(+)}_{n_+}$ throughout the life period. Let $\rho^{(\pm)}_j$, $1\le j\le n_\pm$, denote the corresponding force values. If for any $\sigma\in\{-,+\}$ and $1\le k\le n_\sigma$, $\sum_{j=1}^k\rho^{(\sigma)}_j>-2$, then the process will never reach a continuation threshold, and so its lifetime is ∞, in which case $\lim_{t\to\infty}\eta(t)=\infty$. For $\sigma\in\{+,-\}$ and $k\in\{1,\dots,n_\sigma\}$, if $\sum_{j=1}^k\rho^{(\sigma)}_j>\frac{\kappa}{2}-4$, then a.s. η stays at a positive distance from $v^{(\sigma)}_k$; if $\sum_{j=1}^k\rho^{(\sigma)}_j\ge\frac{\kappa}{2}-2$, then a.s. η does not hit the open interval between $v^{(\sigma)}_k$ and $v^{(\sigma)}_{k+1}$ (with $v^{(\sigma)}_{n_\sigma+1}$ understood as $\sigma\cdot\infty$). If for some t_0, $\widehat{w}(t_0)=\widehat{v}^{(\sigma)}_k(t_0)$, then $\widehat{v}^{(\sigma)}_j(t)=\widehat{v}^{(\sigma)}_k(t)$ for all $1\le j\le k$ and $t\ge t_0$ in the life period, which means that the force point processes $\widehat{v}^{(\sigma)}_j$ for $1\le j\le k$ merge after t_0.

The following proposition will be needed. Recall the one quarter conformal radius $\mathrm{crad}^{(4)}_z(D)$ and boundary Poisson kernel $H_D(z,w)$ defined in Section 2.

Proposition 2.11. *Let $\kappa\in(0,8)$ and $\rho_1,\dots,\rho_m,\rho_{m+1}\in\mathbb{R}$ satisfy $\sum_{j=1}^{m+1}\rho_j=0$, and for any $1\le k\le m$, $\sum_{j=1}^k\rho_j\ge\frac{\kappa}{2}-2$. Suppose $w>v_1>\cdots>v_m>v_{m+1}\in\mathbb{R}$ or $w<v_1<\cdots<v_m<v_{m+1}\in\mathbb{R}$. We also write ρ_∞ for ρ_{m+1}, and v_∞ for v_{m+1}. Let $\mathbb{P}_0$ denote the law of the chordal SLE$_\kappa$ curve in $\mathbb{H}$ from w to ∞. Let $\mathbb{P}_1$ be the law of a chordal SLE$_\kappa(\rho_1,\dots,\rho_m,\rho_{m+1})$ curve in $\mathbb{H}$ started from w with force points $v_1,\dots,v_m,v_{m+1}$. Then*

$$\frac{d\mathbb{P}_1}{d\mathbb{P}_0}=\frac{\mathbf{1}_{E_0}}{\mathcal{Z}}\prod_{j=1}^m\Big(\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(\infty))^{-\frac{\rho_j(\kappa-4)}{2\kappa}}H_{D_\infty}(v_j,v_\infty)^{-\frac{\rho_j(\rho_\infty+\kappa-4)}{4\kappa}}\Big)\prod_{1\le j<k\le m}H_{D_\infty}(v_j,v_k)^{-\frac{\rho_j\rho_k}{4\kappa}},\tag{2.6}$$

where E_0 is the event that $\mathbb{H}\backslash\eta$ contains a connected component, denoted by D_∞, which contains a neighborhood of the line segment $[v_\infty,v_1]$ in $\mathbb{H}$; $\Omega_j(\infty)$ is the connected component of $\mathbb{C}\setminus([v_j,w]\cup\eta\cup\{z\in\mathbb{C}:\overline{z}\in\eta\})$ which contains v_∞, $1\le j\le m$; and $\mathcal{Z}>0$ is given by

$$\mathcal{Z}:=\prod_{j=1}^m\Big(\frac{w-v_j}{w-v_\infty}\Big)^{\frac{\rho_j}{\kappa}}\prod_{j=1}^m|v_j-v_\infty|^{\frac{\rho_j\rho_\infty}{2\kappa}}\prod_{1\le j<k\le m}|v_j-v_k|^{\frac{\rho_j\rho_k}{2\kappa}}.$$

Proof. By symmetry, we may assume that $w>v_1>\cdots>v_m>v_{m+1}$. By [16], before any force point is separated by η from ∞, the law $\mathbb{P}_1$ can be obtained by tilting the law $\mathbb{P}_0$ by the local martingale defined by

$$N(t)=\frac{1}{\mathcal{Z}}\prod_{j=1}^{m+1} g_t'(v_j)^{\frac{\rho_j(\rho_j+4-\kappa)}{4\kappa}}\prod_{j=1}^{m+1}|\widehat w(t)-\widehat v_j(t)|^{\frac{\rho_j}{\kappa}}\prod_{1\le j<k\le m+1}|\widehat v_j(t)-\widehat v_k(t)|^{\frac{\rho_j\rho_k}{2\kappa}}, \tag{2.7}$$

where g_t are the chordal Loewner maps, $\widehat w$ is the driving function, $\widehat v_j(t)=g_t(v_j)$, and the constant $\mathcal{Z}>0$ is such that $N(0)=1$. More specifically, this means that, if τ is a stopping time such that $\tau\le\min\{T_j:1\le j\le m+1\}=T_1$, where T_j is the first time that v_j is swallowed by the process, and $N(t)$, $0\le t<\tau$, is bounded by a uniform constant, then

$$\mathbb{P}_1=N(\tau)\cdot\mathbb{P}_0\quad\text{on }\mathcal{F}_\tau. \tag{2.8}$$

Here if $\tau=T_1$, then $N(\tau)$ is understood as $N(T_1):=\lim_{t\uparrow T_1}N(t)$, which $\mathbb{P}_0$-a.s. converges. This can be also checked directly using Girsanov Theorem and Itô's formula ([12]). For every $n\in\mathbb{N}$, let $\tau_n=\inf(T_1\cup\{t\ge 0:N(t)\ge n\})$. Then (2.8) holds for each τ_n. Since $\mathcal{F}_{T_1}\cap\{T_1=\tau_n\}\subset\mathcal{F}_{\tau_n}$, we then get $\mathbb{P}_1=N(T_1)\cdot\mathbb{P}_0$ on $\mathcal{F}_{T_1}\cap\{T_1=\tau_n\}$. Since this holds for any $n\in\mathbb{N}$, we get $\mathbb{P}_1=N(T_1)\cdot\mathbb{P}_0$ on $\mathcal{F}_{T_1}\cap E_B$, where $E_B:=\bigcup_{n=1}^\infty\{T_1=\tau_n\}=\{\sup_{0\le t<T_1}N(t)<\infty\}$, i.e., the event that N is bounded on $[0,T_1)$. Since $\mathbb{P}_0$-a.s. $\lim_{t\uparrow T_1}N(t)$ converges, we have $\mathbb{P}_0[E_B]=1$.

For $0\le t<T_1$ and $1\le j\le m$, let $D_t=\mathbb{H}\setminus\mathrm{Hull}(\eta([0,t]))$, and let $\Omega_j(t)$ denote the union of D_t, its reflection about $\mathbb{R}$, and the interval $(-\infty,v_j)$. Since g_t maps D_t conformally onto $\mathbb{H}$, we have $H_{D_t}(v_j,v_k)=g_t'(v_j)g_t'(v_k)/|\widehat v_j(t)-\widehat v_k(t)|^2$, $1\le j<k\le m+1$. Since g_t maps $\Omega_j(t)$ conformally onto $\mathbb{C}\setminus[\widehat v_j(t),\infty)$, we get $\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(t))=|\widehat v_j(t)-\widehat v_\infty(t)|/g_t'(v_\infty)$.

Let $\widehat x_j=\widehat w-\widehat v_j$, and $R_j=\frac{\widehat x_j}{\widehat x_\infty}$. Since $\rho_\infty=-\sum_{j=1}^m\rho_j$, $N(t)$ equals

$$\frac{1}{\mathcal{Z}}\prod_{j=1}^{m}\Big(R_j(t)^{\frac{\rho_j}{\kappa}}\,\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(t))^{-\frac{\rho_j(\kappa-4)}{2\kappa}}H_{D_t}(v_j,v_\infty)^{-\frac{\rho_j(\rho_\infty+\kappa-4)}{4\kappa}}\Big)\prod_{1\le j<k\le m}H_{D_t}(v_j,v_k)^{-\frac{\rho_j\rho_k}{4\kappa}}.$$

Suppose E_0 occurs. Then $T_1=\cdots=T_m=T_\infty$. Let $t\uparrow T_1=T_\infty$. Since $D_t\to D_\infty$ and $\Omega_j(t)\to\Omega_j(\infty)$ in the Carathéodory topology, we have $\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(t))\to\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(\infty))$, $1\le j\le m$, and $H_{D_t}(v_j,v_k)\to H_{D_\infty}(v_j,v_k)$, $1\le j<k\le m+1$.

For $0\le t<T_\infty$, let P_t denote the set of prime ends of D_t, which lie on either $[w,\infty)$ or the right side of $\eta([0,t])$, i.e., it is the image of $[\widehat w(t),\infty)$ under g_t^{-1}. Suppose $\kappa\in(0,4]$. Then $T_\infty=\infty$. Let $L>|v_\infty-w|$, and ξ_t be the connected component of $\{|z-w|=L\}\cap D_t$ whose closure contains $w-L$. Since $\eta(t)\to\infty$ as $t\to\infty$, there is $N>0$ such that $|\eta(t)-w|>L$ for $t\ge N$. For those t, since ξ_t separates $[v_\infty,v_j]$ from P_t in D_t, the extremal distance (cf. [1]) between $[v_\infty,v_j]$ and P_t in D_t is by comparison principle at least $\log(L/|v_\infty-w|)/\pi$. Thus, the extremal distance between $[v_\infty,v_j]$ and P_t in D_t tends to ∞ as $t\uparrow T_\infty$. Suppose $\kappa\in(4,8)$. Then $T_\infty<\infty$ and $\eta(T_\infty)\in(-\infty,v_\infty)$. Let $\varepsilon\in(0,|\eta(T_\infty)-v_\infty|)$, and ξ_t be the connected component of $\{|z-\eta(T_\infty)|=\varepsilon\}\cap D_t$ whose closure contains $\eta(T_\infty)+\varepsilon$. Then there is $\delta>0$ such that $|\eta(t)-\eta(T_\infty)|<\varepsilon$ for $t\in[T_\infty-\delta,T_\infty)$. For those t, since ξ_t separates $[v_\infty,v_j]$ from P_t in D_t, the extremal distance between $[v_\infty,v_j]$ and P_t in D_t is at least $\log(|\eta(T_\infty)-v_\infty|/\varepsilon)/\pi$. So we again get that the extremal distance between $[v_\infty,v_j]$ and P_t in D_t tends to ∞ as $t\uparrow T_\infty$.

Since g_t maps D_t conformaly onto $\mathbb{H}$, and takes P_t and $[v_\infty, v_j]$ to $[\widehat{w}(t), \infty)$ and $[\widehat{v}_\infty(t), \widehat{v}_j(t)]$, respectively, by conformal invariance, we get $R_j(t) = \frac{\widehat{w}(t)-\widehat{v}_j(t)}{\widehat{w}(t)-\widehat{v}_\infty(t)} \to 1$ as $t \uparrow T_\infty$.

On the event E_0, since $T_1 = T_\infty$, $N(T_1) = \lim_{t\uparrow T_1} N(t)$ equals the RHS of (2.6). This implies that $E_0 \subset E_B$. By the assumptions on κ and ρ_j's, we know that $\mathbb{P}_1$ is supported by E_0. Since $\mathbb{P}_1 = N(T_1) \cdot \mathbb{P}_0$ on $\mathcal{F}_{T_1} \cap E_B$, and $\mathcal{F}_{T_1}$ agrees with $\mathcal{F}_{T_\infty}$ on the event E_0, we see that $d(\mathbb{P}_1|\mathcal{F}_{T_\infty})/d(\mathbb{P}_0|\mathcal{F}_{T_\infty})$ is given by the RHS of (2.6). If $\kappa \in (0,4]$, then $\mathbb{P}_0$-a.s. $T_\infty = \infty$, and so (2.6) holds. If $\kappa \in (4,8)$, then $\mathbb{P}_0$-a.s. $T_\infty < \infty$. If η follows the law $\mathbb{P}_0$, then by the DMP for chordal SLE$_\kappa$, conditionally on $\mathcal{F}_{T_\infty}$, the part of η after T_∞ is a chordal SLE$_\kappa$ from $\eta(T_\infty)$ to ∞ in $H_{T_\infty} := \mathbb{H} \setminus \mathrm{Hull}(\eta([0, T_\infty]))$. If η follows the law $\mathbb{P}_1$, then by the DMP for chordal SLE$_\kappa(\underline{\rho})$ and the fact that $\sum_{j=1}^m \rho_j + \rho_\infty = 0$, conditionally on $\mathcal{F}_{T_\infty}$, the part of η after T_∞ is also a chordal SLE$_\kappa$ from $\eta(T_\infty)$ to ∞ in H_{T_∞}. So the absolute continuity between $\mathbb{P}_1$ and $\mathbb{P}_0$ on $\mathcal{F}_{T_\infty}$ extends to $\mathcal{F}_\infty$ with the same Radon-Nikodym derivative, and we again have (2.6). $\square$

Remark 2.12. We may express $d\mathbb{P}_1/d\mathbb{P}_0$ in the above theorem in terms of a conformal map from D_∞ onto $\mathbb{H}$. Suppose $w > v_1 > \cdots > v_m > v_{m+1}$. Suppose $\partial D_\infty \cap \mathbb{R} = [x_L, x_R]$. Let g_* be a conformal map from D_∞ onto $\mathbb{H}$ such that $g_*(x_L) = \infty$. By Schwarz reflection principle, g_* extends to a conformal map defined on the union of D_∞, its reflection about $\mathbb{R}$, and (x_L, x_R). Then $H_{D_\infty}(v_j, v_k) = g_*'(v_j) g_*'(v_k)/|g_*(v_j) - g_*(v_k)|^2$, $1 \le j < k \le m+1$. Since g_* maps $\Omega_j(\infty)$ conformally onto $\mathbb{C}\setminus[g_*(v_j), \infty)$, we get $\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(\infty)) = |g_*(v_j) - g_*(v_\infty)|/g_*'(v_\infty)$, $1 \le j \le m$. Combining these formulas with the equality $\sum_{j=1}^{m+1} \rho_j = 0$, we get

$$\frac{d\mathbb{P}_1}{d\mathbb{P}_0} = \mathbf{1}_{E_0} \prod_{j=1}^{m+1} \frac{g_*'(v_j)^{\frac{\rho_j(\rho_j+4-\kappa)}{4\kappa}}}{|w - v_j|^{\frac{\rho_j}{\kappa}}} \prod_{1 \le j < k \le m+1} \Big(\frac{|g_*(v_j) - g_*(v_k)|}{|v_j - v_k|}\Big)^{\frac{\rho_j \rho_k}{2\kappa}}. \tag{2.9}$$

2.4 Multiple hypergeometric functions

Let $\alpha, \beta_1, \dots, \beta_m, \gamma \in \mathbb{R}$, and $\gamma \notin \{0, -1, -2, \dots\}$. We use the Pochhammer symbol $(\alpha)_n$ to denote the rising factorial, i.e., $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \ge 1$. Note that $(1)_n = n!$. Write $\underline{\beta} = (\beta_1, \dots, \beta_m)$. Let $F(\alpha, \beta_1, \dots, \beta_m, \gamma; \underline{x})$, $\underline{x} = (x_1, \dots, x_m)$, be the (first) Appell-Lauricella multiple hypergeometric function defined by (cf. [10])

$$F(\alpha, \underline{\beta}, \gamma; \underline{x}) = \sum_{n_1=0}^\infty \cdots \sum_{n_m=0}^\infty \frac{(\alpha)_{n_1+\cdots+n_m} (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}}{(\gamma)_{n_1+\cdots+n_m} (1)_{n_1} \cdots (1)_{n_m}} x_1^{n_1} \cdots x_m^{n_m}. \tag{2.10}$$

Using Stirling's formula, one easily see that the series itself as well as the series of the partial derivatives to any order converge absolutely and uniformly on $[-r, r]^m$ for any $r \in (0,1)$. Thus, F is C^∞ on $(-1,1)^m$, and one may differentiate the series term by term. For $1 \le j \le m$, let $\underline{e}_j$ denote the vector in $\mathbb{R}^m$, whose j-th component is 1, and other components are 0. Straightforward calculation shows that for any $1 \le j \le m$,

$$\partial_{x_j} F(\alpha, \underline{\beta}, \gamma; \underline{x}) = \frac{\alpha \beta_j}{\gamma} F(\alpha+1, \underline{\beta} + \underline{e}_j, \gamma+1; \underline{x}). \tag{2.11}$$

Since we may change the order of summation, we find that for any $\Lambda \subset \{1, \dots, m\}$,

$$F(\alpha, \underline{\beta}, \gamma; \underline{x}) = \sum_{\underline{n} \in \overline{\mathbb{N}}^{\Lambda}} \frac{(\alpha)_{|\underline{n}|} \prod_{j \in \Lambda} (\beta_j)_{n_j}}{(\gamma)_{|\underline{n}|} \prod_{j \in \Lambda} (1)_{n_j}} \prod_{j \in \Lambda} x_j^{n_j} F(\alpha + |\underline{n}|, \underline{\beta}|_{\Lambda^c}, \gamma + |\underline{n}|; \underline{x}|_{\Lambda^c}), \tag{2.12}$$

where $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\}$, $|\underline{n}| = \sum_{j \in \Lambda} n_j$ for $\underline{n} \in \overline{\mathbb{N}}^{\Lambda}$, and $\Lambda^c := \{1, \dots, m\} \setminus \Lambda$. The equality holds even in the case $\Lambda = \emptyset$ or $\Lambda = \{1, \dots, m\}$. In the former case, there is only one term in the summation, and the equality is trivial; in the latter case, the F-functions on the RHS of (2.12) are understood as constant 1, and the equality reduces to the definition (2.10).

Let $F = F(\alpha, \beta_1, \dots, \beta_m, \gamma; \cdot)$. By (2.10), we have

$$F(0, x_2, \dots, x_m) = F(\alpha, \beta_2, \dots, \beta_m, \gamma; x_2, \dots, x_m); \tag{2.13}$$

$$F(x_1, \dots, x_{m-2}, x_{m-1}, x_{m-1}) = F(\alpha, \beta_1, \dots, \beta_{m-2}, \beta_{m-1} + \beta_m, \gamma; x_1, \dots, x_{m-1}). \tag{2.14}$$

If $\gamma > \alpha + \beta_m$, by Stirling's formula, the series (2.10) converges uniformly on $[0, r]^{m-1} \times [0, 1]$ for any $r \in (0, 1)$. Thus, F extends continuously from $[0, 1)^m$ to $[0, 1)^{m-1} \times [0, 1]$, and by (2.12) and Gauss's Theorem,

$$\begin{aligned} F(x_1, \dots, x_{m-1}, 1) &= \sum_{\underline{n} \in \overline{\mathbb{N}}^{m-1}} \frac{(\alpha)_{|\underline{n}|} \prod_{j=1}^{m-1} (\beta_j)_{n_j}}{(\gamma)_{|\underline{n}|} \prod_{j=1}^{m-1} (1)_{n_j}} \prod_{j=1}^{m-1} x_j^{n_j} F(\alpha + |\underline{n}|, \beta_m, \gamma + |\underline{n}|, 1) \\ &= \sum_{\underline{n} \in \overline{\mathbb{N}}^{m-1}} \frac{(\alpha)_{|\underline{n}|} \prod_{j=1}^{m-1} (\beta_j)_{n_j}}{(\gamma)_{|\underline{n}|} \prod_{j=1}^{m-1} (1)_{n_j}} \prod_{j=1}^{m-1} x_j^{n_j} \frac{\Gamma(\gamma + |\underline{n}|) \Gamma(\gamma - \alpha - \beta_m)}{\Gamma(\gamma + |\underline{n}| - \beta_m) \Gamma(\gamma - \alpha)} \\ &= \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta_m)}{\Gamma(\gamma - \beta_m) \Gamma(\gamma - \alpha)} F(\alpha, \beta_1, \dots, \beta_{m-1}, \gamma - \beta_m; x_1, \dots, x_{m-1}). \end{aligned} \tag{2.15}$$

We are going to derive some PDEs for the multiple hypergeometric functions. Some of them can be found in the literature. But for completeness, we will provide detailed proofs. For $\underline{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $\underline{n} = (n_1, \dots, n_m) \in \overline{\mathbb{N}}^m$, we define $|\underline{n}| = n_1 + \cdots + n_m$ and $\underline{x}^{\underline{n}} = x_1^{n_1} \cdots x_m^{n_m}$. We may express $F(\underline{x})$ as $\sum_{\underline{n} \in \overline{\mathbb{N}}^m} A_{\underline{n}} \underline{x}^{\underline{n}}$, where $A_{\underline{n}} = \frac{(\alpha)_{|\underline{n}|} \prod_{j=1}^m (\beta_j)_{n_j}}{(\gamma)_{|\underline{n}|} \prod_{j=1}^m (1)_{n_j}}$. Then

$$A_{\underline{n} + \underline{e}_j} = \frac{(\alpha + |\underline{n}|)(\beta_j + n_j)}{(\gamma + |\underline{n}|)(1 + n_j)} A_{\underline{n}}. \tag{2.16}$$

Let θ_j denote the partial differential operator $x_j \partial_{x_j}$. Then

$$\theta_j F(\underline{x}) = \sum_{\underline{n} \in \overline{\mathbb{N}}^m} n_j A_{\underline{n}} \underline{x}^{\underline{n}}. \tag{2.17}$$

So

$$(\alpha + \theta_1 + \cdots + \theta_m)(\beta_j + \theta_j) F(\underline{x}) = \sum_{\underline{n} \in \overline{\mathbb{N}}^m} (\alpha + |\underline{n}|)(\beta_j + n_j) A_{\underline{n}} \underline{x}^{\underline{n}};$$

$$\theta_j(\gamma-1+\theta_1+\cdots+\theta_m)F(\underline{x}) = \sum_{\underline{n}\in\overline{\mathbb{N}}^m} n_j(\gamma-1+|\underline{n}|)A_{\underline{n}}\underline{x}^{\underline{n}}$$

$$= \sum_{\underline{n}\in\overline{\mathbb{N}}^m:n_j\geq 1} n_j(\gamma-1+|\underline{n}|)A_{\underline{n}}\underline{x}^{\underline{n}} = x_j\sum_{\underline{n}\in\overline{\mathbb{N}}^m}(n_j+1)(\gamma+|\underline{n}|)A_{\underline{n}+\underline{e}_j}\underline{x}^{\underline{m}}.$$

By (2.16), F satisfies the PDE (cf. [10, Formula (56) of Chapter 9]) $\mathcal{L}_jF=0$, $1\leq j\leq m$, where

$$\mathcal{L}_j := -(\alpha+\theta_1+\cdots+\theta_m)(\beta_j+\theta_j)+\frac{1}{x_j}\theta_j(\gamma-1+\theta_1+\cdots+\theta_m)$$

$$= \sum_{k=1}^m x_k(1-x_j)\partial_{x_j}\partial_{x_k}+[\gamma-(\alpha+1)x_j]\partial_{x_j}-\beta_j\sum_{k=1}^m x_k\partial_{x_k}-\alpha\beta_j.$$

From (2.17) we also know that for $1\leq j\neq k\leq m$,

$$\partial_{x_k}(\beta_j+\theta_j)F(\underline{x}) = \sum_{\underline{n}\in\overline{\mathbb{N}}^m:n_k\geq 1} n_k(\beta_j+n_j)A_{\underline{n}}\underline{x}^{\underline{n}-\underline{e}_k} = \sum_{\underline{n}\in\overline{\mathbb{N}}^m}(1+n_k)(\beta_j+n_j)A_{\underline{n}+\underline{e}_k}\underline{x}^{\underline{n}}.$$

From (2.16) we know that $(1+n_k)(\beta_j+n_j)A_{\underline{n}+\underline{e}_k}=(1+n_j)(\beta_k+n_k)A_{\underline{n}+\underline{e}_j}$. So F satisfies the PDE $\mathcal{L}_{j,k}F=0$ for $1\leq j\neq k\leq m$, where

$$\mathcal{L}_{j,k} := \partial_{x_k}(\beta_j+\theta_j)-\partial_{x_j}(\beta_k+\theta_k) = (x_j-x_k)\partial_{x_j}\partial_{x_k}+\beta_j\partial_{x_k}-\beta_k\partial_{x_j}.$$

If $j=k$, the equality $\mathcal{L}_{j,k}F=0$ trivially holds. Now we let

$$\mathcal{L} = \sum_{j=1}^m\frac{1-x_j}{x_j}\mathcal{L}_j+\sum_{j=1}^m\sum_{k=1}^m\frac{1}{x_j}\mathcal{L}_{j,k}$$

$$= \sum_{j=1}^m\sum_{k=1}^m(1-x_j)(1-x_k)\partial_{x_j}\partial_{x_k}-\sum_{j=1}^m\alpha\beta_j\Big(\frac{1}{x_j}-1\Big)$$

$$+\sum_{j=1}^m(1-x_j)\Big[\frac{\gamma-\sum_{k=1}^m\beta_k}{x_j}-(\alpha+1)+\sum_{k=1}^m\beta_k\Big(\frac{1}{x_k}-1\Big)\Big]\partial_{x_j}, \tag{2.18}$$

and

$$\mathcal{L}^r = \sum_{j=1}^m x_j(1-x_j)\mathcal{L}_j+\sum_{j=1}^m\sum_{k=1}^m x_jx_k^2\mathcal{L}_{j,k}$$

$$= \sum_{j=1}^m\sum_{k=1}^m x_jx_k(1-x_j)(1-x_k)\partial_{x_j}\partial_{x_k}-\sum_{j=1}^m\alpha\beta_jx_j(1-x_j)$$

$$+\sum_{j=1}^m x_j(1-x_j)\Big[(\gamma-\sum_{k=1}^m\beta_k)-(\alpha+1)x_j+\sum_{k=1}^m\beta_k(1-x_k)\Big]\partial_{x_j}. \tag{2.19}$$

In the above equalities, we used

$$\sum_{j=1}^m\sum_{k=1}^m (x_j-x_k)\partial_{x_j}\partial_{x_k}=\sum_{j=1}^m\sum_{k=1}^m (x_j-x_k)x_jx_k(1-x_j-x_k)\partial_{x_j}\partial_{x_k}=0.$$

Then we have

$$\mathcal{L}F=0 \quad \text{on } (0,1)^m; \quad \mathcal{L}^r F=0 \quad \text{on } (-1,1)^m. \tag{2.20}$$

We now study the positiveness and continuation of the multiple hypergeometric function. We make some assumptions on the parameters.

Definition 2.13. For $m\in\mathbb{N}$, we say that $\alpha,\beta_1,\dots,\beta_m,\gamma\in\mathbb{R}$ satisfy the parameter assumption if $\gamma>0\vee\alpha$ and $\gamma>(0\vee\alpha)+\sum_{j=k}^m\beta_j$ for any $1\le k\le m$.

From now on, we fix $\alpha,\beta_1,\dots,\beta_m,\gamma\in\mathbb{R}$, and let $F=F(\alpha,\beta_1,\dots,\beta_m,\gamma;\cdot)$. Let Δ_m denote the set $\{\underline{x}\in\mathbb{R}^m: 0\le x_1\le\cdots\le x_m<1\}$.

Lemma 2.14. *If $\alpha,\beta_1,\dots,\beta_m,\gamma\in\mathbb{R}$ satisfy the parameter assumption, F is positive on Δ_m.*

Proof. We prove by induction on m, and use the idea in the proof of Lemma 3.1 of [17]. First, consider the case $m=1$. In this case, the multiple hypergeometric function reduces to a single-variable hypergeometric function ${}_2F_1(\alpha,\beta_1,\gamma;x_1)$, and $\Delta_1=[0,1)$. Since $\gamma>\alpha+\beta_1$, by Gauss's Theorem (cf. [10, Formula (20) of Section 1.2]), F extends continuously to $[0,1]$ with $F(1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1)}$. Since $\gamma,\gamma-\alpha,\gamma-\beta_1,\gamma-\alpha-\beta_1>0$, we get $F(1)>0$. We also note that $F(0)=1>0$.

If $\alpha\wedge\beta_1\ge 0$, then we have $F\ge 1$ on $[0,1)$ since every term in (2.10) is nonnegative for $x_1\in[0,1)$, and the first term ($n_1=0$) is 1. Now suppose $\alpha\wedge\beta_1<0$. Let n be the smallest integer such that $n+(\alpha\wedge\beta_1)\ge 0$. We write $F_j=F(\alpha+j,\beta_1+j,\gamma+j;\cdot)$. By (2.11),

$$F_j'=\frac{(\alpha+j)(\beta_1+j)}{\gamma+j}F_{j+1}, \quad j\ge 0. \tag{2.21}$$

We claim that for any $0\le j\le n-1$, $\alpha+j,\beta_1+j,\gamma+j$ satisfy the parameter assumption. In fact, since α,β_1,γ satisfy the parameter assumption, the only way that $\alpha+j,\beta_1+j,\gamma+j$ could fail to satisfy the parameter assumption is that $(\gamma+j)\le(\alpha+j)+(\beta_1+j)$, which implies that $\gamma\le(\alpha+\beta_1)+j$. Since $j<n$, by the definition of n, we have $j<-(\alpha\wedge\beta_1)$. So we get $\gamma<(\alpha+\beta_1)-(\alpha\wedge\beta_1)=\alpha\vee\beta_1$, which contradicts that $\gamma>\alpha\vee\beta_1$. By this claim and the statement in the last paragraph, for each $0\le j\le n-1$, F_j extends continuously to $[0,1]$, and is positive at 0 and 1. Since $\alpha+n,\beta_1+n\ge 0$ and $\gamma+n>0$, by (2.10) $F_n>0$ on $[0,1)$. By (2.21), F_{n-1} is monotone on $[0,1]$. Since $F_{n-1}(1),F_{n-1}(0)>0$, in either case $F_{n-1}>0$ on $[0,1]$. Then we may use the same argument to show that $F_{n-2}>0$ on $[0,1]$ (if $n\ge 2$). Iterating the argument, we get $F=F_0>0$ on $[0,1]$.

Suppose $m\ge 2$ and the lemma holds for $m-1$. First assume that $\alpha\wedge\beta_1\ge 0$. By (2.12),

$$F(\underline{x})=\sum_{n_1=0}^\infty\frac{(\alpha)_{n_1}(\beta_1)_{n_1}}{(\gamma)_{n_1}(1)_{n_1}}x_1^{n_1}F(\alpha+n_1,\beta_2,\dots,\beta_m,\gamma+n_1;x_2,\dots,x_m). \tag{2.22}$$

Since for any $n_1 \geq 0$, $\alpha+n_1, \beta_2, \dots, \beta_m, \gamma+n_1$ satisfy the parameter assumption, by induction hypothesis, $F(\alpha+n_1, \beta_2, \dots, \beta_m, \gamma+n_1; \cdot)$ is positive on Δ_{m-1}. Since $\alpha, \beta_1 \geq 0$ and $\gamma > 0$, every term in the series of (2.22) is nonnegative, and the first term ($n_1 = 0$) is positive on Δ_m. Thus, F is positive on Δ_m. Now we assume that $\alpha \wedge \beta_1 < 0$. Let n be the first integer such that $n + (\alpha \wedge \beta_1) \geq 0$. For each j, let $F_j = F(\alpha+j, \beta_1+j, \beta_2, \dots, \beta_m, \gamma+j; \cdot)$. By (2.11),

$$\partial_{x_1} F_j = \frac{(\alpha+j)(\beta_1+j)}{\gamma+j} F_{j+1}, \quad j \geq 0. \tag{2.23}$$

By (2.12) we get

$$F_n(\underline{x}) = \sum_{n_1=0}^{\infty} \frac{(\alpha+n)_{n_1}(\beta_1+n)_{n_1}}{(\gamma+n)_{n_1}(1)_{n_1}} x_1^{n_1} F(\alpha+n+n_1, \beta_2, \dots, \beta_m, \gamma+n+n_1; x_2, \dots, x_m). \tag{2.24}$$

Since $\alpha+n+n_1, \beta_2, \dots, \beta_m, \gamma+n+n_1$ satisfy the parameter assumption, by induction hypotheses, $F(\alpha+n+n_1, \beta_2, \dots, \beta_m, \gamma+n+n_1; \cdot)$ is positive on Δ_{m-1}. Since $\alpha+n, \beta_1+n \geq 0$ and $\gamma+n > 0$, every term in the series of (2.24) is nonnegative, and the first term is positive on Δ_m. Thus, $F_n > 0$ on Δ_m. From (2.23) we then know that F_{n-1} is monotone in x_1 on Δ_m. Now for every fixed $(x_2, \dots, x_m) \in \Delta_{m-1}$, the x_1 such that $(x_1, x_2, \dots, x_m) \in \Delta_m$ is $[0, x_2]$. By (2.13,2.14),

$$F_j(0, x_2, \dots, x_m) = F(\alpha+j, \beta_2, \dots, \beta_m, \gamma+j; x_2, \dots, x_m),$$

$$F_j(x_2, x_2, \dots, x_m) = F(\alpha+j, \beta_1+j+\beta_2, \beta_3 \dots, \beta_m, \gamma+j; x_2, \dots, x_m).$$

Since $\alpha, \beta_1, \beta_2, \dots, \beta_m, \gamma$ satisfy the parameter assumption, so do $\alpha+j, \beta_2, \dots, \beta_m, \gamma+j$. Thus by induction hypothesis, $F_j(0, x_2, \dots, x_m) > 0$ for $(x_2, \dots, x_m) \in \Delta_{m-1}$. We claim that for any $0 \leq j \leq n-1$, $\alpha+j, \beta_1+j+\beta_2, \beta_3, \dots, \beta_m, \gamma+j$ satisfy the parameter assumption. This holds because the only way that they could fail to satisfy the parameter assumption is $\gamma + j \leq (\alpha+j) + (j + \sum_{s=1}^m \beta_s)$, i.e., $\gamma \leq \alpha + \sum_{s=1}^m \beta_s + j$. Since $j < n$, by the definition of n, $j < -(\alpha \wedge \beta_1)$. So $\gamma < \alpha + \sum_{s=1}^m \beta_s - \alpha \wedge \beta_1 = \alpha \vee \beta_1 + \sum_{s=2}^m \beta_s$, which contradicts that $\gamma > (\sum_{s=1}^m \beta_s) \vee (\alpha + \sum_{s=2}^m \beta_s)$. So the claim is proved, which implies by induction hypothesis that $F_j(x_2, x_2, \dots, x_m) > 0$ for $(x_2, \dots, x_m) \in \Delta_{m-1}$ and $0 \leq j \leq n-1$. Since F_{n-1} is monotone in x_1 on Δ_m and is positive when $x_1 \in \{0, x_2\}$, we see that $F_{n-1} > 0$ on Δ_m. Applying the same argument to F_{n-1}, we get $F_{n-2} > 0$ on Δ_m. Iterating, we get $F = F_0 > 0$ on Δ_m. $\square$

Lemma 2.15. *If $0, \beta_1, \dots, \beta_m, \gamma$ satisfy the parameter assumption, and $\gamma > \alpha$, then (i) F is positive on Δ_m; and (ii) F is monotone in x_j on Δ_m for every $1 \leq j \leq m$.*

Proof. (i) If $\alpha \leq 0$, then $\alpha, \beta_1, \dots, \beta_m, \gamma$ satisfy the parameter assumption, so by Lemma 2.14, $F > 0$ on Δ_m. Now suppose $\alpha > 0$. Let Λ denote the set of all j such that $\beta_j > 0$. Order the elements in $\{1, \dots, m\} \setminus \Lambda$ by $t_1 < \cdots < t_k$, where $k = m - |\Lambda|$. Since $\gamma > \alpha \vee 0$, and $\beta_{t_1}, \dots, \beta_{t_k} \leq 0$, we see that for any $n \geq 0$, $\alpha+n, \beta_{t_1}, \dots, \beta_{t_k}, \gamma+n$ satisfy the parameter assumption. So by Lemma 2.14, $F(\alpha+n, \beta_{t_1}, \dots, \beta_{t_k}, \gamma+n; \cdot) > 0$ on Δ_k. Since α, γ, β_j, $j \in \Lambda$, are positive, for any $\underline{n} \in \overline{\mathbb{N}}^\Lambda$, $\frac{(\alpha)_{|\underline{n}|} \prod_{j \in \Lambda} (\beta_j)_{n_j}}{(\gamma)_{|\underline{n}|} \prod_{j \in \Lambda} (1)_{n_j}} > 0$. By (2.12), $F > 0$ on Δ_m.

(ii) By (2.11), $\partial_{x_j} F = \frac{\alpha \beta_j}{\gamma} F_1$, where $F_1 := F(\alpha+1, \underline{\beta}+\underline{e}_j, \gamma+1; \cdot)$. Note that $\alpha+1, \underline{\beta}+\underline{e}_j, \gamma+1$ satisfy the assumption of the lemma. By (i) $F_1 > 0$ on Δ_m. So the conclusion holds. $\square$

Theorem 2.16. *If $\alpha, \beta_1, \dots, \beta_m, \gamma \in \mathbb{R}$ satisfy the parameter assumption, then F extends to a positive continuous function on $\overline{\Delta_m} = \{(x_1, \dots, x_m) : 0 \le x_1 \le \cdots < x_m \le 1\}$.*

Proof. We prove the theorem by induction. If $m = 1$, then $\overline{\Delta}_1 = [0, 1]$. The statement holds by Lemma 2.14 and Gauss's Theorem. Note that $F(1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1)} > 0$. Now suppose $m \ge 2$, and the statement holds for $m - 1$. Let $S_L = \{\underline{x} \in \Delta_m : x_m = x_{m-1}\}$ and $S_U = \Delta_{m-1} \times \{1\}$. We define P_L and P_U by

$$P_L(x_1, \dots, x_{m-1}, x_m) = (x_1, \dots, x_{m-1}, x_{m-1}), \quad P_U(x_1, \dots, x_{m-1}, x_m) = (x_1, \dots, x_{m-1}, 1).$$

Then for each $\underline{x} \in \overline{\Delta_m}$, $P_L(\underline{x}) \in \overline{S_L}$, $P_U(\underline{x}) \in \overline{S_U}$, and $\underline{x} \in [P_L(\underline{x}), P_U(\underline{x})]$.

By (2.14) and induction hypothesis, $F|_{S_L}$ extends to a positive continuous function on $\overline{S_L}$, and we denote it by F_L. Since $\gamma > \alpha + \beta_m$, by (2.15) and the induction hypothesis, $F|_{S_U}$ extends to a positive continuous function F_U on $\overline{S_U}$.

Suppose (x^n) is a sequence in Δ_m with $x^n \to x^0 \in \overline{\Delta}_m$. To prove the existence of the continuation of F on $\overline{\Delta_m}$, we need to show that $(F(x^n))$ converges. If $x^0_{m-1} < 1$, then this is true since F extends continuously to $[0, 1)^{m-1} \times [0, 1] \supset \Delta_m \cup S_U \ni x^0$. Now suppose $x^0_{m-1} = 1$. Then $P_U(x^n), P_L(x^n) \to x^0$. So we have $F(P_U(x^n)) = F_U(P_U(x^n)) \to F_U(x^0)$ and $F(P_L(x^n)) = F_L(P_L(x^n)) \to F_L(x^0)$. By Lemma 2.15 (ii), F is monotone in x_m on Δ_m. So $F(x^n)$ lies between $F(P_U(x^n))$ and $F(P_L(x^n))$. For the existence of the limit, it remains to show that $F_U = F_L$ on $\overline{S_U} \cap \overline{S_L} = \overline{\Delta_{m-2}} \times \{1\} \times \{1\}$, which follows easily from (2.14,2.15). Finally, since F_U and F_L are respectively positive on $\overline{S_U}$ and $\overline{S_L}$, and F is monotone in x_m, we conclude that F is positive on $\overline{\Delta_m}$. □

3 Intermediate SLE$_\kappa(\underline{\rho})$ Processes

3.1 Forward curves

Fix $\kappa \in (0, 8)$. Let $m \in \mathbb{N}$, $\mathbb{N}_m = \{n \in \mathbb{N} : 1 \le n \le m\}$, and $\mathbb{N}_m^\infty = \mathbb{N}_m \cup \{\infty\}$. Let $\rho_j \in \mathbb{R}$, $j \in \mathbb{N}_m^\infty$, satisfy that $\sum_{j\in\mathbb{N}_m^\infty} \rho_j = 0$, and for $k \in \mathbb{N}_m$, $\sum_{j=1}^k \rho_j > \max\{-2, \frac{\kappa}{2} - 4\}$. Let $w \in \mathbb{R}$. Let $v_1, \dots, v_m, v_\infty \in \mathbb{R}_w \cup \{+\infty, -\infty\}$ be such that either $w^- \ge v_1 \ge \cdots \ge v_m \ge v_\infty$ or $w_+ \le v_1 \le \cdots \le v_m \le v_\infty$. Let $\underline{\rho} = (\rho_1, \dots, \rho_m)$ and $\underline{v} = (v_1, \dots, v_m, v_\infty)$. We are going to define an intermediate SLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from w to ∞ with force points $\underline{v}$. By symmetry, we only need to deal with the case that $w^- \ge v_1 \ge \cdots \ge v_m \ge v_\infty$.

Let η be a chordal SLE$_\kappa(\underline{\rho}, \rho_\infty)$ curve in $\mathbb{H}$ started from $w \in \mathbb{R}$ with force points $\underline{v}$. Since $\sum_{j=1}^k \rho_j > -2$ for $1 \le k \le m$, and $\sum_{j\in\mathbb{N}_m^\infty} \rho_j = 0 > -2$, there is no continuation threshold for η, so the lifetime of η is ∞, and $\eta(t) \to \infty$ as $t \to \infty$. Since $\sum_{j\in\mathbb{N}_m^\infty} \rho_j = 0 > \frac{\kappa}{2} - 4$ and $\sum_{j=1}^k \rho_j > \frac{\kappa}{2} - 4$ for $1 \le k \le m$, η a.s. does not visit any of v_j, $j \in \mathbb{N}_m^\infty$, which is different from w^- or $-\infty$. Let (K_t) be the chordal Loewner hulls generated by η, let $\widehat{w}$ be the chordal Loewner driving function for η, and let $\widehat{v}_j$ be the force point function started from v_j, $j \in \mathbb{N}_m^\infty$. Then $\widehat{v}_j(t) = g^w_{K_t}(v_j)$ (Definition 2.6), and $\widehat{w} \ge \widehat{v}_1 \ge \cdots \widehat{v}_m \ge \widehat{v}_\infty$. Moreover, for some standard

Brownian motion B, $\widehat{w}$ and $\widehat{v}_j$ satisfy the SDE:

$$d\widehat{w}(t)=\sqrt{\kappa}dB(t)+\sum_{k=1}^m\Big(\frac{\rho_k}{\widehat{w}(t)-\widehat{v}_k(t)}-\frac{\rho_k}{\widehat{w}(t)-\widehat{v}_\infty(t)}\Big)dt,\quad \widehat{w}(0)=w; \tag{3.1}$$

$$d\widehat{v}_j(t)=\frac{2}{\widehat{v}_j(t)-\widehat{w}(t)}\,dt,\quad \widehat{v}_j(0)=v_j,\quad j\in\mathbb{N}_m^\infty. \tag{3.2}$$

For $j\in\mathbb{N}_m^\infty$, let $\widehat{x}_j=\widehat{w}_j-\widehat{v}_j$. Then $0\le\widehat{x}_1\le\cdots\le\widehat{x}_m\le\widehat{x}_\infty$. If $v_j=-\infty$, then $\widehat{x}_j\equiv+\infty$; otherwise $\widehat{x}_j$ is finite and satisfies the SDE

$$d\widehat{x}_j(t)=\sqrt{\kappa}dB(t)+\sum_{k=1}^m\Big(\frac{\rho_k}{\widehat{x}_k(t)}-\frac{\rho_k}{\widehat{x}_\infty(t)}\Big)dt+\frac{2}{\widehat{x}_j(t)}\,dt. \tag{3.3}$$

For $j\in\mathbb{N}_m^\infty$, let T_j denote the first time that $\widehat{x}_j=0$. Then $T_1\le\cdots\le T_m\le T_\infty$. In the case that $v_j=w^-$, we have $T_j=0$. Define continuous processes I_j, $j\in\mathbb{N}_m$, on $[0,\infty]$ by

$$I_j(t)=\exp\Big(\int_0^t \mathbf{1}_{\{\widehat{x}_j\widehat{x}_\infty\ne0\}}(s)\Big(\frac{2}{\widehat{x}_\infty(s)^2}-\frac{2}{\widehat{x}_j(s)\widehat{x}_\infty(s)}\Big)\,ds\Big). \tag{3.4}$$

Note that the set of t such that any $\widehat{x}_j(t)$ equals 0 has Lebesgue measure zero. Since $0\le\widehat{x}_j\le\widehat{x}_\infty$, I_j is nonnegative and decreasing. Since $\widehat{x}_j=\widehat{x}_\infty$ after T_∞, I_j is constant on $[T_\infty,\infty]$. If $v_j=v_\infty$, then $\widehat{x}_j\equiv\widehat{x}_\infty$, and so $I_j\equiv1$. If $v_\infty=-\infty$, then $I_j\equiv1$ for all j. Now suppose $v_j\ne v_\infty$ and $v_\infty\ne-\infty$. For $0\le t<T_\infty$, we define $\Omega_j(t)$ to be the union of $\mathbb{H}\setminus K_t$, $(-\infty,v_j)$, and the reflection of $\mathbb{H}\setminus K_t$ about $\mathbb{R}$. Then g_t maps $\Omega_j(t)$ conformally onto $\mathbb{C}\setminus[\widehat{v}_j(t),\infty)$, and takes $v_\infty\in\Omega_j(t)$ to $\widehat{v}_\infty(t)$. By chordal Loewner equation and (3.2),

$$\frac{dg_t'(v_j)}{g_t'(v_j)}=-\frac{2}{\widehat{x}_j(t)^2}\,dt,\quad j\in\mathbb{N}_m^\infty;\quad \frac{d|\widehat{v}_j(t)-\widehat{v}_\infty(t)|}{|\widehat{v}_j(t)-\widehat{v}_\infty(t)|}=-\frac{2}{\widehat{x}_j(t)\widehat{x}_\infty(t)}\,dt,\quad j\in\mathbb{N}_m. \tag{3.5}$$

So we get

$$I_j(t)=\frac{|\widehat{v}_j(t)-\widehat{v}_\infty(t)|}{g_t'(v_\infty)|v_j-v_\infty|}=\frac{\operatorname{crad}^{(4)}_{v_\infty}(\Omega_j(t))}{\operatorname{crad}^{(4)}_{v_\infty}(\Omega_j(0))},\quad 0\le t<T_\infty. \tag{3.6}$$

Since $\operatorname{dist}(v_\infty,\eta)>0$, $\mathbb{H}\setminus\eta$ contains a connected component, denoted by D_∞, whose boundary contains v_∞. For $1\le j\le m$, let $\Omega_j(\infty)$ denote the union of D_∞, the reflection of D_∞ about $\mathbb{R}$, and the real interval $(\eta(T_\infty),v_j)$ if $T_\infty<\infty$ or $(-\infty,v_j)$ if $T_\infty=\infty$. Then $\Omega_j(\infty)$ is a simply connected domain containing v_∞. As $t\uparrow T_\infty$, $\Omega_j(t)\to\Omega_j(\infty)$ in the Carathéodory topology. So $\operatorname{crad}^{(4)}_{v_\infty}(\Omega_j(t))\to\operatorname{crad}^{(4)}_{v_\infty}(\Omega_j(\infty))\in(0,\infty)$, which implies that $I_j(T_\infty)=\lim_{t\uparrow T_\infty}I_j(t)$ is a finite positive number.

For $j\in\mathbb{N}_m$, define $R_j=\widehat{x}_j/\widehat{x}_\infty$ on $[0,T_\infty)$, and $R_j\equiv1$ on $[T_\infty,\infty]$. Here if $v_\infty=-\infty$, then R_j is understood as constant 0 if $v_j\ne-\infty$, and constant 1 if $v_j=-\infty$. Then $0\le R_1\le$

$\cdots \le R_m \le 1$. If $v_\infty = w^-$, then $T_\infty = 0$, and all $R_j \equiv 1$. Suppose now $v_\infty \notin \{w^-, -\infty\}$. Then $T_\infty > 0$, and each R_j satisfies the following SDE up to T_∞

$$dR_j = \frac{1-R_j}{\widehat{x}_\infty}\sqrt{\kappa}dB + \frac{1-R_j}{\widehat{x}_\infty^2}\Big[\frac{2}{R_j} + 2 - \kappa + \sum_{k=1}^m \rho_k\Big(\frac{1}{R_k} - 1\Big)\Big]dt. \tag{3.7}$$

Since $\rho_1 + \cdots + \rho_m + \rho_\infty = 0 > \frac{\kappa}{2} - 4$, either $\kappa \in (0,4]$, $T_\infty = \infty$, and $\lim_{t\to T_\infty}\eta(t) = \infty$; or $\kappa \in (4,8)$, $T_\infty < \infty$, and $\eta(T_\infty) \in (-\infty, v_\infty)$. Using the same extremal distance argument as in the proof of Proposition 2.11 except with $[v_\infty, v_j \wedge \min(\eta([0,t]) \cap \mathbb{R})]$ in place of $[v_\infty, v_j]$, we get $R_j(t) \to 1$ as $t \uparrow T_\infty$. Thus, R_j is continuous on $[0,\infty]$. Also note that in any case, (3.7) holds throughout $[0,\infty)$ because of the factor $1 - R_j$ on its RHS.

Define parameters

$$\alpha = 1 - \frac{4}{\kappa}, \quad \beta_j = \frac{2\rho_j}{\kappa}, \quad j \in \mathbb{N}_m, \quad \gamma = \frac{4}{\kappa} + \sum_{k\in\mathbb{N}_m} \beta_k, \tag{3.8}$$

and $F = F(\alpha, \beta_1, \dots, \beta_k, \gamma; \cdot)$. Let $\underline{R}(t) = (R_1(t), \cdots, R_m(t)) \in \overline{\Delta_m}$, and

$$M(t) = \frac{F(\underline{R}(t))}{F(\underline{R}(0))}\prod_{j\in\mathbb{N}_m} I_j(t)^{\frac{\alpha\rho_j}{2}}, \quad t \in [0,\infty]. \tag{3.9}$$

Lemma 3.1. *M is a uniformly integrable positive continuous martingale.*

Proof. It is easy to see that $\alpha, \beta_1, \dots, \beta_k, \gamma$ satisfy the parameter assumption in Definition 2.13. By Theorem 2.16, F extends to a positive continuous function on $\overline{\Delta_m}$. Since $\underline{R}$ is continuous and takes values in $\overline{\Delta_m}$, $F(\underline{R})$ is positive and continuous. We also know that I_j, $1 \le j \le m$, are positive and continuous. So M is positive and continuous.

Now we prove the martingale property. If all v_j's are equal to v_∞, then all R_j's and I_j's are constant 1, and so is M. If $v_\infty = -\infty$, then all I_j's are constant 1, and $\underline{R}$ is constant, and so M is again constant 1. Now we suppose that not all v_j, $1 \le j \le m$, are equal to v_∞, and $v_\infty \ne -\infty$. Let m' be the biggest $j \le m$ such that $v_j \ne v_\infty$. Then $v_j \ne v_\infty$, $1 \le j \le m'$, and $v_j = v_\infty$, $m'+1 \le j \le m$. So η is a chordal $\mathrm{SLE}_\kappa(\rho_1, \dots, \rho_{m'}, \rho'_\infty)$ curve in $\mathbb{H}$ started from w with force points $v_1, \dots, v_{m'}, v_\infty$, where $\rho'_\infty = \rho_\infty + \sum_{j=m'+1}^m \rho_j = -\sum_{j=1}^{m'} \rho_j$. We have $I_j = R_j = 1$ for $m'+1 \le j \le m$. Let $\widetilde{\gamma} = \frac{4}{\kappa} + \sum_{j=1}^{m'} \beta_j$ and $\widetilde{F} = F(\alpha, \beta_1, \dots, \beta_{m'}, \widetilde{\gamma}; \cdot)$. Then by (2.15), $F(x_1, \dots, x_{m'}, 1, \dots, 1)$ equals a constant times $\widetilde{F}(x_1, \dots, x_{m'})$. Let $\underline{\widetilde{R}} = (R_1, \dots, R_{m'})$. Then $M(t) = \frac{\widetilde{F}(\underline{\widetilde{R}}(t))}{\widetilde{F}(\underline{\widetilde{R}}(0))}\prod_{j=1}^{m'} I_j(t)^{\frac{\alpha\rho_j}{2}}$, which is the M defined for the chordal $\mathrm{SLE}_\kappa(\rho_1, \dots, \rho_{m'}, \rho'_\infty)$ curve. So by replacing m by m' we may assume below that $v_j \ne v_\infty$ for all $1 \le j \le m$.

Since $\underline{R}(t) \in [0,1)^m$ for $t < T_\infty$, from (2.18,2.20,3.4,3.7) and Itô's formula, we see that $M(t)$ is a local martingale up to T_∞. Here we used the fact that the set of t such that any $R_j(t)$ equals 0 has Lebesgue measure zero. Since M is constant on $[T_\infty, \infty]$, it is a local martingale throughout $[0,\infty]$. To show that M is uniformly integrable, it suffices to show that $\sup_{0\le t<T_\infty} M(t)$ is integrable. By Theorem 2.16, $|\log(F(\underline{R}(t)))|$ is bounded by a constant

depending only on κ and ρ_j's. So we only have to control the size of $\prod_{j=1}^m I_j^{\frac{\alpha\rho_j}{2}}$. From (3.6) and (3.4), we easily get

$$I_k \ge I_j \ge \frac{|v_k - v_\infty|}{|v_j - v_\infty|} I_k, \quad \text{on } [0, T_\infty), \quad 1 \le j \le k \le m. \tag{3.10}$$

Let $\rho_\Sigma = \sum_{j=1}^m \rho_j$. By (3.10), it now suffices to show that $\sup_{0\le t<T_\infty} I_m(t)^{\frac{\alpha\rho_\Sigma}{2}}$ is integrable. Since I_m is decreasing, if $\alpha\rho_\Sigma \ge 0$, then $\sup_{0\le t<T_\infty} I_m(t)^{\frac{\alpha\rho_\Sigma}{2}}$ is bounded by 1, and so is integrable. Now we assume that $\alpha\rho_\Sigma < 0$. Then $I_m(t)^{\frac{\alpha\rho_\Sigma}{2}}$ is increasing.

Let $\tau_n = T_\infty \wedge \inf\{t \in [0, T_\infty) : M(t) \ge n\}$, $n \in \mathbb{N}$. Then (τ_n) is an increasing sequence of stopping times tending to T_∞, and for each n, $M(t \wedge \tau_n)$ is a bounded martingale. By Optional Stopping Theorem, $\mathbb{E}[M(\tau_n)] = M(0) = 1$. By Theorem 2.16 and (3.10), $M(\tau_n) \asymp I_m(\tau_n)^{\frac{\alpha\rho_\Sigma}{2}}$, with the implicit constants depending only on $\kappa, \rho_1, \dots, \rho_m, v_1, \dots, v_m, v_\infty$. Thus, $\mathbb{E}[I_m(\tau_n)^{\frac{\alpha\rho_\Sigma}{2}}]$ is bounded by a constant. Since $I_m(t)^{\frac{\alpha\rho_\Sigma}{2}}$ is increasing, and $\tau_n \uparrow T_\infty$, by monotone convergence theorem, $\mathbb{E}[\sup_{0\le t<T_\infty} I_m(t)^{\frac{\alpha\rho_\Sigma}{2}}] < \infty$. So the proof is done. □

By this lemma, we know that $\mathbb{E}[M(T_\infty)] = M(0) = 1$. So we may define another probability measure by weighting the law of η by $M(T_\infty)$.

Definition 3.2. A (forward) intermediate $\text{SLE}_\kappa(\underline{\rho})$ ($\text{iSLE}_\kappa(\underline{\rho})$ for short) curve in $\mathbb{H}$ from w to ∞ with force points $\underline{v}$ is a random curve η, whose law is absolutely continuous w.r.t. that of a chordal $\text{SLE}_\kappa(\underline{\rho}, \rho_\infty)$ curve in $\mathbb{H}$ from w to ∞ with force points $\underline{v}$, and the Radon-Nikodym derivative is $M(T_\infty)$. We extend the definition to general simply connected domains via conformal maps.

We now describe some properties of the $\text{iSLE}_\kappa(\underline{\rho})$ curve. Because of the absolute continuity, it satisfies every almost sure property of the chordal $\text{SLE}_\kappa(\underline{\rho}, \rho_\infty)$ curve. For example, it a.s. ends at its target, and does not visit any of its force points not immediately next to any of its endpoints. If $\kappa \le 4$, the curve is simple, does not visit the boundary arc between its two endpoints which does not contain any force point, and does not visit the boundary arc between its target point and its last force point which does not contain its initial point. In the case that the domain is $\mathbb{H}$, and the force points are on the left of the initial point w, these two boundary arcs that will not be visited are $(w, +\infty)$ and $(-\infty, v_\infty)$.

There are some degenerate cases. If all v_j's are equal to v_∞, then since $\sum_{j=1}^m \rho_j + \rho_\infty = 0$, and M is constant 1, the $\text{iSLE}_\kappa(\underline{\rho})$ curve is just a chordal SLE_κ curve in $\mathbb{H}$ from w to ∞. If $v_\infty = -\infty$, then M is again constant 1, and the $\text{iSLE}_\kappa(\underline{\rho})$ curve is a chordal $\text{SLE}_\kappa(\underline{\rho})$ curve with force points $v_1, \dots, v_m$. So Theorem 1.1 is a special case of Theorem 1.2.

Now we assume that not all v_j's are equal to v_∞, and $v_\infty \ne -\infty$. By merging force points as we did in the proof of Theorem 3.1, we may assume that $v_j \ne v_\infty$ for all $j \in \mathbb{N}_m$. We now derive a formula of $M(T_\infty)$ in terms of conformal radius. Since $\Omega_j(t) \to \Omega_j(\infty)$ in the Carathéodory topology, by (3.6) we have $I_j(t) \to \text{crad}^{(4)}_{v_\infty}(\Omega_j(\infty)) / \text{crad}^{(4)}_{v_j}(\Omega_j(0))$ as $t \uparrow T_\infty$.

Recall that $R_j(t)\to 1$ as $t\uparrow T_\infty$. Let $\underline{1}=(1,\dots,1)\in\mathbb{R}^m$. Then we get

$$M(T_\infty)=\frac{F(\underline{1})}{F(\underline{R}(0))}\prod_{j\in\mathbb{N}_m}\Big(\frac{\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(\infty))}{\mathrm{crad}^{(4)}_{v_\infty}(\Omega_j(0))}\Big)^{\frac{\rho_j(\kappa-4)}{2\kappa}}. \tag{3.11}$$

3.2 Reversed curves

Let $\kappa,\rho_1,\dots,\rho_m,\rho_\infty,\underline{\rho}$ be as in Section 3.1. Let $w^r\in\mathbb{R}$ and $v^r_1,\dots,v^r_m,v^r_\infty\in\mathbb{R}_{w^r}\cup\{+\infty,-\infty\}$ be such that either $(w^r)^+\le v^r_\infty\le v^r_m\le\cdots\le v^r_1$ or $(w^r)^-\ge v^r_\infty\ge v^r_m\ge\cdots\ge v^r_1$. Let $\underline{v}^r=(v^r_1,\dots,v^r_m,v^r_\infty)$. We will define a reversed intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from w^r to ∞ with force points $\underline{v}^r$. By symmetry, we only need to deal with the case that $(w^r)^+\le v^r_\infty< v^r_m\le\cdots\le v^r_1$.

Let $\rho^r_j=-\rho_j$, $j\in\mathbb{N}_m^\infty$, and $\underline{\rho}^r=(\rho^r_1,\dots,\rho^r_m)$. Let η^r be a chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r,\rho^r_\infty)$ curve in $\mathbb{H}$ started from w^r with force points $\underline{v}^r$. By the assumptions on ρ_j's, we have $\rho^r_\infty=\sum_{j=1}^m\rho_j>-2$ and for any $k\in\mathbb{N}_m$, $\rho^r_\infty+\sum_{j=k}^m\rho^r_j=\sum_{j=1}^{k-1}\rho_j>-2$. So there is no continuation threshold for η^r. Thus, the lifetime of η^r is ∞, and $\eta^r(t)\to\infty$ as $t\to\infty$. Similarly, we have $\rho^r_\infty>\frac{\kappa}{2}-4$ and for any $k\in\mathbb{N}_m$, $\rho^r_\infty+\sum_{j=k}^m\rho^r_j>\frac{\kappa}{2}-4$. So η^r a.s. does not visit any of its force point other than $(w^r)^+$ or $+\infty$.

Let (K^r_t) be the chordal Loewner hulls generated by η^r, let $\widehat{w}^r$ be the driving function, and let $\widehat{v}^r_j$ be the force point function started from v^r_j, $j\in\mathbb{N}_m^\infty$. Then $\widehat{v}^r_j(t)=g^{w^r}_{K^r_t}(v^r_j)$, $\widehat{w}^r\le\widehat{v}^r_\infty\le\widehat{v}^r_m\le\cdots\le\widehat{v}^r_1$, and for some standard Brownian motion B^r, $\widehat{w}^r$ and $\widehat{v}^r_j$, $j\in\mathbb{N}_m^\infty$, satisfy the SDE:

$$d\widehat{w}^r(t)=\sqrt{\kappa}dB^r(t)+\sum_{k\in\mathbb{N}_m}\Big(\frac{\rho^r_k}{\widehat{w}^r(t)-\widehat{v}^r_k(t)}-\frac{\rho^r_k}{\widehat{w}^r(t)-\widehat{v}^r_\infty(t)}\Big)dt,\quad \widehat{w}^r(0)=w^r; \tag{3.12}$$

$$d\widehat{v}^r_j(t)=\frac{2}{\widehat{v}^r_j(t)-\widehat{w}^r(t)}\,dt,\quad \widehat{v}^r_j(0)=v^r_j,\quad j\in\mathbb{N}_m^\infty. \tag{3.13}$$

For $j\in\mathbb{N}_m^\infty$, let $\widehat{x}^r_j=\widehat{w}^r_j-\widehat{v}^r_j$. Then $0\ge\widehat{x}^r_\infty\ge\widehat{x}^r_m\ge\cdots\ge\widehat{x}^r_1$, and each finite function $\widehat{x}^r_j$ ($v^r_j\ne+\infty$) satisfies the SDE

$$d\widehat{x}^r_j(t)=\sqrt{\kappa}dB^r(t)+\sum_{k\in\mathbb{N}_m}\Big(\frac{\rho^r_k}{\widehat{x}^r_k(t)}-\frac{\rho^r_k}{\widehat{x}^r_\infty(t)}\Big)dt+\frac{2}{\widehat{x}^r_j(t)}\,dt. \tag{3.14}$$

For $j\in\mathbb{N}_m^\infty$, let T^r_j denote the first time that $\widehat{x}^r_j=0$. Then $T^r_\infty\le T^r_m\le\cdots\le T^r_1$. Now Equation (3.5) holds here with additional superscripts "r".

Define I^r_j, $j\in\mathbb{N}_m$, on $[0,\infty]$ by

$$I^r_j(t)=\exp\Big(\int_0^t\mathbf{1}_{\{\widehat{x}^r_j\widehat{x}^r_\infty\ne0\}}(s)\Big(\frac{2}{\widehat{x}^r_j(s)^2}-\frac{2}{\widehat{x}^r_j(s)\widehat{x}^r_\infty(s)}\Big)\,ds\Big). \tag{3.15}$$

Since $\widehat{x}^r_j\ge\widehat{x}^r_\infty\ge0$, I^r_j is continuous and decreasing. Since $\widehat{x}^r_j=\widehat{x}^r_\infty$ on $[T^r_j,\infty)$, I^r_j takes constant value on $[T^r_j,\infty]$. If $v^r_j=v^r_\infty$, then $\widehat{v}^r_j\equiv\widehat{v}^r_\infty$, and so $I^r_j\equiv1$. If $v^r_j=+\infty$, we also get $I_j^r \equiv 1$. Let Λ denote the set of $j \in \mathbb{N}_m$ such that $+\infty > v_j^r > v_\infty^r$. By chordal Loewner equation and (3.13), we find that, for $j \in \Lambda$, up to T_j^r,

$$I_j^r(t) = \frac{|\widehat{v}_j^r(t) - \widehat{v}_\infty^r(t)|}{(g_{K_t^r})'(v_j^r)|v_j^r - v_\infty^r|} = \frac{\operatorname{crad}^{(4)}_{v_j^r}(\Omega^r(t))}{\operatorname{crad}^{(4)}_{v_j^r}(\Omega^r(0))} \geq \frac{|v_j^r - v_\infty^r| \wedge \operatorname{dist}(v_j^r, \eta^r([0,t]))}{4|v_j^r - v_\infty^r|}, \tag{3.16}$$

where $\Omega^r(t)$ is the union of $\mathbb{H} \setminus K_t^r$, the interval $(v_\infty^r \vee \max(\overline{K_t^r} \cap \mathbb{R}), \infty)$, and the reflection of $\mathbb{H} \setminus K_t^r$ about $\mathbb{R}$. The second "=" follows from the fact that $g_{K_t^r}$ maps $\Omega^r(t)$ conformally onto $\mathbb{C} \setminus (-\infty, \widehat{v}_\infty^r(t)]$, and takes v_j^r to $\widehat{v}_j^r(t)$; and the "$\geq$" follows from Koebe's 1/4 theorem. Since I_j^r stays constant on $[T_j^r, \infty]$, and η^r does not get closer to v_j^r after T_j^r, we get

$$1 = I_j^r(0) \geq I_j^r(t) \geq (1 \wedge (\operatorname{dist}(v_j^r, \eta^r([0,t]))/|v_j^r - v_\infty^r|))/4, \quad t \in [0,\infty], \quad j \in \Lambda. \tag{3.17}$$

Since $I_j^r \equiv 1$ for $j \in \mathbb{N}_m \setminus \Lambda$, and $\operatorname{dist}(v_j^r, \eta^r) > 0$, we get $I_j^r > 0$ on $[0,\infty]$ for all $j \in \mathbb{N}_m$.

For $1 \leq j \leq m$, define R_j^r on $[0,\infty]$ such that $R_j^r = \widehat{x}_\infty^r / \widehat{x}_j^r$ on $[0, T_j^r)$, and $R_j^r \equiv 1$ on $[T_j^r, \infty]$. Here if $v_j^r = v_\infty^r = +\infty$, then R_j^r is understood as constant 1; and if $v_j^r = +\infty > v_\infty^r$, then R_j^r is understood as constant 0. Then $0 \leq R_1^r \leq \cdots \leq R_m^r \leq 1$. Let $\underline{R}^r = (R_1^r, \dots, R_m^r) \in \overline{\Delta_m}$. If $v_j^r \neq +\infty$, R_j^r satisfies the following SDE up to T_j^r:

$$dR_j^r = \frac{R_j^r(1 - R_j^r)}{\widehat{x}_\infty^r} \sqrt{\kappa} dB^r + \frac{R_j^r(1 - R_j^r)}{(\widehat{x}_\infty^r)^2} \Big[2 + (2-\kappa)R_j^r + \sum_{k=1}^m \rho_k(1 - R_k^r)\Big] dt. \tag{3.18}$$

The same extremal distance argument as before shows that in the case that for $j \in \Lambda$, as $t \uparrow T_j^r$, $R_j^r \to 1$. Thus, for all $j \in \mathbb{N}_m$, R_j^r is continuous on $[0,\infty]$. Also note that in any case (3.18) holds throughout $[0,\infty]$ because of the factor $R_j^r(1 - R_j^r)$ on the RHS.

Let F be the multiple hypergeometric function as in the last subsection. Define the M^r on $[0,\infty]$ by

$$M^r(t) = \frac{F(\underline{R}^r(t))}{F(\underline{R}^r(0))} \prod_{j \in \mathbb{N}_m} I_j^r(t)^{\frac{\alpha\rho_j}{2}} = \frac{F(\underline{R}^r(t))}{F(\underline{R}^r(0))} \prod_{j \in \Lambda} I_j^r(t)^{\frac{\rho_j(\kappa-4)}{2\kappa}}. \tag{3.19}$$

Lemma 3.3. *M^r is a positive continuous local martingale.*

Proof. The continuity and positiveness of M follows from the continuity and positiveness of $F(\underline{R}^r)$ and I_j^r, $1 \leq j \leq m$. Here we use the continuity of $\underline{R}^r$ and the continuity and positiveness of F on $\overline{\Delta_m}$. Now we check the local martingale property of M^r.

If $v_j^r = v_\infty^r$, then R_j^r is constant 1. If $v_j^r = +\infty > v_\infty^r$, then R_j^r is constant 0. Thus, if $\Lambda = \emptyset$, then $F(\underline{R}^r)$ is constant, and so is M^r. Suppose now $\Lambda = \{j \in \mathbb{N}_m : m_1 \leq j \leq m_2\} \neq \emptyset$, where $m_1 \leq m_2 \in \mathbb{N}_m$. By (2.13,2.15), $F(0,\dots,0,x_{m_1},\dots,x_{m_2},1,\dots,1)$ equals a constant times $\widetilde{F}(x_{m_1},\dots,x_{m_2})$, where $\widetilde{F}$ is the multiple hypergeometric function $F(\alpha, \beta_{m_1}, \dots, \beta_{m_2}, \gamma - \sum_{k=m_2+1}^m \beta_k; \cdot)$. So $M^r(t) = \frac{\widetilde{F}(\underline{R}^c(t)|_\Lambda)}{\widetilde{F}(\underline{R}^c(0)|_\Lambda)} \prod_{j \in \Lambda} I_j^r(t)^{\frac{\alpha\rho_j}{2}}$.

For $j \in \Lambda$, we have $R_j^c(t) < 1$ before $T_{m_2}^r$. By (2.19,2.20,3.15,3.18) and Itô's formula, we find that M^r is a local martingale up to $T_{m_2}^r$. Conditionally on $\widehat{w}^r(t)$, $t \leq T_{m_2}^r$, the process

$\widetilde{w} := \widehat{w}(T^r_{m_2} + \cdot)$ is the driving function of a chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r, \rho^r_\infty)$ curve in $\mathbb{H}$ started from $\widehat{w}^r(T^r_{m_2})$ with force points $\widehat{v}^r_j(T^r_{m_2})$ and force point processes $\widetilde{v}^r_j := \widehat{v}^r_j(T^r_{m_2} + \cdot)$, $j \in \mathbb{N}^\infty_m$. If we define $\widetilde{M}^r$ for this process, then from what we have proved, $\widetilde{M}^r$ is a local martingale up to the first time that any force point $\widehat{v}^r_j(T^r_{m_2})$, which lies strictly between $\widehat{w}^r(T^r_{m_2})$ and $+\infty$, is separated from ∞. Moreover, $\widetilde{M}^r = M^r(T^r_{m_2} + \cdot)/M^r(T^r_{m_2})$. So M^r is a local martingale at least up to $T^r_{m_2-1}$. Repeating this argument, we conclude that M^r is a local martingale up to T^r_1. Since every R^r_j is constant 1 on $[T^r_1, \infty]$, and every I^r_j's takes (random) constant value on $[T^r_1, \infty]$, so does M^r. Thus, M^r is a local martingale throughout $[0, \infty]$. □

Definition 3.4. A reversed intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ ($\mathrm{iSLE}^r_\kappa(\underline{\rho})$ for short) curve in $\mathbb{H}$ from w^r to ∞ with force points $\underline{v}^r$ is a random curve, whose law is obtained by locally weighting the law of a chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r, \rho^r_\infty)$ curve in $\mathbb{H}$ started from w^r with force points $\underline{v}^r$ by the positive continuous local martingale M^r (which is then a supermartingale) as in Lemma A.1. We extend the definition to general simply connected domains via conformal maps.

Remark 3.5. By Lemma A.1, the law of the $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ curve is absolutely continuous w.r.t. the chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r, \rho^r_\infty)$ curve if and only if M^r is uniformly integrable w.r.t. the latter law, and then the Radon-Nikodym derivative is $M^r(\infty)$. We will see that this holds if κ and $\rho_1, \dots, \rho_m$ satisfies Condition (I) or (II) in Theorem 1.1.

We now describe some properties of the $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ curve. Because of the local absolute continuity, it satisfies every local almost sure property of the chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r, \rho^r_\infty)$ curve. For example, before the end of its lifetime, it a.s. does not visit any of its force points not immediately next to its initial point. If $\kappa \le 4$, then the curve is simple. If, in addition, its law is (globally) absolutely continuous w.r.t. that of the chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r, \rho^r_\infty)$ curve, then it a.s. do not accumulate at any of its force points not immediately next to any of its endpoints. The following lemma provides us the converse statement.

Lemma 3.6. *Let $\mathbb{P}_r$ denote the law of an $iSLE^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from w^r to ∞ with force points $\underline{v}^r$. Let $\mathbb{P}_c$ denote the law of a chordal $SLE_\kappa(\underline{\rho}^r, \rho^r_\infty)$ curve in $\mathbb{H}$ started from w^r with force points $\underline{v}^r$. Let $\mathcal{F}^r$ be the filtration. Let $S = \{v_j : j \in \Lambda\}$. Then we have the following.*

(i) Let τ be an $\mathcal{F}^r$-stopping time such that $\mathrm{dist}(\eta([0,\tau)), S)$ is bounded from below by a positive constant, then $M^r(\cdot \wedge \tau)$ is uniformly bounded. If $\mathbb{P}_c$ is supported by the space of curves whose lifetimes are strictly greater than τ, then so is $\mathbb{P}_r$.

(ii) $\mathbb{P}_r$ restricted to the event $\{\mathrm{dist}(\eta, S) > 0\}$ is absolutely continuous w.r.t. $\mathbb{P}_c$.

(iii) If $\mathbb{P}_r$ is supported by $\{\mathrm{dist}(\eta, S) > 0\}$, then $\mathbb{P}_r \ll \mathbb{P}_c$.

(iv) $\mathbb{P}_r$ is supported by the set of curves that have zero spherical distance from $S \cup \{\infty\}$.

Here we use the convention that if $S = \emptyset$, then $\mathrm{dist}(\eta, \emptyset) = \infty$.

Proof. (i) By (3.19,3.17) and the fact that F is continuous and positive on the compact set $\overline{\Delta_m}$, $M^r(\cdot \wedge \tau)$ is uniformly bounded. If $\mathbb{P}_c[T_\Sigma > \tau] = 1$, then by Lemma A.1 (iii), $\mathbb{P}_r[T_\Sigma > \tau] = \mathbb{E}_c[\mathbf{1}_{\{\tau<\infty\}} M^r(\tau)] = \mathbb{E}_c[M^r(\tau)] = M^r(0) = 1$.

(ii) For each $n \in \mathbb{N}$, let τ_n be the first t such that $\mathrm{dist}(\eta([0,t]), S) \le 1/n$, which satisfies the assumption in (i). By (i) and Lemma A.1 (iv), $\mathbb{P}_r$ restricted to $\mathcal{F}^r_{\tau_n}$ is absolutely continuous w.r.t. $\mathbb{P}_c$ restricted to $\mathcal{F}^r_{\tau_n}$. Since $\mathcal{F}^r_{\tau_n}$ agrees with $\mathcal{F}^r_\infty$ on the event $\{\tau_n = \infty\}$, the restriction of $\mathbb{P}_r$ to $\{\tau_n = \infty\}$ is absolutely continuous w.r.t. $\mathbb{P}_c$. Since $\{\mathrm{dist}(\eta, S) > 0\} = \bigcup_{n\in\mathbb{N}} \{\tau_n = \infty\}$, we get (ii). Finally, (iii) and (iv) follow immediately from (ii) and the fact that $\mathbb{P}_c$ is supported by the curves that end at ∞. □

Remark 3.7. In the case that $m = 1$, the $\mathrm{iSLE}_\kappa(\rho)$ and $\mathrm{iSLE}^r_\kappa(\rho)$ curves both agree with the intermediate $\mathrm{SLE}_\kappa(\rho)$ curve defined in [21]. So Theorems 1.1 and 1.2 extend the reversibility results there. For $m \ge 2$, an $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ curve is in general different from an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve.

3.3 Driving functions

Define G_j, $1 \le j \le m$, on $\overline{\Delta_m}$ by

$$G_j(\underline{x}) = \mathbf{1}_{\{x_j \ne 1\}} x_j \cdot \frac{\partial_{x_j} F(\underline{x})}{F(\underline{x})}. \tag{3.20}$$

We know that $\partial_{x_j} F$ is well defined on $(-1,1)^m$. Since by (2.15), $F(x_1,\dots,x_{m'},1,\dots,1)$ equals some constant times $F(\alpha_1,\beta_1,\dots,\beta_{m'},\gamma-\sum_{k=m'+1}^m \beta_k; x_1,\dots,x_{m'})$, $\partial_{x_j} F$ is also well defined on $\overline{\Delta_m} \cap \{\underline{x} \in \mathbb{R}^m : x_j < 1\}$. Since F is positive on $\overline{\Delta_m}$, G_j is well defined on $\overline{\Delta_m}$. By Girsanov Theorem we see that the driving function $\widehat{w}$ for the $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from w to ∞ with force points $v_1,\dots,v_m,v_\infty$, which generates chordal Loewner hulls (K_t), satisfies the SDE

$$d\widehat{w}(t) = \sqrt{\kappa} dB(t) + \sum_{j=1}^m \Big(\frac{1}{\widehat{w}(t)-\widehat{v}_j(t)} - \frac{1}{\widehat{w}(t)-\widehat{v}_\infty(t)}\Big)[\rho_j + \kappa G_j(\underline{R}(t))]dt, \tag{3.21}$$

where B is a standard Brownian motion; $\widehat{v}_j(t) = g^w_{K_t}(v_j)$, $j \in \mathbb{N}^\infty_m$; $\underline{R} = (R_1,\dots,R_m)$; and $R_j(t) = \frac{\widehat{w}(t)-\widehat{v}_j(t)}{\widehat{w}(t)-\widehat{v}_\infty(t)}$ before the first time that $\widehat{w}(t) = \widehat{v}_\infty(t)$, and equals 1 after that time.

Similarly, the driving function $\widehat{w}^r$ for an $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from w^r to ∞ with force points $v^r_1,\dots,v^r_m,v^r_\infty$, which generates chordal Loewner hulls (K^r_t), satisfies the SDE

$$d\widehat{w}^r(t) = \sqrt{\kappa} dB^r(t) - \sum_{j=1}^m \Big(\frac{1}{\widehat{w}^r(t)-\widehat{v}^r_j(t)} - \frac{1}{\widehat{w}^r(t)-\widehat{v}^r_\infty(t)}\Big)[\rho_j + \kappa G_j(\underline{R}^r(t))]dt, \tag{3.22}$$

where B^r is a standard Brownian motion, $\widehat{v}^r_j(t) = g^{w^r}_{K^r_t}(v^r_j)$, $j \in \mathbb{N}^\infty_m$; $\underline{R}^r = (R^r_1,\dots,R^r_m)$; and $R^r_j(t) = \frac{\widehat{w}^r(t)-\widehat{v}^r_\infty(t)}{\widehat{w}^r(t)-\widehat{v}^r_j(t)}$ before the first time that $\widehat{w}^r(t) = \widehat{v}^r_\infty(t)$, and equals 1 after that time.

From the SDEs for driving functions, we see that both $\mathrm{iSLE}_\kappa(\underline{\rho})$ and $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ processes satisfy DMP. We now provide a proof for the DMP of $\mathrm{iSLE}_\kappa(\rho)$. Suppose by symmetry that $\widehat{v}_j \le \widehat{w}$ for all j. We claim that, for any $j \in \mathbb{N}_m^\infty$ and $\tau, t \ge 0$,

$$\widehat{v}_j(\tau+t) = \begin{cases} g^{\widehat{w}(\tau)}_{K_{\tau+t}/K_\tau}(\widehat{v}_j(\tau)), & \text{if } \widehat{v}_j(\tau) < \widehat{w}(\tau); \\ g^{\widehat{w}(\tau)}_{K_{\tau+t}/K_\tau}(\widehat{w}(\tau)^-), & \text{if } \widehat{v}_j(\tau) = \widehat{w}(\tau). \end{cases} \tag{3.23}$$

If $\tau \cdot t = 0$, the statement is trivial. Suppose now $\tau, t > 0$. Let $\widetilde{v}_j(\tau+t)$ denote the RHS of (3.23). Since $\widehat{w}(\tau) \in \overline{K_{\tau+t}/K_\tau}$ and $\widehat{w}(\tau) \ge c_{K_\tau}$, we see that $g_{K_{\tau+t}/K_\tau}$ maps $\mathbb{C}\setminus((K_{\tau+t}/K_\tau)^{\mathrm{doub}} \cup [\widehat{v}_j(\tau),\infty))$ conformally onto $\mathbb{C}\setminus[\widetilde{v}_j(\tau+t),\infty)$. By the definition of $\widehat{v}_j$, $g_{K_{\tau+t}}$ and g_{K_τ} maps $\mathbb{C}\setminus(K^{\mathrm{doub}}_{\tau+t} \cup [v_j,\infty))$ conformally onto $\mathbb{C}\setminus[\widehat{v}_j(\tau+t),\infty)$ and $\mathbb{C}\setminus((K_{\tau+t}/K_\tau)^{\mathrm{doub}} \cup [\widehat{v}_j(\tau),\infty))$, respectively. Thus, $g_{K_{\tau+t}/K_\tau}$ maps $\mathbb{C}\setminus((K_{\tau+t}/K_\tau)^{\mathrm{doub}} \cup [\widehat{v}_j(\tau),\infty))$ conformally onto $\mathbb{C}\setminus[\widehat{v}_j(\tau+t),\infty)$. So we have $\widetilde{v}_j(\tau+t) = \widehat{v}_j(\tau+t)$, as desired.

Suppose now τ is an $\mathcal{F}$-stopping time. On the event $\tau < \infty$, define $B^\tau(t) = B(\tau+t) - B(\tau)$, $\widehat{w}^\tau = \widehat{w}(\tau+\cdot)$, and $\widehat{v}^\tau_j = \widehat{v}_j(\tau+\cdot)$, $j \in \mathbb{N}_m^\infty$. Then B^τ is a Brownian motion conditionally on $\mathcal{F}_\tau$ and the event $\{\tau<\infty\}$; and $\widehat{w}^\tau$, $\widehat{v}^\tau_j$, $j \in \mathbb{N}_m^\infty$, and B^τ solve (3.21). The chordal Loewner hulls generated by $\widehat{w}^\tau$ are $K^\tau_t := K_{\tau+t}/K_\tau$, $t \ge 0$. By (3.23), $\widehat{v}^\tau_j(t) = g^{\widehat{w}(\tau)}_{K^\tau}(\widehat{v}^\tau_j(0))$, where $\widehat{v}^\tau_j(0) = \widehat{v}_j(\tau)$ is understood as $\widehat{w}(\tau)^-$ if $\widehat{v}_j(\tau) = \widehat{w}(\tau)$. Thus, $\widehat{w}^\tau$ generates a chordal Loewner curve η^τ, whose law conditional on $\mathcal{F}_\tau$ is an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from $\widehat{w}(\tau)$ to ∞ with force points $\widehat{v}_j(\tau)$, $j \in \mathbb{N}_m^\infty$, where if any $\widehat{v}_j(\tau)$ equals to $\widehat{w}(\tau)$, then as a force point it is treated as $\widehat{w}(\tau)^-$. Since $g^{-1}_{K_\tau}$ maps $\mathbb{H}$ conformally onto $\mathbb{H}\setminus K_\tau$, and maps $\widehat{w}(\tau)$ to $\eta(\tau)$, and $\eta^\tau(t)$ to $\eta(\tau+\cdot)$, the conditional law of $\eta(\tau+\cdot)$ given $\mathcal{F}_\tau$ and the event $\{\tau<\infty\}$ is an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}\setminus K_\tau$ from $\eta(\tau)$ to ∞ with force points $\min(\{v_j\} \cup \eta([0,\tau]) \cap \mathbb{R})$, $j \in \mathbb{N}_m^\infty$. A similar statement with max in place of min holds if $\widehat{v}_j \ge \widehat{w}$ for all j.

At the end of this subsection, we describe the driving function for a forward or reversed intermediate $\mathrm{SLE}_\kappa(\underline{\rho})$ curves in $\mathbb{H}$ when the target is not ∞. Let κ, ρ_j and ρ^r_j, $j \in \mathbb{N}_m^\infty$, $\underline{\rho}$ and $\underline{\rho}^r$ be as before. Let $w_- < w_+ \in \mathbb{R}$. Let $v_\infty \le v_m \le \cdots \le v_1 \in \{w_-^+, w_+^-\} \cup (w_-, w_+)$, and $\underline{v} = (v_1,\dots,v_m,v_\infty)$. Let η be an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from w_+ to w_- with force points $\underline{v}$. Then the part of η up to the first time that it separates w_- from ∞ is a chordal Loewner curve with some speed. After normalization, we make this part of η a chordal Loewner curve (with speed 1), and call it an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from w_+ to w_- with force points $\underline{v}$. We similarly define an $\mathrm{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from w_- to w_+ with force points $\underline{v}$. Following the argument in [16] we obtain the proposition below. We leave the proof to the interested reader.

Proposition 3.8. *(i) The driving process $\widehat{w}_+$ of an $iSLE_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from w_+ to w_- with force points $\underline{v}$, which generates chordal Loewner hulls $(K_+(t))$, satisfies the SDE*

$$d\widehat{w}_+ = \sqrt{\kappa}dB_+ + \frac{\kappa-6}{\widehat{w}_+ - \widehat{w}_-^+}\,dt + \sum_{j=1}^m \Big(\frac{1}{\widehat{w}_+ - \widehat{v}_j^+} - \frac{1}{\widehat{w}_+ - \widehat{v}_\infty^+}\Big)[\rho_j + \kappa G_j(\underline{R}^+)]dt, \tag{3.24}$$

where B_+ is a standard Brownian motion; $\widehat{w}_-^+(t) = g^{w_+}_{K_+(t)}(w_-)$ and $\widehat{v}_j^+(t) = g^{w_+}_{K_+(t)}(v_j)$, $j \in \mathbb{N}_m^\infty$; $\underline{R}^+ = (R_1^+, \dots, R_m^+)$, and $R_j^+ = \frac{\widehat{w}_+ - \widehat{v}_j^+}{\widehat{w}_+ - \widehat{v}_\infty^+} \cdot \frac{\widehat{w}_-^+ - \widehat{v}_\infty^+}{\widehat{w}_-^+ - \widehat{v}_j^+}$ before the first time that the denominator vanishes, and equals 1 after that time.

(ii) *The driving process $\widehat{w}_-$ of an iSLE$_\kappa^r(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from w_- to w_+ with force points $\underline{v}$, which generates chordal Loewner hulls $(K_-(t))$, satisfies the SDE*

$$d\widehat{w}_- = \sqrt{\kappa} dB_- + \frac{\kappa - 6}{\widehat{w}_- - \widehat{w}_+^-} dt - \sum_{j=1}^m \Big(\frac{1}{\widehat{w}_- - \widehat{v}_j^-} - \frac{1}{\widehat{w}_- - \widehat{v}_\infty^-} \Big) [\rho_j + \kappa G_j(\underline{R}^-)] dt, \quad (3.25)$$

where B_- is a standard Brownian motion; $\widehat{w}_+^-(t) = g^{w_-}_{K_-(t)}(w_+)$ and $\widehat{v}_j^-(t) = g^{w_-}_{K_-(t)}(v_j)$, $j \in \mathbb{N}_m^\infty$; $\underline{R}^- = (R_1^-, \dots, R_m^-)$, and $R_j^- = \frac{\widehat{w}_- - \widehat{v}_\infty^-}{\widehat{w}_- - \widehat{v}_j^-} \cdot \frac{\widehat{w}_+^- - \widehat{v}_j^-}{\widehat{w}_+^- - \widehat{v}_\infty}$ before the first time that the denominator vanishes, and equals 1 after that time.

4 Commutation Coupling

We are going to construct a commutation coupling of an iSLE$_\kappa(\underline{\rho})$ curve with an iSLE$_\kappa^r(\underline{\rho})$ curve in the sense of [2]. More specifically, we will prove the following theorem.

Theorem 4.1. *Let $\kappa \in (0, 8)$. Let $\rho_1, \dots, \rho_m, \rho_\infty \in \mathbb{R}$ satisfies that $\sum_{j=1}^k \rho_j > (-2) \vee (\frac{\kappa}{2} - 4)$ for any $k \in \mathbb{N}_m$, and $\sum_{j \in \mathbb{N}_m^\infty} \rho_j = 0$. Let $w_+ > w_- \in \mathbb{R}$. Let $v_1 > \dots > v_m > v_\infty \in (w_-, w_+) \cup \{w_-^+, w_+^-\}$. Let $\underline{\rho} = (\rho_1, \dots, \rho_m)$ and $\underline{v} = (v_1, \dots, v_m, v_\infty)$. Then there are a pair of random curves $\eta_+(t_+)$, $0 \le t_+ < T_+$, and $\eta_-(t_-)$, $0 \le t_- < T_-$, defined on the same probability space such that η_+ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from w_+ to w_- with force points $\underline{v}$, η_- is an iSLE$_\kappa^r(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from w_- to w_+ with force points $\underline{v}$, and they commute with each other in the following sense. Let $\mathcal{F}^\pm$ and $K_\pm(\cdot)$ be the filtration and chordal Loewner hulls, respectively, generated by $\eta_\pm$.*

(i) *If τ_- is an $\mathcal{F}^-$-stopping time, then conditionally on $\mathcal{F}^-_{\tau_-}$ and the event that $\tau_- < T_-$, up to a time-change, the law of η_+ up to the time that it hits $\eta_-([0, \tau_-])$ is that of an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H} \backslash K_-(\tau_-)$ from w_+ to $\eta_-(\tau_-)$ with force points $v_j \vee \max(\eta_-([0, \tau_-]) \cap \mathbb{R})$, $j \in \mathbb{N}_m^\infty$, up to the time that it hits $\eta_-([0, \tau_-])$ or separates $\eta_-([0, \tau_-])$ from ∞.*

(ii) *If τ_+ is an $\mathcal{F}^+$-stopping time, then conditionally on $\mathcal{F}^+_{\tau_+}$ and the event that $\tau_+ < T_+$, up to a time-change, the law of η_- up to the time that it hits $\eta_+([0, \tau_+])$ is that of an iSLE$_\kappa^r(\underline{\rho})$ curve in $\mathbb{H} \backslash K_+(\tau_+)$ from w_- to $\eta_+(\tau_+)$ with force points $v_j \wedge \min(\eta_+([0, \tau_+]) \cap \mathbb{R})$, $j \in \mathbb{N}_m^\infty$, up to the time that it hits $\eta_+([0, \tau_+])$ or separates $\eta_+([0, \tau_+])$ from ∞.*

Let $\mathbb{P}_\pm$ denote the marginal law of $\eta_\pm$ in the theorem. We call the joint law of η_+ and η_- a (global) commutation coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$. We now introduce local commutation couplings.

For $\sigma \in \{+,-\}$, let Ξ_σ denote the space of crosscuts ξ in $\mathbb{H}$, which have positive distance from $\{w_+, w_-\}$, and separate w_σ from both ∞ and $w_{-\sigma}$. So for each $\xi \in \Xi_\sigma$, $\mathrm{Hull}(\xi)$ contains a neighborhood of w_σ in $\mathbb{H}$, and does not have $w_{-\sigma}$ on its closure. Let Ξ be the set of $(\xi_+, \xi_-) \in \Xi_+ \times \Xi_-$ such that $\overline{\mathrm{Hull}(\xi_+)} \cap \overline{\mathrm{Hull}(\xi_-)} = \emptyset$. For each $\sigma \in \{+,-\}$ and $\xi \in \Xi_\sigma$, let τ^σ_ξ denote the first t such that $\eta_\sigma(t) \in \overline{\xi}$. If such t does not exist, then we set $\tau^\sigma_\xi = T_\sigma$.

For $(\xi_+, \xi_-) \in \Xi$, a coupling of a curve η_+ with law $\mathbb{P}_+$ and a curve η_- with law $\mathbb{P}_-$ is called a locally commutation coupling within (ξ_+, ξ_-) if Theorem 4.1 (i) holds up to $\tau^+_{\xi_+}$ with the additional assumption that $\tau_- \le \tau^-_{\xi_-}$, and Theorem 4.1 (ii) holds up to $\tau^-_{\xi_-}$ with the additional assumption that $\tau_+ \le \tau^+_{\xi_+}$.

This section is devoted to the proof of this theorem. The construction of the coupling follows the procedure in [23, 22]. We first study how two deterministic/random chordal Loewner curves interact with each other. Then use that to construct local commutation couplings, and finally extend the local couplings to a global commutation coupling.

4.1 Deterministic ensemble

Let w_+, w_- and v_j, $j \in \mathbb{N}_m^\infty$, be as in Theorem 4.1. Let $\eta_+(t)$, $0 \le t < T_+$, and $\eta_-(t)$, $0 \le t < T_-$, be two chordal Loewner curves in $\mathbb{H}$ with $\eta_\pm(0) = w_\pm$, which respectively generate chordal Loewner hulls $(K_+(t))$ and $(K_-(t))$. Suppose further that for $\sigma \in \{+,-\}$, η_σ does not visit $\{v_j : j \in \mathbb{N}_m^\infty\} \setminus \{w_+^-, w_-^+\}$, $w_{-\sigma} \notin \overline{K_\sigma(t)}$ for $0 \le t < T_\sigma$, and that the Lebesgue measure of $\eta_\sigma \cap \mathbb{R}$ is zero. We remark here that these properties are almost surely satisfied if η_σ follows the law $\mathbb{P}_\sigma$. Let $\widehat{w}_+$ and $\widehat{w}_-$ be their driving functions. Then $\widehat{w}_\pm(0) = w_\pm$, and we have chordal Loewner equations:

$$\partial_t g_{K_\pm(t)}(z) = \frac{2}{g_{K_\pm(t)}(z) - \widehat{w}_\pm(t)}, \quad g_{K_\pm(0)}(z) = z. \tag{4.1}$$

Let

$$\mathcal{D} = \{(t_+, t_-) \in [0, T_+) \times [0, T_-) : \overline{K_+(t_+)} \cap \overline{K_-(t_-)} = \emptyset\}. \tag{4.2}$$

For $\sigma \in \{+,-\}$, let $T^{\mathcal{D}}_\sigma : [0, T_{-\sigma}) \to (0, T_\sigma]$ be such that $T^{\mathcal{D}}_\sigma(t_{-\sigma})$ is the supremum of t_σ such that $(t_+, t_-) \in \mathcal{D}$. For a function X defined on $\mathcal{D}$ and $s \in [0, T_+)$ (resp. $[0, T_-)$), we use $X|^+_s$ (resp. $X|^-_s$) to denote the function obtained from X by restricting the first (resp. second) variable to be s. For example, $X|^-_s$ is the function $t \mapsto X(t, s)$ with definition domain $[0, T^{\mathcal{D}}_+(s))$. We also view $X|^\pm_0$ as functions defined on $\mathcal{D}$. For example, $X|^+_0(t_+, t_-) = X(0, t_-)$.

For each $(t_+, t_-) \in \mathcal{D}$, we define $K(t_+, t_-) = K_+(t_+) \cup K_-(t_-)$, which is an $\mathbb{H}$-hull, and

$$K_{\sigma, t_{-\sigma}}(t_\sigma) = K(t_+, t_-)/K_{-\sigma}(t_{-\sigma}) = g_{K_{-\sigma}(t_{-\sigma})}(K_\sigma(t_\sigma)), \quad \sigma \in \{+,-\}.$$

Then we have

$$g_{K_{+,t_-}(t_+)} \circ g_{K_-(t_-)} = g_{K(t_+,t_-)} = g_{K_{-,t_+}(t_-)} \circ g_{K_+(t_+)}. \tag{4.3}$$

Let $\eta_{\sigma, t_{-\sigma}}(t_\sigma) = g_{K_{-\sigma}(t_{-\sigma})}(\eta_\sigma(t_\sigma))$. Then $(K_{\sigma, t_{-\sigma}}(t_\sigma))$ are the chordal Loewner hulls generated by $\eta_{\sigma, t_{-\sigma}}$, $\sigma \in \{+,-\}$.

Fix $\sigma \ne \nu \in \{+,-\}$ and $t_\nu \in [0,T_\nu)$. Let $\mathrm{m}_{\sigma,t_\nu}(t_\sigma)=\mathrm{hcap}_2(K_{\sigma,t_\nu}(t_\sigma))$. Since $g_{K_\nu(t_\nu)}$ maps $\mathbb{H}\setminus K_\nu(t_\nu)$ conformally onto $\mathbb{H}$, by Proposition 2.8, $\eta_{\sigma,t_\nu}(t_\sigma)$, $0\le t_\sigma < T^{\mathcal{D}}_\sigma(t_\nu)$, is a chordal Loewner curve with speed $d\,\mathrm{m}_{\sigma,t_\nu}$. For $t'_\sigma > t_\sigma \in [0,T^{\mathcal{D}}_\sigma(t_\nu))$, by (4.3) we have

$$K_{\sigma,t_\nu}(t'_\sigma)/K_{\sigma,t_\nu}(t_\sigma) = g_{K_{\sigma,t_\nu}(t_\sigma)}(K_{\sigma,t_\nu}(t'_\sigma)\setminus K_{\sigma,t_\nu}(t_\sigma)) = g_{K_{\sigma,t_\nu}(t_\sigma)}\circ g_{K_\nu(t_\nu)}(K_\sigma(t'_\sigma)\setminus K_\sigma(t_\sigma))$$
$$= g_{K_{\nu,t_\sigma}(t_\nu)}\circ g_{K_\sigma(t_\sigma)}(K_\sigma(t'_\sigma)\setminus K_\sigma(t_\sigma)) = g_{K_{\nu,t_\sigma}(t_\nu)}(K_\sigma(t'_\sigma)/K_\sigma(t_\sigma)).$$

By Proposition 2.8, $\bigcap_{t'_\sigma:t'_\sigma>t_\sigma}\overline{K_\sigma(t'_\sigma)/K_\sigma(t_\sigma)}=\{\widehat{w}_\sigma(t_\sigma)\}$. Thus, the chordal Loewner driving function with speed $d\,\mathrm{m}_{\sigma,t_\nu}$ for η_{σ,t_ν} is

$$W_\sigma(t_+,t_-):=g_{K_{\nu,t_\sigma}}(\widehat{w}_\sigma(t_\sigma)),\quad 0\le t_\sigma<T^{\mathcal{D}}_\sigma(t_\nu). \tag{4.4}$$

Note that $W_\sigma|^\nu_0=\widehat{w}_\sigma$. Since $\mathrm{hcap}_2(K_\sigma(t'_\sigma)/K_\sigma(t_\sigma))=t'_\sigma-t_\sigma$, and $\mathrm{hcap}_2(K_{\sigma,t_\nu}(t'_\sigma)/K_{\sigma,t_\nu}(t_\sigma))=\mathrm{m}_{\sigma,t_\nu}(t'_\sigma)-\mathrm{m}_{\sigma,t_\nu}(t_\sigma)$, by sending $t'_\sigma\to t_\sigma^+$, we use Proposition 2.5 to conclude that $\mathrm{m}_{\sigma,t_\nu}$ has a right-hand derivative at t_σ, which is equal to $g'_{K_{\nu,t_\sigma}}(\widehat{w}_\sigma(t_\sigma))^2$. Since $g_{K_{\nu,t_\sigma}}$ and $\widehat{w}_\sigma$ are continuous in t_σ, the right-hand derivatives are actually two-sided derivatives. We now define

$$A_{\sigma,n}(t_+,t_-)=g^{(n)}_{K_{\nu,t_\sigma}}(\widehat{w}_\sigma(t_\sigma)),\quad n=1,2,3;\quad A_{\sigma,S}=\frac{A_{\sigma,3}}{A_{\sigma,1}}-\frac{3}{2}\Big(\frac{A_{\sigma,2}}{A_{\sigma,1}}\Big)^2, \tag{4.5}$$

where the superscript (n) stands for n-th derivative. So $A_{\sigma,S}(t_+,t_-)$ is the Schwarzian derivative of $g_{K_{\nu,t_\sigma}}$ at $\widehat{w}_\sigma(t_\sigma)$. Then we get

$$\partial_{t_\sigma} g_{K_{\sigma,t_\nu}(t_\sigma)}(z)=\frac{2A_{\sigma,1}(t_+,t_-)^2}{g_{K_{\sigma,t_\nu}(t_\sigma)}(z)-W_\sigma(t_+,t_-)}; \tag{4.6}$$

Let $X_{\sigma,\nu}=W_\sigma-W_\nu$ and $X^{A:}_{\sigma,\nu}=A_{\sigma,1}/X_{\sigma,\nu}$. Setting $z=\widehat{w}_\nu(t_\nu)$ in (4.6), and using (4.4), we get

$$\partial_{t_\sigma} W_\nu = -2A_{\sigma,1}^2/X_{\sigma,\nu}=-2A_{\sigma,1}X^{A:}_{\sigma,\nu}. \tag{4.7}$$

Differentiating (4.6) w.r.t. z, we get

$$\partial_{t_\sigma}\log(g'_{K_{\sigma,t_\nu}(t_\sigma)}(z))=\frac{\partial_{t_\sigma} g'_{K_{\sigma,t_\nu}(t_\sigma)}(z)}{g'_{K_{\sigma,t_\nu}(t_\sigma)}(z)}=-\frac{2A_{\sigma,1}(t_+,t_-)^2}{(g_{K_{\sigma,t_\nu}(t_\sigma)}(z)-W_\sigma(t_+,t_-))^2}. \tag{4.8}$$

Setting $z=\widehat{w}_\nu(t_\nu)$ in (4.8), we get by (4.4,4.5)

$$\partial_{t_\sigma}A_{\nu,1}/A_{\nu,1}=-2(X^{A:}_{\sigma,\nu})^2. \tag{4.9}$$

Differentiating (4.8) further w.r.t. z twice and setting $z=\widehat{w}_\nu(t_\nu)$, we get by (4.4,4.5)

$$\partial_{t_\sigma}A_{\nu,S}=-12(X^{A:}_{\sigma,\nu})^2(X^{A:}_{\nu,\sigma})^2. \tag{4.10}$$

Define I_S on $\mathcal{D}$ by

$$I_S(t_+,t_-)=\exp\Big(-12\int_0^{t_+}\int_0^{t_-} X^{A:}_{\sigma,\nu}(s_+,s_-)^2X^{A:}_{\nu,\sigma}(s_+,s_-)^2ds_-ds_+\Big). \tag{4.11}$$

By (4.10) and that $A_{\sigma,S}|_0^\nu \equiv 1$ we get

$$\partial_{t_\sigma} I_S / I_S = A_{\sigma,S}. \tag{4.12}$$

Differentiating (4.3) w.r.t. t_σ, using (4.1,4.6,4.4,4.5) and setting $\zeta = g_{K_\sigma(t_\sigma)}(z)$, we get

$$\partial_{t_\sigma} g_{K_{\nu,t_\sigma}(t_\nu)}(\zeta) = \frac{2g'_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)}{\zeta - \widehat{w}_\sigma(t_\sigma)} - \frac{2g'_{K_{\nu,t_\sigma}(t_\nu)}(\widehat{w}_\sigma(t_\sigma))^2}{g_{K_{\nu,t_\sigma}(t_\nu)}(\zeta) - g_{K_{\nu,t_\sigma}(t_\nu)}(\widehat{w}_\sigma(t_\sigma))}. \tag{4.13}$$

Differentiating the above formula w.r.t. ζ, we get

$$\partial_{t_\sigma} g'_{K_{\nu,t_\sigma}(t_\nu)}(\zeta) = \frac{2g''_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)}{\zeta - \widehat{w}_\sigma(t_\sigma)} - \frac{2g'_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)}{(\zeta - \widehat{w}_\sigma(t_\sigma))^2} + \frac{2g'_{K_{\nu,t_\sigma}(t_\nu)}(\widehat{w}_\sigma(t_\sigma))^2 g'_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)}{(g_{K_{\nu,t_\sigma}(t_\nu)}(\zeta) - g_{K_{\nu,t_\sigma}(t_\nu)}(\widehat{w}_\sigma(t_\sigma)))^2}. \tag{4.14}$$

Sending $\zeta \to \widehat{w}_\sigma(t_\sigma)$ in (4.13) and (4.14) respectively, we get

$$\partial_{t_\sigma} g_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)|_{\zeta=\widehat{w}_\sigma(t_\sigma)} = -3A_{\sigma,2}(t_+,t_-); \tag{4.15}$$

$$\frac{\partial_{t_\sigma} g'_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)}{g'_{K_{\nu,t_\sigma}(t_\nu)}(\zeta)}\Big|_{\zeta=\widehat{w}_\sigma(t_\sigma)} = \frac{1}{2}\Big(\frac{A_{\sigma,2}(t_+,t_-)}{A_{\sigma,1}(t_+,t_-)}\Big)^2 - \frac{4}{3}\frac{A_{\sigma,3}(t_+,t_-)}{A_{\sigma,1}(t_+,t_-)}. \tag{4.16}$$

Recall the g_K^w in Definition 2.6. For $j \in \mathbb{N}_m^\infty$ and $\sigma \in \{+,-\}$, we call $\widehat{v}_j^\sigma(t_\sigma) := g^{w_\sigma}_{K_\sigma(t_\sigma)}(v_j)$, $0 \le t_\sigma < T_\sigma$, the force point process started from v_j driven by η_σ. We are going to define the force point process started from v_j jointly driven by η_+ and η_-, which is a function V_j defined on $\mathcal{D}$. We need the following proposition.

Proposition 4.2. *For any $(t_+,t_-) \in \mathcal{D}$ and $v \in (w_-,w_+) \cup \{w_-^+, w_+^-\}$,*

$$g^{\eta_{+,t_-}(0)}_{K_{+,t_-}(t_+)} \circ g^{\eta_-(0)}_{K_-(t_-)}(v) = g^{\eta_{-,t_+}(0)}_{K_{-,t_+}(t_-)} \circ g^{\eta_+(0)}_{K_+(t_+)}(v). \tag{4.17}$$

Proof. Suppose $v \in (w_-,w_+) \cup \{w_-^+, w_+^-\}$. Since $\overline{K_+(t_+)} \cap \overline{K_-(t_-)} = \emptyset$, there are three cases. Case 1. $v \notin \overline{K_+(t_+)} \cup \overline{K_-(t_-)}$. In this case, $g^{\eta_-(0)}_{K_-(t_-)}(v) = g_{K_-(t_-)}(v)$, which is not contained in the closure of $g_{K_-(t_-)}(K_+(t_+)) = K_{+,t_-}(t_+)$, and so $g^{\eta_{+,t_-}(0)}_{K_{+,t_-}(t_+)} \circ g^{\eta_-(0)}_{K_-(t_-)}(v) = g_{K_{+,t_-}(t_+)} \circ g_{K_-(t_-)}(v)$. Symmetrically, the RHS equals $g_{K_{-,t_+}(t_-)} \circ g_{K_+(t_+)}(v)$. So we get (4.17) by (4.3). Case 2. $v \in \overline{K_+(t_+)}$. Then $g^{\eta_+(0)}_{K_+(t_+)}(v) = c_{K_+(t_+)}$. Since $\overline{K_+(t_+)} \cap \overline{K_-(t_-)} = \emptyset$, $[c_{K_+(t_+)}, d_{K_+(t_+)}]$ is disjoint from the closure of $g_{K_+(t_+)}(K_-(t_-)) = K_{-,t_+}(t_-)$. Thus,

$$g^{\eta_{-,t_+}(0)}_{K_{-,t_+}(t_-)} \circ g^{\eta_+(0)}_{K_+(t_+)}(v) = g_{K_{-,t_+}(t_-)}(c_{K_+(t_+)}) = \lim_{x \uparrow a_{K_+(t_+)}} g_{K_{-,t_+}(t_-)} \circ g_{K_+(t_+)}(x).$$

On the other hand, since $v \notin \overline{K_-(t_-)}$, $g^{\eta_-(0)}_{K_-(t_-)}(v) = g_{K_-(t_-)}(v)$, which is contained in the closure of $g_{K_-(t_-)}(K_+(t_+)) = K_{+,t_-}(t_+)$, and is less than $g_{K_-(t_-)}(w_+) = \eta_{+,t_-}(0)$. Thus,

$$g^{\eta_{+,t_-}(0)}_{K_{+,t_-}(t_+)} \circ g^{\eta_-(0)}_{K_-(t_-)}(v) = c_{K_{+,t_-}(t_+)} = \lim_{y \uparrow a_{K_{+,t_-}(t_+)}} g_{K_{+,t_-}(t_+)}(y)$$

$$= \lim_{x \uparrow a_{K_+(t_+)}} g_{K_{+,t_-}(t_+)} \circ g_{K_-(t_-)}(x),$$

where we used $g_{K_-(t_-)}(a_{K_+(t_+)}) = a_{K_{+,t_-}(t_+)}$. Combining the above two displayed formulas with (4.3), we get (4.17) in Case 2. The last case, i.e., $v \in \overline{K_2(t_2)}$, is symmetric. □

Because of the proposition, we define $g^{(w_+,w_-)}_{K(t_+,t_-)}$ on $(w_-, w_+) \cup \{w_-^+, w_+^-\}$ by

$$g^{(w_+,w_-)}_{K(t_+,t_-)} = g^{\eta_{+,t_-}(0)}_{K_{+,t_-}(t_+)} \circ g^{\eta_-(0)}_{K_-(t_-)} = g^{\eta_{-,t_+}(0)}_{K_{-,t_+}(t_-)} \circ g^{\eta_+(0)}_{K_+(t_+)}, \quad (t_+, t_-) \in \mathcal{D}. \tag{4.18}$$

By Remark (2.7), $g^{(w_+,w_-)}_{K(t_+,t_-)}$ is nondecreasing and a contraction for any $(t_+, t_-) \in \mathcal{D}$. We now define V_j on $\mathcal{D}$, $j \in \mathbb{N}_m^\infty$, by $V_j(t_+, t_-) = g^{(w_+,w_-)}_{K(t_+,t_-)}(v_j)$. Then $W_- \le V_\infty \le V_m \le \cdots \le V_1 \le W_+$. Let $\sigma \ne \nu \in \{+, -\}$ be fixed as before. Then

$$\widehat{v}_j^\nu = V_j|_0^\sigma, \quad V_j(t_+, t_-) = g_{K_{\sigma,t_\nu}(t_\sigma)}(\widehat{v}_j^\nu(t_\nu)) \tag{4.19}$$

Let $X_{\sigma,j} = W_\sigma - V_j$ and $X_{j,k} = V_j - V_k$. Let $X^{A:}_{\sigma,j} = \mathbf{1}_{\{X_{\sigma,j} \ne 0\}} A_{\sigma,1}/X_{\sigma,j}$. Since the Lebesuge measure of $\eta_\sigma \cap \mathbb{R}$ is zero, the Lebesgue measure of $\eta_{\sigma,\tau_\nu} \cap \mathbb{R}$ is also zero. By Proposition 2.10 and (4.18,4.19), for any $t_\nu \in [0, T_\nu)$, $V_j|^\nu_{t_\nu}$ is absolutely continuous with

$$\partial_{t_\sigma} V_j = -2A_{\sigma,1}^2/X_{\sigma,j} = -2A_{\sigma,1} X^{A:}_{\sigma,j} \quad a.e. \tag{4.20}$$

Combining (4.7) and (4.20), we get

$$\partial_{t_\sigma} X_{\nu,j} = -2X_{\nu,j} X^{A:}_{\sigma,\nu} X^{A:}_{\sigma,j}, \quad \partial_{t_\sigma} X_{j,k} = -2X_{j,k} X^{A:}_{\sigma,j} X^{A:}_{\sigma,k}, \quad a.e. \tag{4.21}$$

Define $Y_{\nu,j}$ on $\mathcal{D}$ by

$$Y_{\nu,j}(t_+, t_-) = \begin{cases} X_{\nu,j}(t_+, t_-)/X_{\nu,j}|_0^\sigma(t_\nu), & \text{if } \widehat{w}_\nu(t_\nu) \ne \widehat{v}_j^\nu(t_\nu); \\ g'_{K_{\sigma,t_\nu}(t_\sigma)}(\widehat{w}_\nu(t_\nu)) = A_{\nu,1}(t_+, t_-), & \text{if } \widehat{w}_\nu(t_\nu) = \widehat{v}_j^\nu(t_\nu). \end{cases} \tag{4.22}$$

Recall that $\widehat{w}_\nu(t_\nu) = W_\nu|_0^\sigma(t_\nu) \notin \overline{K_{\sigma,t_\nu}(t_\sigma)}$. By (4.4,4.19), $Y_{\nu,j}$ is well defined, continuous, and positive on $\mathcal{D}$. By (4.9,4.21),

$$\partial_{t_\sigma} Y_{\nu,j}/Y_{\nu,j} = -2X^{A:}_{\sigma,\nu} X^{A:}_{\sigma,j} \quad a.e. \tag{4.23}$$

We then define $E_{\nu,j}$ on $\mathcal{D}$ by $E_{\nu,j} = \frac{Y_{\nu,j}}{Y_{\nu,j}|_0^\nu}$. Let $X^{A:0}_{\sigma,\nu} = X^{A:}_{\sigma,\nu}|_0^\nu$ and $X^{A:0}_{\sigma,j} = X^{A:}_{\sigma,j}|_0^\nu$. By (4.23),

$$\partial_{t_\sigma} E_{\nu,j}/E_{\nu,j} = -2X^{A:}_{\sigma,\nu} X^{A:}_{\sigma,j} + 2X^{A:0}_{\sigma,\nu} X^{A:0}_{\sigma,j}, \quad a.e. \tag{4.24}$$

If $\{v_j, v_k\} \not\subset \overline{K_\nu(t_\nu)}$, then $\{\widehat{v}_j^\sigma(t_\sigma), \widehat{v}_k^\sigma(t_\sigma)\} \not\subset \overline{K_{\nu,t_\sigma}(t_\nu)}$, and we define $Y^\sigma_{j,k}$ on $\mathcal{D}$ by

$$Y^\sigma_{j,k}(t_+, t_-) = \begin{cases} X_{j,k}(t_+, t_-)/X_{j,k}|_0^\nu(t_\sigma), & \text{if } \widehat{v}_j^\sigma(t_\sigma) \ne \widehat{v}_k^\sigma(t_\sigma); \\ g'_{K_{\nu,t_\sigma}(t_\nu)}(\widehat{v}_j^\sigma(t_\sigma)), & \text{if } \widehat{v}_j^\sigma(t_\sigma) = \widehat{v}_k^\sigma(t_\sigma). \end{cases} \tag{4.25}$$

By (4.19), $Y^\sigma_{j,k}$ is well defined, continuous, and positive on the set of $(t_+,t_-)\in\mathcal{D}$ such that $\{v_j,v_k\}\not\subset\overline{K_\nu(t_\nu)}$. By (4.21)

$$\partial_{t_\sigma} Y^\nu_{j,k}/Y^\nu_{j,k}=-2X^{A:}_{\sigma,j}X^{A:}_{\sigma,k}\quad a.e.;\quad \partial_{t_\sigma} Y^\sigma_{j,k}/Y^\sigma_{j,k}=-2X^{A:}_{\sigma,j}X^{A:}_{\sigma,k}+2X^{A:0}_{\sigma,j}X^{A:0}_{\sigma,k}\quad a.e. \tag{4.26}$$

We then define $E_{j,k}$ on $\mathcal{D}$ by

$$E_{j,k}(t_+,t_-)=\begin{cases} Y^+_{j,k}(t_+,t_-)/Y^+_{j,k}(0,t_-), & \text{if } \{v_j,v_k\}\not\subset\overline{K_-(t_-)};\\ Y^-_{j,k}(t_+,t_-)/Y^-_{j,k}(t_+,0), & \text{if } \{v_j,v_k\}\not\subset\overline{K_+(t_+)}.\end{cases} \tag{4.27}$$

The $E_{j,k}$ is well defined because if $\{v_j,v_k\}\not\subset\overline{K_+(t_+)}$ and $\{v_j,v_k\}\not\subset\overline{K_-(t_-)}$ both hold, then if $v_j\ne v_k$, both lines of the RHS of (4.27) equal $\frac{X_{j,k}(t_+,t_-)X_{j,k}(0,0)}{X_{j,k}(t_+,0)X_{j,k}(0,t_-)}$, and if $v_j=v_k$, both lines equal $\frac{g'_{K(t_+,t_-)}(v_j)}{g'_{K_+(t_+)}(v_j)g'_{K_-(t_-)}(v_j)}$. Moreover, $E_{j,k}$ is positive and continuous on $\mathcal{D}$ because both $Y^+_{j,k}$ and $Y^-_{j,k}$ are positive and continuous on their respective domains. By (4.26), we have

$$\partial_{t_\sigma}E_{j,k}/E_{j,k}=-2X^{A:}_{\sigma,j}X^{A:}_{\sigma,k}+2X^{A:0}_{\sigma,j}X^{A:0}_{\sigma,k}\quad a.e. \tag{4.28}$$

Proposition 4.3. *Let $(\xi_+,\xi_-)\in\Xi$. There is $C\in(1,\infty)$ depending only on ξ_+,ξ_- such that the restrictions of $A_{\sigma,1}$, $X_{+,-}$, I_S, $E_{\sigma,j}$, and $E_{j,k}$, $\sigma\in\{+,-\}$, $j,k\in\mathbb{N}^\infty_m$, to $[0,\tau^+_{\xi_+})\times[0,\tau^-_{\xi_-})$, are all bounded from above by C and from below by $1/C$.*

Proof. Throughout the proof, a constant is a number depending only on ξ_+,ξ_-. By symmetry, assume that $\sigma=+$. Fix $(t_+,t_-)\in[0,\tau^+_{\xi_+})\times[0,\tau^-_{\xi_-})$. Let $x_\pm$ be the endpoint of $\xi_\pm$ that lies on (w_-,w_+), and $x_0=(x_++x_-)/2$. Since $K(t_+,t_-)\subset \mathrm{Hull}(\xi_+\cup\xi_-)$, we have $1\ge g'_{K(t_+,t_-)}(x_0)\ge g'_{\mathrm{Hull}(\xi_+\cup\xi_-)}(x_0)>0$, which implies that $|\log g'_{K(t_+,t_-)}(x_0)|$ is bounded by a constant. By (4.3) and that $K(t_+,0)=K_+(t_+)$, $|\log g'_{K_{-,t_+}(t_-)}(g_{K_+(t_+)}(x_0))|$ is bounded by a constant. Since $g'_{K_{-,t_+}(t_-)}\in(0,1]$, and is increasing on $[b_{K_{-,t_+}(t_-)},\infty)$, we see that $|\log g'_{K_{-,t_+}(t_-)}|$ is bounded by a constant on $[g_{K_+(t_+)}(x_0),\infty)=:I_+$. By Proposition 2.9, $\widehat{w}_+(t_+)\in I_+$. Since $A_{+,1}(t_+,t_-)=g'_{K_{-,t_+}(t_-)}(\widehat{w}_+(t_+))$, we see that $|\log A_{+,1}(t_+,t_-)|$ is bounded by a constant.

The quantity $X_{+,-}(t_+,t_-)=W_+(t_+,t_-)-W_-(t_+,t_-)$ is bounded from above by $d_{K(t_+,t_-)}-c_{K(t_+,t_-)}$, which is further bounded by the constant $d_{\mathrm{Hull}(\eta_+\cup\eta_-)}-c_{\mathrm{Hull}(\eta_+\cup\eta_-)}$ by Proposition 2.2. For the lower bound, pick any $x_1<x_2\in(x_-,x_+)$. Then $X_{+,-}(t_+,t_-)\ge g_{K(t_+,t_-)}(x_2)-g_{K(t_+,t_-)}(x_1)$, which is further bounded from below by the positive constant $g_{\mathrm{Hull}(\eta_+\cup\eta_-)}(x_2)-g_{\mathrm{Hull}(\eta_+\cup\eta_-)}(x_1)$ due to the fact that $g'_{\mathrm{Hull}(\eta_+\cup\eta_-)/K(t_+,t_-)}|_{[x_1,x_2]}\in(0,1]$.

From what we have proved, $|X^{A:}_{+,-}|=|A_{+,1}/X_{+,-}|$ and $|X^{A:}_{-,+}|=|A_{-,1}/X_{-,+}|$ are uniformly bounded by a constant on $[0,t_+]\times[0,t_-]$. We also know that $t_\pm$ is bounded by the constant $\mathrm{hcap}_2(\mathrm{Hull}(\xi_\pm))$. By (4.11) we see that $|\log I_S(t_+,t_-)|$ is bounded by a constant.

For $E_{+,j}$, consider two cases. Case 1. $v_j\ge x_0$. We have seen that $|\log g'_{K_{-,t_+}(t_-)}|$ is bounded by a constant on $I_+=[g_{K_+(t_+)}(x_0),\infty)$, and $\widehat{w}_+(t_+)\in I_+$. Since $\widehat{v}^+_j(t_+)=g^{w_+}_{K_+(t_+)}(v_j)\in I_+$, by (4.22), $|\log Y_{+,j}(t_+,t_-)|$ is bounded by a constant.

Case 2. $v_j \le x_0$. Let $y_+ = (x_0 + x_+)/2$. Then

$$\widehat{v}_j^+(t_+) = g_{K_+(t_+)}(v_j) \le g_{K_+(t_+)}(x_0) < g_{K_+(t_+)}(y_+) \le c_{K_+(t_+)} \le \widehat{w}_+(t_+),$$

which implies by (4.3,4.4,4.19) that $V_j(t_+,t_-) \le g_{K(t_+,t_-)}(x_0) < g_{K(t_+,t_-)}(y_+) \le W_+(t_+,t_-)$. Since $g'_{K(t_+,t_-)} \ge g'_{\mathrm{Hull}(\xi_+\cup\xi_-)} > 0$, $g_{K(t_+,t_-)}(y_+) - g_{K(t_+,t_-)}(x_0)$ is bounded from below by the positive constant $g_{\mathrm{Hull}(\xi_+\cup\xi_-)}(y_+) - g_{\mathrm{Hull}(\xi_+\cup\xi_-)}(x_0)$. So $X_{+,j}(t_+,t_-) = W_+(t_+,t_-) - V_j(t_+,t_-)$ is bounded from below by a positive constant. On the other hand, by Proposition 2.2, $X_{+,j}(t_+,t_-)$ is bounded from above by the constant $d_{\mathrm{Hull}(\xi_+\cup\xi_-)} - c_{\mathrm{Hull}(\xi_+\cup\xi_-)}$. These properties are also satisfied by $X_{+,j}(t_+,0)$. By (4.22), $|\log Y_{+,j}(t_+,t_-)|$ is bounded by a constant.

Thus, in both cases, $|\log Y_{+,j}(t_+,t_-)|$ is bounded by a constant. This property is also satisfied by $|\log Y_{+,j}(0,t_-)|$. Since $E_{+,j}(t_+,t_-) = \frac{Y_{+,j}(t_+,t_-)}{Y_{+,j}(0,t_-)}$, $|\log E_{+,j}(t_+,t_-)|$ is also bounded by a constant.

For $E_{j,k}$, by symmetry and relabeling, we may assume that $v_j \ge v_k \vee x_0$. Let $y_- = (x_0 + x_-)/2$. Consider two cases. Case 1. $v_k \ge y_-$. Using the proof of Case 1 for $E_{+,j}$ with y_- in place of x_0, we see that $|\log g'_{K_{-,t_+}(t_-)}|$ is bounded by a constant on $[g_{K_+(t_+)}(y_-),\infty)$, which contains both $\widehat{v}_j^+(t_+)$ and $\widehat{v}_k^+(t_+)$. So by (4.25), $|\log Y^+_{j,k}(t_+,t_-)|$ is bounded by a constant. Case 2. $v_k \le y_-$. Using the proof of Case 2 for $E_{+,j}$ with x_0 and y_- in place of y_+ and x_0, respectively, we see that $g_{\mathrm{Hull}(\xi_+\cup\xi_-)}(x_0) - g_{\mathrm{Hull}(\xi_+\cup\xi_-)}(y_-) \le X_{j,k}(t_+,t_-) \le d_{\mathrm{Hull}(\xi_+\cup\xi_-)} - c_{\mathrm{Hull}(\xi_+\cup\xi_-)}$, which implies that $|\log Y^+_{j,k}(t_+,t_-)|$ is again bounded by a constant. Thus, $|\log E_{j,k}(t_+,t_-)|$ is also bounded by a constant by (4.27). □

For $j \in \mathbb{N}_m$, define R_j on $\mathcal{D}$ by $R_j = \frac{X_{+,j}}{X_{+,\infty}} \cdot \frac{X_{-,\infty}}{X_{-,j}}$ if $X_{+,\infty}X_{-,j} \ne 0$; and $R_j = 1$ if $X_{+,\infty}X_{-,j} = 0$. Let $\underline{R} = (R_1,\dots,R_m)$. It is clear that $0 \le R_1 \le \cdots \le R_m \le 1$. So $\underline{R}$ takes values in $\overline{\Delta_m}$.

Lemma 4.4. *Every R_j is continuous on $\mathcal{D}$, and so is $\underline{R}$.*

Proof. Fix $j \in \mathbb{N}_m$. Let T_∞^+ be the first time that η_+ reaches $(-\infty, v_\infty]$. We understand T_∞^+ as T_+ if such time does not exist; and as 0 if $v_\infty = w_+^-$. Similarly, let T_j^- be the first time that η_- reaches $[v_j,\infty)$. Let $\mathcal{D}_j = \mathcal{D} \cap ([0,T_\infty^+) \times [0,T_j^-))$. Then $X_{+,\infty}X_{-,j} \ne 0$ on $\mathcal{D}_j$, and so R_j is continuous on $\mathcal{D}_j$. If $T_\infty^+ < T_+$, then $\widehat{w}_+(T_\infty^+) = \widehat{v}_\infty^+(T_\infty^+)$, which implies that $\widehat{v}_j^+ \equiv \widehat{v}_\infty^+$ on $[T_\infty^+, T_+)$. By (4.19), we see that, on $\mathcal{D} \cap \{(t_+,t_-) : t_+ \ge T_\infty^+\}$, $V_j \equiv V_\infty$, which implies that $R_j \equiv 1$. Similarly, we have $R_j \equiv 1$ on $\mathcal{D} \cap \{(t_+,t_-) : t_- \ge T_j^-\}$.

If T_∞^+ or T_j^- equals 0, then R_j is constant 1, and its continuity is trivial. Suppose T_∞^+ and T_j^- are both positive. Then $(T_\infty^+, T_\infty^-) \notin \mathcal{D}$ because $\overline{K_+(T_\infty^+)}$ and $\overline{K_-(T_j^-)}$ both contain $[v_\infty, v_j]$. It suffices to show that (i) if $0 < T_\infty^+ < T_+$, then as $t_+ \uparrow T_\infty^+$, $R_j(t_+,t_-) \to 1$ uniformly in $t_- \in [0, T_-^{\mathcal{D}}(T_\infty^+))$; and (ii) if $0 < T_j^- < T_-$, then as $t_- \uparrow T_j^-$, $R_j(t_+,t_-) \to 1$ uniformly in $t_+ \in [0, T_+^{\mathcal{D}}(T_j^-))$. They follow from an extremal distance argument shown below.

(i) Since η_+ does not visit v_∞, $x_0 := \eta_+(T_\infty^+) \in (w_-, v_\infty)$. Let $0 < \delta < |x_0 - v_\infty|$. Suppose $t_+ < T_\infty^+$ is such that $\mathrm{diam}(\eta_+([t_+, T_\infty^+])) < \delta$. Then for $t_- \in [0, T_-^{\mathcal{D}}(T_\infty^+))$, any curve in

$\mathbb{H}\setminus K(t_+,t_-)$ connecting the line segment $[v_\infty, v_j \wedge \min(\eta_+([0,t_+])\cap\mathbb{R})]$, denoted by I, and the union of the right side of $\eta_+([0,t_+])$, $[w_+,\infty]$, $(-\infty,w_-]$, and the left side of $\eta_-([0,t_-])$, denoted by U, must cross the semi-annulus $\{z\in\mathbb{H}:\delta<|z-x_0|<|v_\infty-x_0|\}$. By comparison principle of extremal length, the extremal distance between I and U in $\mathbb{H}\setminus K(t_+,t_-)$ is at least $\log(|v_\infty-x_0|/\delta)$. Since $g_{K(t_+,t_-)}$ maps $\mathbb{H}\setminus K(t_+,t_-)$ conformally onto $\mathbb{H}$, and maps I and U respectively to $[V_\infty(t_+,t_-),V_j(t_+,t_-)]$ and $(-\infty,W_-(t_+,t_-)]\cup[W_+(t_+,t_-),\infty)$, the extremal distance between the latter two sets in $\mathbb{H}$, which can be expressed as a function f of $R_j(t_+,t_-)$, is at least $\log(|v_\infty-x_0|/\delta)$. Since the function f is bounded on $(0,1-\varepsilon]$ for any $\varepsilon>0$, we then finish the proof of (i). The proof of (ii) is similar. $\square$

4.2 Stochastic ensemble

We adopt the assumption and notation in the previous subsection. Let $\kappa>0$ and ρ_j, $j\in\mathbb{N}_m^\infty$, be as in Theorem 4.1. Let $\widetilde{\rho}_j=\rho_j/\kappa$, $j\in\mathbb{N}_m^\infty$.

Suppose $\widehat{w}_+(t_+)$ and $\widehat{w}_-(t_-)$ are independent semimartingales with quadratic variation being $\langle\widehat{w}_\pm\rangle_t=\kappa t$, $0\le t<T_\pm$. Let $\mathcal{F}^\pm$ be the filtration generated by $\widehat{w}_\pm$. Fix $\sigma\ne\nu\in\{+,-\}$ and two $\mathcal{F}^\nu$-stopping times τ_ν and τ_ν' with $\tau_\nu\le\tau_\nu'$ and $\tau_\nu<T_\nu$. Since $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent, $\widehat{w}_\sigma(t_\sigma)$ is also an $(\mathcal{F}^\sigma_{t_\sigma}\times\mathcal{F}^\nu_{\tau_\nu'})_{t_\sigma\ge 0}$-semimartingale. We will repeatedly apply Itô's formula in this subsection, where the underlying filtration is alwarys $(\mathcal{F}^\sigma_{t_\sigma}\times\mathcal{F}^\nu_{\tau_\nu'})_{t_\sigma\ge0}$, the time parameter t_ν is fixed to be τ_ν, and the time parameter t_σ runs from 0 to $T^{\mathcal{D}}_\sigma(\tau_\nu)$. By (4.4) and (4.15), W_σ satisfies the SDE

$$\partial_\sigma W_\sigma = A_{\sigma,1}\partial\widehat{w}_\sigma+\Big(\frac{\kappa}{2}-3\Big)A_{\sigma,2}\partial t_\sigma. \tag{4.29}$$

By (4.5,4.16), we get

$$\frac{\partial_\sigma A_{\sigma,1}}{A_{\sigma,1}}=\frac{A_{\sigma,2}}{A_{\sigma,1}}\partial\widehat{w}_\sigma+\frac12\Big(\frac{A_{\sigma,2}}{A_{\sigma,1}}\Big)^2\partial t_\sigma+\Big(\frac{\kappa}{2}-\frac43\Big)\frac{A_{\sigma,3}}{A_{\sigma,1}}.$$

Let $A_\sigma^{2:1}=A_{\sigma,2}/A_{\sigma,1}$ and

$$\mathrm{b}=\frac{6-\kappa}{2\kappa},\quad \mathrm{c}=\frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

The previous formula implies that

$$\partial_\sigma A_{\sigma,1}^{\mathrm{b}}/A_{\sigma,1}^{\mathrm{b}}=\mathrm{b}A_\sigma^{2:1}\partial\widehat{w}_\sigma+(\mathrm{c}/6)\cdot A_{\sigma,S}\partial t_\sigma. \tag{4.30}$$

From (4.7,4.20,4.29), we get

$$\partial_\sigma X_{\sigma,\nu}/X_{\sigma,\nu}=X^{A:}_{\sigma,\nu}\partial\widehat{w}_\sigma-\kappa\,\mathrm{b}\,X^{A:}_{\sigma,\nu}A_\sigma^{2:1}\partial t_\sigma+2(X^{A:}_{\sigma,\nu})^2\partial t_\sigma; \tag{4.31}$$

$$\partial_\sigma X_{\sigma,j}/X_{\sigma,j}=X^{A:}_{\sigma,j}\partial\widehat{w}_\sigma-\kappa\,\mathrm{b}\,X^{A:}_{\sigma,j}A_\sigma^{2:1}\partial t_\sigma+2(X^{A:}_{\sigma,j})^2\partial t_\sigma. \tag{4.32}$$

Here (4.31) holds throughout, and (4.32) holds up to the time that $X_{\sigma,j}=0$.

Define positive continuous functions $E_+, E_-, E_{+,-}$ on $\mathcal{D}$ by $E_\sigma = \frac{A_{\sigma,1}}{A_{\sigma,1}|_0^\sigma}$, $\sigma \in \{+,-\}$, and $E_{+,-}(t_+,t_-) = \frac{X_{+,-}(t_+,t_-)X_{+,-}(0,0)}{X_{+,-}(t_+,0)X_{+,-}(0,t_-)}$. By (4.9,4.30,4.31),

$$\partial_\sigma E_\nu^{\rm b}/E_\nu^{\rm b} = -2\,{\rm b}((X^{A:}_{\sigma,\nu})^2 - (X^{A:0}_{\sigma,\nu})^2)\partial t_\sigma; \quad \partial_\sigma E_\sigma^{\rm b}/E_\sigma^{\rm b} = {\rm b}A_\sigma^{2:1}\partial\widehat{w}_\sigma + ({\rm c}/6)\cdot A_{\sigma,S}\partial t_\sigma. \quad (4.33)$$

$$\partial_\sigma E_{+,-}^{-2\,{\rm b}}/E_{+,-}^{-2\,{\rm b}} = -2\,{\rm b}(X^{A:}_{\sigma,\nu} - X^{A:0}_{\sigma,\nu}\Big)\partial\widehat{w}_\sigma + 2\kappa\,{\rm b}^2\,X^{A:}_{\sigma,\nu}A_\sigma^{2:1}\partial t_\sigma$$
$$+ 2\,{\rm b}((X^{A:}_{\sigma,\nu})^2 - (X^{A:0}_{\sigma,\nu})^2)\partial t_\sigma - 4\kappa\,{\rm b}^2\,X^{A:0}_{\sigma,\nu}(X^{A:}_{\sigma,\nu} - X^{A:0}_{\sigma,\nu})\partial t_\sigma. \quad (4.34)$$

Let $E_{S,{\rm b}} = I_S^{-\frac{\rm c}{6}} E_+^{\rm b} E_-^{\rm b} E_{+,-}^{-2\,{\rm b}}$. Combining (4.33,4.34) with (4.12), we get

$$\partial_\sigma E_{S,{\rm b}}/E_{S,{\rm b}} = ({\rm b}\,A_\sigma^{2:1} - 2\,{\rm b}(X^{A:}_{\sigma,\nu} - X^{A:0}_{\sigma,\nu}))(\partial\widehat{w}_\sigma + 2\kappa\,{\rm b}\,X^{A:0}_{\sigma,\nu}\partial t_\sigma). \quad (4.35)$$

Recall the R_j, $E_{\pm,j}$, and $E_{j,k}$ defined before. Since $R_j = (\frac{X_{\sigma,j}}{X_{\sigma,\infty}}\cdot\frac{X_{\nu,\infty}}{X_{\nu,j}})^{\sigma\cdot 1}$, by (4.21,4.32), R_j satisfies the following SDE up to the time that it equals 1:

$$\partial_\sigma R_j/R_j = \sigma(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})\partial\widehat{w}_\sigma + \sigma(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})(-\kappa\,{\rm b}\,A_\sigma^{2:1}\partial t_\sigma + 2X^{A:}_{\sigma,\nu}\partial t_\sigma)$$
$$+ \sigma(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})((2-\kappa/2+\sigma\kappa/2)X^{A:}_{\sigma,j} + (2-\kappa/2-\sigma\kappa/2)X^{A:}_{\sigma,\infty})\partial t_\sigma. \quad (4.36)$$

Since $E_{\sigma,j} = \frac{Y_{\sigma,j}}{Y_{\sigma,j}|_0^\sigma}$ and $Y_{\sigma,j} = \frac{X_{\sigma,j}}{X_{\sigma,j}|_0^\nu}$ (the generic case), by (4.24,4.32), for $j \in \mathbb{N}_m^\infty$,

$$\frac{\partial_\sigma(E_{\sigma,j}/E_{\nu,j})}{E_{\sigma,j}/E_{\nu,j}} = (X^{A:}_{\sigma,j} - X^{A:0}_{\sigma,j})\partial\widehat{w}_\sigma - \kappa\,{\rm b}\,X^{A:}_{\sigma,j}A_\sigma^{2:1}\partial t_\sigma + 2(X^{A:}_{\sigma,\nu}X^{A:}_{\sigma,j} - X^{A:0}_{\sigma,\nu}X^{A:0}_{\sigma,j})\partial t_\sigma$$
$$+ (2(X^{A:}_{\sigma,j})^2 - 2(X^{A:0}_{\sigma,j})^2)\partial t_\sigma - \kappa X^{A:0}_{\sigma,j}(X^{A:}_{\sigma,j} - X^{A:0}_{\sigma,j})\partial t_\sigma. \quad (4.37)$$

Recall that $\widetilde{\rho}_j = \rho_j/\kappa$. Define

$$E_{j,\infty,\widetilde{\rho}} = \prod_{s\in\{+,-\}}\Big(\frac{E_{s,j}}{E_{s,\infty}}\Big)^{s\widetilde{\rho}_j} = \Big(\frac{E_{\sigma,j}/E_{\nu,j}}{E_{\sigma,\infty}/E_{\nu,\infty}}\Big)^{\sigma\widetilde{\rho}_j}, \quad 1 \le j \le m.$$

By (4.37),

$$\frac{\partial_\sigma E_{j,\infty,\widetilde{\rho}}}{E_{j,\infty,\widetilde{\rho}}} = \sigma\widetilde{\rho}_j[(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - (X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})]\partial\widehat{w}_\sigma - \sigma\widetilde{\rho}_j\kappa\,{\rm b}\,A_\sigma^{2:1}(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})\partial t_\sigma$$
$$+2\sigma\widetilde{\rho}_j[X^{A:}_{\sigma,\nu}(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - X^{A:0}_{\sigma,\nu}(X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})]\partial t_\sigma$$
$$+2\sigma\widetilde{\rho}_j[((X^{A:}_{\sigma,j})^2 - (X^{A:}_{\sigma,\infty})^2) - ((X^{A:0}_{\sigma,j})^2 - (X^{A:0}_{\sigma,\infty})^2)]\partial t_\sigma$$
$$-\sigma\widetilde{\rho}_j\kappa[X^{A:}_{\sigma,\infty}(X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty}) + X^{A:0}_{\sigma,\infty}(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})]\partial t_\sigma$$
$$-\sigma\widetilde{\rho}_j\kappa[(X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - ((X^{A:0}_{\sigma,j})^2 - (X^{A:0}_{\sigma,\infty})^2)]\partial t_\sigma$$
$$-\kappa\sigma\widetilde{\rho}_j(X^{A:}_{\sigma,\infty} - X^{A:0}_{\sigma,\infty})[(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - (X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})]\partial t_\sigma$$

$$+ (\kappa/2)\sigma\widetilde{\rho}_j(\sigma\widetilde{\rho}_j - 1)[(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - (X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})]^2\partial t_\sigma. \tag{4.38}$$

Let $\alpha, \beta_1, \dots, \beta_m, \gamma$ be defined by (3.8). Then they satisfy the parameter assumption. Let F be the hypergeometric function $F(\alpha, \beta_1, \dots, \beta_m, \gamma; \cdot)$. By Theorem 2.16, it extends to a positive continuous function on $\overline{\Delta_m}$. So $F(\underline{R})$ is a positive continuous function defined on $\mathcal{D}$. Since F is smooth in $(-1,1)^m$, $F(\underline{R})$ is a local martingale up to the first time that $\underline{R}$ exits $(-1,1)^m$. Recall the G_j defined by (3.20). Combining (2.18,2.19,2.20) with (4.36), we see that $F(\underline{R})$ satisfies the following SDE up to the first time that $\underline{R}$ exits $(-1,1)^m$:

$$\frac{\partial_\sigma F(\underline{R})}{F(\underline{R})} = \sigma\sum_j (X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})G_j(\underline{R})\partial\widehat{w}_\sigma - \sigma\sum_j \kappa\,\mathrm{b}\,A^{2:1}_\sigma(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})G_j(\underline{R})\partial t_\sigma$$

$$-\sigma\sum_j (X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})\Big(-2\kappa\,\mathrm{b}\,X^{A:}_{\sigma,\nu} + \sigma\sum_k \rho_k(X^{A:}_{\sigma,k} - X^{A:}_{\sigma,\infty})\Big)G_j(\underline{R})\partial t_\sigma$$

$$-\sigma\sum_j \frac{\rho_j(\kappa-4)}{2\kappa}(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})[2X^{A:}_{\sigma,\nu} + \sigma(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - (X^{A:}_{\sigma,j} + X^{A:}_{\sigma,\infty})]\partial t_\sigma. \tag{4.39}$$

We now argue that $F(\underline{R})$ satisfies (4.39) throughout $[0, T^{\mathcal{D}}_\sigma(\tau_\nu))$. We need to deal with the case that some R_j equals 1. In fact, if on some interval there is $m' < m$ such that $R_j|^{\nu}_{\tau_\nu} = 1$ for $m'+1 \le j \le m$, and $R_j|^{\nu}_{\tau_\nu} < 1$ for $1 \le j \le m'$, then by (2.15), $F(\underline{R})$ equals some constant times $\widetilde{F}(\underline{\widetilde{R}})$, where $\underline{\widetilde{R}} = (R_1, \dots, R_{m'})$ and $\widetilde{F}$ is the hypergeometric function $F(\alpha, \beta_1, \dots, \beta_{m'}, \gamma - \sum_{j=m'+1}^m \beta_m; \cdot)$. So on that interval $\frac{\partial_\sigma F(\underline{R})}{F(\underline{R})} = \frac{\partial_\sigma \widetilde{F}(\underline{\widetilde{R}})}{\widetilde{F}(\underline{\widetilde{R}})}$, and we may get (4.39) by applying (2.18,2.19,2.20) to $\widetilde{F}$ and using the facts that for $j \le m'$, R_j satisfies (4.36), and the terms on the RHS of (4.39) for $j > m'$ vanish because $G_j(\underline{R})$ and $X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}$ both vanish.

Define another positive continuous function F_R on $\mathcal{D}$ by $F_R(t_+, t_-) = \frac{F(\underline{R}(t_+,t_-))F(\underline{R}(0,0))}{F(\underline{R}(t_+,0))F(\underline{R}(0,t_-))}$. By (4.39), F_R satisfies the following SDEs:

$$\frac{\partial_\sigma F_R}{F_R} = \sigma\sum_j \Big[(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})G_j(\underline{R}) - (X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})G_j(\underline{R}|^{\nu}_0)\Big]\partial\widehat{w}_\sigma$$

$$-\sigma\sum_j \kappa\,\mathrm{b}(A^{2:1}_\sigma - 2(X^{A:}_{\sigma,\nu} - X^{A:0}_{\sigma,\nu}))(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})G_j(\underline{R})\partial t_\sigma$$

$$-\sigma\sum_j (X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})\Big(\sum_k \rho_k(X^{A:}_{\sigma,k} - X^{A:}_{\sigma,\infty})\Big)G_j(\underline{R})\partial t_\sigma$$

$$+\sigma\sum_j (X^{A:0}_{\sigma,j} - X^{A:0}_{\sigma,\infty})\Big(\sum_k \rho_k(X^{A:0}_{\sigma,k} - X^{A:0}_{\sigma,\infty})\Big)G_j(\underline{R}|^{\nu}_0)\partial t_\sigma$$

$$-\sigma\sum_j \frac{\rho_j(\kappa-4)}{2\kappa}(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty})[2X^{A:}_{\sigma,\nu} + \sigma(X^{A:}_{\sigma,j} - X^{A:}_{\sigma,\infty}) - (X^{A:}_{\sigma,j} + X^{A:}_{\sigma,\infty})]\partial t_\sigma$$

$$+\sigma\sum_j \frac{\rho_j(\kappa-4)}{2\kappa}(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})[2X^{A:0}_{\sigma,\nu}+\sigma(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})-(X^{A:0}_{\sigma,j}+X^{A:0}_{\sigma,\infty})]\partial t_\sigma$$

$$-\kappa\sum_{j,k}(X^{A:0}_{\sigma,k}-X^{A:0}_{\sigma,\infty})G_j(\underline{R}|^\nu_0)\Big[(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})G_j(\underline{R})-(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})G_j(\underline{R}|^\nu_0)\Big]\partial t_\sigma. \quad (4.40)$$

Here the terms in the last line of (4.40) come from the quadratic covariation, and other terms on the RHS of (4.40) come from the difference between the RHS of (4.39) and the function obtained by replacing the τ_ν by 0 in the RHS of (4.39). Note that $A^{2:1}_\sigma|^\nu_0\equiv 0$.

Define M on $\mathcal{D}$ by

$$M=F_R E_{S,\mathrm{b}}\prod_{j\in\mathbb{N}_m}[E_{j,\infty,\widetilde{\rho}}\cdot(E_{j,j}E_{\infty,\infty}/E^2_{j,\infty})^{\frac{\rho_j(4-\kappa)}{4\kappa}}]\cdot\prod_{j,k\in\mathbb{N}_m^\infty}E_{j,k}^{\frac{\rho_j\rho_k}{4\kappa}}. \quad (4.41)$$

Lemma 4.5. *(i) The function M is positive and continuous on $\mathcal{D}$, and takes value 1 on $[0,T_+)\times\{0\}$ and $\{0\}\times[0,T_-)$. (ii) For any $(\xi_+,\xi_-)\in\Xi$, $|\log M|$ on $[0,\tau^+_{\xi_+})\times[0,\tau^-_{\xi_-})$ is uniformly bounded by a constant depending only on ξ_+,ξ_-.*

Proof. (i) This holds because every factor on the RHS of (4.41) is positive and continuous on $\mathcal{D}$. (ii) This follows from Proposition 4.3 and the fact that F is continuous and positive on the compact set $\overline{\Delta_m}$. □

We may now calculate that M satisfies the following SDE:

$$\frac{\partial_\sigma M}{M}=\Big(\sigma\sum_j(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})G_j(\underline{R})-\sigma\sum_j(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})G_j(\underline{R}|^\nu_0)$$

$$+\mathrm{b}\,A^{2:1}_\sigma-2\,\mathrm{b}(X^{A:}_{\sigma,\nu}-X^{A:0}_{\sigma,\nu})+\sigma\sum_j\widetilde{\rho}_j[(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})-(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})]\Big)\times$$

$$\times\Big(\partial\widehat{w}_\sigma+2\kappa\,\mathrm{b}\,X^{A:0}_{\sigma,\nu}\partial t_\sigma-\sigma\sum_k(X^{A:0}_{\sigma,k}-X^{A:0}_{\sigma,\infty})[\rho_k+\kappa G_j(\underline{R}|^\nu_0)]\partial t_\sigma\Big). \quad (4.42)$$

The computation is tedious but straightforward. First, we note that the coefficients of $\partial\widehat{w}_\sigma$ in the SDEs (4.35,4.38,4.40) sum up to the coefficients of $\partial\widehat{w}_\sigma$ in (4.42). Since $\partial\langle\widehat{w}_\sigma\rangle=\kappa\partial t_\sigma$, the SDEs contribute the following covariation terms:

$$\Big(\sigma\kappa\,\mathrm{b}(A^{2:1}_\sigma-2(X^{A:}_{\sigma,\nu}-X^{A:0}_{\sigma,\nu}))+\sum_k\rho_k[(X^{A:}_{\sigma,k}-X^{A:0}_{\sigma,k})-(X^{A:0}_{\sigma,\infty}-X^{A:0}_{\sigma,\infty})]\Big)\times$$

$$\times\sum_j\Big[(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})G_j(\underline{R})-(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})G_j(\underline{R}|^\nu_0)\Big]\partial t_\sigma$$

$$+\sigma\,\mathrm{b}(A^{2:1}_\sigma-2(X^{A:}_{\sigma,\nu}-X^{A:0}_{\sigma,\nu}))\sum_j\rho_j[(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})-(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})]\partial t_\sigma. \quad (4.43)$$

By (4.28) and the fact that $\sum_{j\in\mathbb{N}_m^\infty}\rho_j=0$, we have

$$\frac{\partial_\sigma\prod_j(E_{j,j}E_{\infty,\infty}/E_{j,\infty}^2)^{\frac{\rho_j(4-\kappa)}{4\kappa}}}{(\prod_j E_{j,j}E_{\infty,\infty}/E_{j,\infty}^2)^{\frac{\rho_j(4-\kappa)}{4\kappa}}}=-\sum_j\frac{\rho_j(4-\kappa)}{2\kappa}\Big[(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})^2-(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})^2\Big]\partial t_\sigma; \quad (4.44)$$

$$\frac{\partial_\sigma\prod_{j,k\in\mathbb{N}_m^\infty}E_{j,k}^{\frac{\rho_j\rho_k}{4\kappa}}}{\prod_{j,k\in\mathbb{N}_m^\infty}E_{j,k}^{\frac{\rho_j\rho_k}{4\kappa}}}=-\frac{1}{2\kappa}\Big(\sum_j\rho_j(X^{A:}_{\sigma,j}-X^{A:}_{\sigma,\infty})\Big)^2\partial t_\sigma+\frac{1}{2\kappa}\Big(\sum_j\rho_j(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})\Big)^2\partial t_\sigma. \quad (4.45)$$

It remains to show that the sum of the coefficients of ∂t_σ in (4.35,4.38,4.40,4.43,4.44,4.45) is equal to the sum of the coefficients of ∂t_σ in (4.42). For that purpose, the interested reader may first compare all terms containing the factor $G_j(\underline{R})$ or $G_j(\underline{R}|_0^\nu)$, and then all remaining terms containing the factor $A^{2:1}_\sigma$, $X^{A:}_{\sigma,\nu}$, or $X^{A:0}_{\sigma,\nu}$, and finally all other terms.

4.3 Construction of the couplings

Suppose η_+ follows the law $\mathbb{P}_+$, η_- follows the law $\mathbb{P}_-$, and η_+ and η_- are independent. Then they almost surely satisfy the assumptions in the previous subsections, and we then adopt the notation there.

By Proposition 3.8, $\widehat{w}_+$ and $\widehat{w}_-$ satisfy (3.24) and (3.25) for a pair of independent Brownian motions B_+ and B_-. Let $\sigma\ne\nu\in\{+,-\}$. We may rewrite the SDEs as:

$$d\widehat{w}_\sigma=\sqrt{\kappa}dB_\sigma-2\kappa\,\mathrm{b}\,X^{A:0}_{\sigma,\nu}dt_\sigma+\sigma\sum_j(X^{A:0}_{\sigma,j}-X^{A:0}_{\sigma,\infty})[\rho_j+\kappa G_j(\underline{R}|_0^\nu)]dt_\sigma. \quad (4.46)$$

Combining (4.42) and (4.46), we obtain the following lemma.

Lemma 4.6. *Let $\sigma\ne\nu\in\{+,-\}$. Then for any $\mathcal{F}^\nu$-stopping times τ_ν and τ'_ν with $\tau_\nu\le\tau'_\nu$ and $\tau_\nu<T_\nu$, $M|^\nu_{\tau_\nu}$ is an $(\mathcal{F}^\sigma_{t_\sigma}\vee\mathcal{F}^\nu_{\tau'_\nu})_{t_\sigma\ge0}$-local martingale up to $T^{\mathcal{D}}_\sigma(\tau_\nu)$.*

Let $\mathbb{P}^i$ denote the law of (η_+,η_-). Since η_+ and η_- are independent, $\mathbb{P}^i=\mathbb{P}_+\times\mathbb{P}_-$ is the independent coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$. Fix $\underline{\xi}=(\xi_+,\xi_-)\in\Xi$. Combining Lemmas 4.6 and 4.5, we find that for any $\sigma\ne\nu\in\{+,-\}$, $t_\sigma\mapsto M(t_+\wedge\tau^+_{\xi_+},t_-\wedge\tau^-_{\xi_-})$ is a bounded $(\mathcal{F}^\sigma_{t_\sigma}\vee\mathcal{F}^\nu_{t_\nu})_{t_\sigma\ge0}$-martingale. Since $M(t_+\wedge\tau^+_{\xi_+},t_-\wedge\tau^-_{\xi_-})\to M(\tau^+_{\xi_+},\tau^-_{\xi_-})$ as $t_+,t_-\to\infty$, by dominated convergence theorem, we get $M(t_+\wedge\tau^+_{\xi_+},t_-\wedge\tau^-_{\xi_-})=\mathbb{E}^i[M(\tau^+_{\xi_+},\tau^-_{\xi_-})|\mathcal{F}^+_{t_+}\vee\mathcal{F}^-_{t_-}]$ for $t_+,t_-\ge0$. This means that $(t_+,t_-)\mapsto M(t_+\wedge\tau^+_{\xi_+},t_-\wedge\tau^-_{\xi_-})$ is an $(\mathcal{F}^+_{t_+}\vee\mathcal{F}^-_{t_-})$-martingale closed by $M(\tau^+_{\xi_+},\tau^-_{\xi_-})$. In particular, we have $\mathbb{E}^i[M(\tau^+_{\xi_+},\tau^-_{\xi_-})]=M(0,0)=1$. So we may define another probability measure $\mathbb{P}^c_{\underline{\xi}}$ by $d\mathbb{P}^c_{\underline{\xi}}=M(\tau^+_{\xi_+},\tau^-_{\xi_-})d\mathbb{P}^i$.

Suppose now (η_+,η_-) follows the law $\mathbb{P}^c_{\underline{\xi}}$ instead of $\mathbb{P}^i$. We now describe the properties of (η_+,η_-). By the martingale property of M, we have $\mathbb{E}^i[M(\tau^+_{\xi_+},\tau^-_{\xi_-})|\mathcal{F}^\sigma_\infty]=M|^\nu_0(\tau^\sigma_{\xi_\sigma})=1$, which implies that $\mathbb{P}^c_{\underline{\xi}}$ is also a coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$.

Fix $\sigma \neq \nu \in \{+,-\}$. Let τ_ν be an $\mathcal{F}^\nu$-stopping time with $\tau_\nu \le \tau^\nu_{\xi_\nu}$. By the martingale property of M, we see that for any $t_\sigma \ge 0$,

$$d(\mathbb{P}^c_{(\xi_+,\xi_-)}|\mathcal{F}^\sigma_{t_\sigma\wedge\tau^\sigma_{\xi_\sigma}} \vee \mathcal{F}^\nu_{\tau_\nu}) = M|^\nu_{\tau_\nu}(t_\sigma \wedge \tau^\sigma_{\xi_\sigma}) d(\mathbb{P}^i|\mathcal{F}^\sigma_{t_\sigma\wedge\tau^\sigma_{\xi_\sigma}} \vee \mathcal{F}^\nu_{\tau_\nu}).$$

By (4.42,4.46) and Girsanov Theorem, there is an $(\mathcal{F}^\sigma_{t_\sigma\wedge\tau^\sigma_{\xi_\sigma}} \vee F^\nu_{\tau_\nu})_{t_\sigma\ge 0}$-Brownian motion $\widetilde{B}^{\tau_\nu}_\sigma$ under $\mathbb{P}^c_{(\xi_+,\xi_-)}$ such that $\widehat{w}_\sigma$ satisfies the following SDE up to $\tau^\sigma_{\xi_\sigma}$:

$$d\widehat{w}_\sigma = \sqrt{\kappa} d\widetilde{B}^{\tau_\nu}_\sigma + \Big(\kappa\, \mathrm{b}\, A^{2:1}_\sigma|^\nu_{\tau_\nu} - 2\kappa\, \mathrm{b}\, X^{A:}_{\sigma,\nu}|^\nu_{\tau_\nu} + \kappa \sum_j (X^{A:}_{\sigma,j}|^\nu_{\tau_\nu} - X^{A:}_{\sigma,\infty}|^\nu_{\tau_\nu})[\rho_j + \kappa G_j(\underline{R}|^\nu_{\tau_\nu})]\Big) dt_\sigma.$$

By (4.29), W_σ satisfies the following SDE up to $\tau^\sigma_{\xi_\sigma}$ (with the variable t_ν fixed being τ_ν):

$$\partial_\sigma W_\sigma = A_{\sigma,1}\sqrt{\kappa}\partial \widetilde{B}^{\tau_\nu}_\sigma + \frac{(\kappa-6)A^2_{\sigma,1}}{W_\sigma - W_\nu}\partial t_\sigma + \sum_j \Big(\frac{A^2_{\sigma,1}}{W_\sigma - V_\sigma} - \frac{A^2_{\sigma,1}}{W_\sigma - V_\infty}\Big)[\rho_j + \kappa G_j(\underline{R})]\partial t_\sigma.$$

Note that η_{σ,τ_ν} and $(K_{\sigma,\tau_\nu}(\cdot))$ are chordal Loewner curve and chordal Loewner hulls, respectively, driven by $W_\sigma|^\nu_{\tau_\nu}$ with speed $A^2_{\sigma,1}|^\nu_{\tau_\nu}$, the Brownian motion $\widetilde{B}^\nu_{\tau_\nu}$ is independent of $\mathcal{F}^\nu_{\tau_\nu}$, and the processes $W_\nu|^\nu_{\tau_\nu}$ and $V_j|^\nu_{\tau_\nu}$ are force point processes for this family of Loewner hulls started from $\widehat{w}_\nu(\tau_\nu)$ and $\widehat{v}^\nu_j(\tau_\nu)$. By Proposition 3.8, we see that, conditionally on $\mathcal{F}^\nu_{\tau_\nu}$, after a reparametrization by half-plane capacity, the law of the part of η_{σ,τ_ν} up to the time that it hits $\overline{g_{K_\nu(\tau_\nu)}(\xi_\sigma)}$ agrees with that of an iSLE$_\kappa(\underline{\rho})$ (if $\sigma = +$) or iSLE$^r_\kappa(\underline{\rho})$ (if $\sigma = -$) curve in $\mathbb{H}$ under chordal coordinate from $g_{K_\nu(\tau_\nu)}(w_\sigma)$ to $\widehat{w}_\nu(\tau_\nu)$ with force points $\widehat{v}^\nu_j(\tau_\nu)$, $j \in \mathbb{N}^\infty_m$, up to the same hitting time. Applying the conformal map $g^{-1}_{K_\nu(\tau_\nu)}$, we then conclude that η_σ satisfies Theorem 4.1 (i) (if $\sigma = +$) or Theorem 4.1 (ii) (if $\sigma = -$) up to $\tau^\sigma_{\xi_\sigma}$ with the additional assumption that $\tau_\nu \le \tau^\nu_{\xi_\nu}$. So $\mathbb{P}^c_{\underline{\xi}}$ is a local commutation coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$ within $\underline{\xi}$.

Lemma 4.7. *Let η_+ and η_- be two random Loewner curves started from w_+ and w_-, respectively. Let $(\xi_+,\xi_-) \in \Xi$. Let $\sigma \neq \nu \in \{+,-\}$. Suppose that the law of η_ν restricted to $\mathcal{F}^\nu_{\tau^\nu_{\xi_\nu}}$ agrees with $\mathbb{P}_\nu$, and conditionally on $\mathcal{F}^\nu_{\tau^\nu_{\xi_\nu}}$, η_σ satisfies Theorem 4.1 (i) up to $\tau^+_{\xi_+}$ if $\sigma = +$, or Theorem 4.1 (ii) up to $\tau^-_{\xi_-}$ if $\sigma = -$. Then the law of η_σ restricted to $\mathcal{F}^\sigma_{\tau^\sigma_{\xi_\sigma}}$ agrees with $\mathbb{P}_\sigma$.*

Proof. We know the law of $\eta_\nu|_{[0,\tau^\nu_{\xi_\nu}]}$ and the conditional law of $\eta_\sigma|_{[0,\tau^\sigma_{\xi_\sigma}]}$ given $\eta_\nu|_{[0,\tau^\nu_{\xi_\nu}]}$, which together determine the joint law of $\eta_\sigma|_{[0,\tau^\sigma_{\xi_\sigma}]}$ and $\eta_\nu|_{[0,\tau^\nu_{\xi_\nu}]}$. So the joint law of η_+ and η_- restricted to $\mathcal{F}^+_{\tau^+_{\xi_+}} \vee \mathcal{F}^-_{\tau^-_{\xi_-}}$ is also determined, which has to agree with the local commutation coupling $\mathbb{P}^c_{\underline{\xi}}$. So the law of η_σ restricted to $\mathcal{F}^\sigma_{\tau^\sigma_{\xi_\sigma}}$ agrees with $\mathbb{P}_\sigma$. $\square$

We now use the local commutation couplings to construct a global commutation coupling, and finish the proof of Theorem 4.1. First, we observe that, for any $(\xi_+,\xi_-) \in \Xi$, if any coupling $\mathbb{P}$ of $\mathbb{P}_+$ and $\mathbb{P}_-$ agrees with $\mathbb{P}^c_{(\xi_+,\xi_-)}$ on $\mathcal{F}^+_{\tau^+_{\xi_+}} \vee \mathcal{F}^-_{\tau^-_{\xi_-}}$, then $\mathbb{P}$ is also a local commutation coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$ within (ξ_+,ξ_-).

Let $\underline{\xi}^k = (\xi^k_+, \xi^k_-) \in \Xi$, $1 \le k \le n$. By [23, Theorem 6.1], there is a bounded positive continuous $(\mathcal{F}^+_{t_+} \vee \mathcal{F}^-_{t_-})_{(t_+,t_-)\in\mathbb{R}^2_+}$-martingale $M^n(t_+, t_-)$, $t_+, t_- \ge 0$, such that $M^n(t, 0) = M^n(0, t) = 1$ for any $t \ge 0$, and for any $1 \le k \le n$, M^n agrees with M on $[0, \tau^+_{\xi^k_+}] \times [0, \tau^-_{\xi^k_-}]$. Moreover, M^n takes random constant value, denoted by $M^n(\underline{\infty})$, if $t_+ \ge \tau^+_{\xi^k_+}$ and $t_- \ge \tau^-_{\xi^k_-}$ for all $1 \le k \le n$. So we have $\mathbb{E}^i[M^n(\underline{\infty})] = M^n(0, 0) = 1$, and may define another probability measure $\mathbb{P}^n$ by $d\mathbb{P}^n = M^n(\underline{\infty})d\mathbb{P}^i$. By the martingale property of M^n, $\mathbb{P}^n$ is also a coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$, and for any $1 \le k \le n$,

$$d(\mathbb{P}^n|\mathcal{F}^+_{\tau^+_{\xi^k_+}} \vee \mathcal{F}^-_{\tau^-_{\xi^k_-}})/d(\mathbb{P}^i|\mathcal{F}^+_{\tau^+_{\xi^k_+}} \vee \mathcal{F}^-_{\tau^-_{\xi^k_-}}) = M^n(\tau^+_{\xi^k_+}, \tau^-_{\xi^k_-}) = M(\tau^+_{\xi^k_+}, \tau^-_{\xi^k_-}).$$

Thus, $\mathbb{P}^n$ agrees with $\mathbb{P}^c_{\underline{\xi}^k}$ on $\mathcal{F}^+_{\tau^+_{\xi^k_+}} \vee \mathcal{F}^-_{\tau^-_{\xi^k_-}}$ for $1 \le k \le n$, which implies that $\mathbb{P}^n$ is a local commutation coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$ within $\underline{\xi}^k$ for any $1 \le k \le n$.

We may pick a countable subset Ξ^* of Ξ such that for every $(\xi_+, \xi_-) \in \Xi$, there is $(\xi^*_+, \xi^*_-) \in \Xi^*$ such that for $\sigma \in \{+, -\}$, $\mathrm{Hull}(\xi_\sigma) \subset \mathrm{Hull}(\xi^*_\sigma)$, which then implies that $\tau^\sigma_{\xi_\sigma} \le \tau^\sigma_{\xi^*_\sigma}$. Enumerate Ξ^* by $\{\underline{\xi}^k : k \in \mathbb{N}\}$. By the previous paragraph, for each $n \in \mathbb{N}$, there is a coupling $\mathbb{P}^n$ of $\mathbb{P}_+$ and $\mathbb{P}_-$, which is a local commutation coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$ within $\underline{\xi}^k$ for any $1 \le k \le n$. We let $\mathbb{P}^\infty$ be a subsequential weak limit of the sequence $\mathbb{P}^n$ in some suitable topology. Then $\mathbb{P}^\infty$ is still a coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$, and for every $k \in \mathbb{N}$, it is a local commutation coupling of $\mathbb{P}_+$ and $\mathbb{P}_-$ within $\underline{\xi}^k$. Finally, if (ξ_+, ξ_-) follows the law $\mathbb{P}^\infty$, then Theorem 4.1 (i) and (ii) both hold. This is because for $\sigma \ne \nu \in \{+, -\}$ and $\tau_\nu < T_\nu$, $T^{\mathcal{D}}_\sigma(\tau_\nu) = \sup\{\tau^\sigma_{\xi_\sigma} : (\xi_+, \xi_-) \in \Xi^*, \tau_\nu < \tau^\nu_{\xi_\nu}\}$.

5 Proofs of the Main Theorems

In this section, we will prove Theorem 1.2, which contains Theorem 1.1 as a special case. We work on the cases $\kappa \in (0, 4]$ and $\kappa \in (4, 8)$ separately. Let $\mathbb{N}_m = \{1, \dots, m\}$, $\mathbb{N}^\infty_m = \mathbb{N}_m \cup \{\infty\}$, $\underline{\rho} = (\rho_1, \dots, \rho_m)$, $\rho^r_j = -\rho_j$, $j \in \mathbb{N}^\infty_m$, $\underline{\rho}^r = (\rho^r_1, \dots, \rho^r_m)$, $\underline{v} = (v_1, \dots, v_m, v_\infty)$, $v^r_j = J(v_j)$, $j \in \mathbb{N}^\infty_m$, and $\underline{v}^r = (v^r_1, \dots, v^r_m, v^r_\infty)$. By symmetry, we assume that $\sigma = -$.

We only need to show that the time-reversal of $J(\eta)$ has the same law as η^r after a time-change because the absolute continuity statement then follows from Lemma 3.6 (iii) and the fact that η a.s. does not visit any force point other than $0^\pm$ and $\pm\infty$.

5.1 The simple curve case

Proof of Theorem 1.2 in the case $\kappa \le 4$. Since $\kappa \le 4$ and $\sigma = -$, η a.s. does not intersect $(0, \infty)$. Let $f(z) = 1/(1 - z)$. Let $u_j = f(v_j)$, $j \in \mathbb{N}^\infty_m$, and $\underline{u} = (u_1, \dots, u_m, u_\infty)$. We use the convention that $f(0^-) = 1^-$ and $f(-\infty) = 0^+$. Then $u_1 > \dots > u_m > u_\infty \in (0, 1) \cup \{0^+, 1^-\}$, and $f(\eta)$ does not intersect $(-\infty, 0)$. So $f(\eta)$ does not separate 0 from ∞ before it ends. Thus, we may reparametrize the complete $f(\eta)$ by half-plane capacity to get an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from $f(0) = 1$ to $f(\infty) = 0$ with force points $\underline{u}$.

Similarly, η^r a.s. does not intersect $(-\infty,0)$. Let $f^r(z)=f\circ J(z)=z/(z+1)$. Then $u_j=f^r(v_j^r)$, $j\in\mathbb{N}_m^\infty$, and $f^r(\eta^r)$ does not intersect $(1,\infty)$. So $f^r(\eta^r)$ does not separate 1 from ∞ before it ends. Thus, we may reparametrize the complete $f^r(\eta^r)$ by half-plane capacity to get an $\mathrm{iSLE}_\kappa^r(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from $f^r(0)=0$ to $f^r(\infty)=1$ with force points $\underline{u}$. Thus, $f(\eta)$ and $f^r(\eta^r)$ have the same laws as the η_+ and η_- in Theorem 4.1, respectively, with $\underline{u}$ in place of $\underline{v}$. It now suffices to show that the η_+ and η_- in Theorem 4.1 are time-reversal of each other in the case $\kappa\le 4$.

Suppose $\tau_-<T_-$ is an $\mathcal{F}^-$-stopping time. Then given $\mathcal{F}^-_{\tau_-}$, up to a time-change, the part of η_+ up to $T_+^{\mathcal{D}}(\tau_-)$, which is the first time that it intersects $\overline{K_-(\tau_-)}$ or separates $\overline{K_-(\tau_-)}$ from ∞, is an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}\setminus K_-(\tau_-)$ from w_+ to $\eta_-(\tau_-)$ with force points $v_j\vee\max\{\overline{K_-(\tau_-)}\cap\mathbb{R}\}$, $j\in\mathbb{N}_m^\infty$, also up to $T_+^{\mathcal{D}}(\tau_-)$. Since η_+ a.s. ends at w_-, if the $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve $\eta_+|_{[0,T_+^{\mathcal{D}}(\tau_-))}$ in $\mathbb{H}\setminus K_-(\tau_-)$ does not land at its target $\eta_-(\tau_-)$, then $\eta_+(T_+^{\mathcal{D}}(\tau_-))$ belongs to one of the following boundary arcs of $\mathbb{H}\setminus K_-(\tau_-)$: (i) P_R, the part of $\partial K_-(\tau_-)$ on the right of $\eta_-(\tau_-)$, (ii) P_L, the part of $\partial K_-(\tau_-)$ on the left of $\eta_-(\tau_-)$, and (iii) $P_{\mathbb{R}}$, the real interval $(-\infty,\min\{\overline{K_-(\tau_-)}\cap\mathbb{R}\}]$. The $g_{K_-(\tau_-)}$-image of $\eta_+|_{[0,T_+^{\mathcal{D}}(\tau_-))}$ is an $\mathrm{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from $W_+(0,\tau_-)$ to $W_-(0,\tau_-)$ with force points $V_j(0,\tau_-)$, $j\in\mathbb{N}_m^\infty$, up to some time. Since $W_-\le V_\infty\le\cdots\le V_1\le W_+$, this curve a.s. does not visit the intervals $(W_-(0,\tau_-),V_\infty(0,\tau_-)]$ and $(-\infty,W_-(0,\tau_-))$, which are respectively the $g_{K_-(\tau_-)}$-images of P_R and $P_L\cup P_{\mathbb{R}}$. So η_+ a.s. does not visit $P_R\cup P_L\cup P_{\mathbb{R}}$ at $T_+^{\mathcal{D}}(\tau_-)$, which implies that η_+ a.s. visits $\eta_-(\tau_-)$ at $T_+^{\mathcal{D}}(\tau_-)$.

Consider countably many $\mathcal{F}^-$-stopping times: $q\wedge\tau^-_{\xi_-}$, where $q\in\mathbb{Q}_+$ and $\xi_-\in\Xi^*_-$, which is the projection of Ξ^* to Ξ_-. Then a.s. η_+ visits $\eta_-(q\wedge\tau^-_{\xi_-})$ for every $q\in\mathbb{Q}_+$ and $\xi_-\in\Xi^*_-$. By the denseness of $\mathbb{Q}_+$ in $\mathbb{R}_+$ and the continuity of η_+ and η_-, we know that a.s. $\eta_-([0,\tau^-_{\xi_-}])\subset\eta_+([0,T_+))$ for every $\xi_-\in\Xi^*_-$, which further implies that a.s. $\eta_-([0,T_-))\subset\eta_+([0,T_+])$. Since η_+ a.s. does not visit (w_+,∞), η_- does not either. So η_- is a time-change of a complete $\mathrm{iSLE}_\kappa^r(\underline{\rho})$ curve. Since η_+ a.s. does not visit any of its force points other than w_+^- or w_-^+, η_- has the same property. By Lemma 3.6 (iii), the law of η_- is absolutely continuous w.r.t. that of a chordal $\mathrm{SLE}_\kappa(\underline{\rho}^r,\rho^r_\infty)$ curve in $\mathbb{H}$ under chordal coordinate from w_- to w_+ with force points $\underline{v}$. Thus, η_- a.s. ends at w_+, and we get $\eta_-([0,T_-])=\eta_+([0,T_+])$. From this we then conclude that η_- is a time-reversal of η_+. □

5.2 The non-simple curve case

The argument in the previous subsection does not work for $\kappa\in(4,8)$ because for a commuting pair of nonsimple curves, if we condition on a part of one curve, the first point that the second curve will hit the given part of the first curve may not be the tip point.

Proof of Theorem 1.2 in the case that $\kappa\in(4,8)$. We now have $\sum_{j=1}^k\rho_j\ge\frac{\kappa}{2}-2$ for any $1\le k\le m$. Let $\mathbb{P}_2$ and $\mathbb{P}_2^r$ denote the laws of η and η^r, respectively, in the theorem. Let $\mathcal{R}$ denote the space of chordal Loewner curves γ from 0 to ∞, such that the time-reversal of $J(\gamma)$ could be parametrized to be a chordal Loewner curve, which will be denoted by $J_*(\gamma)$. We also use

J_* to denote the pushforward map induced by J_*. Our goal is to show that $\mathbb{P}_2$ is supported by $\mathcal{R}$, and $J_*(\mathbb{P}_2)=\mathbb{P}_2^r$.

We first consider the case that all force points take values in $(-\infty,0)$, i.e., there are no degenerate force points. Let $\mathbb{P}_0$ denote the law of a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. By reversibility of chordal SLE$_\kappa$ for $\kappa\in(4,8)$ (cf. [7]), $\mathbb{P}_0$ is supported by $\mathcal{R}$, and $J_*(\mathbb{P}_0)=\mathbb{P}_0$. We will use an idea in [17], which is to show that the both $\mathbb{P}_2$ and $\mathbb{P}_2^r$ are absolutely continuous w.r.t. $\mathbb{P}_0$, and the Radon-Nikodym derivatives are related by the map J_*.

Let $\mathbb{P}_1$ denote the law of the chordal SLE$_\kappa(\underline{\rho},\rho_\infty)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{v}$. By Proposition 2.11, $\mathbb{P}_1\ll\mathbb{P}_0$, and $d\mathbb{P}_1/d\mathbb{P}_0$ is given by (2.6). By the definition of iSLE$_\kappa(\underline{\rho})$ curve, $\mathbb{P}_2\ll\mathbb{P}_1$, and $d\mathbb{P}_2/d\mathbb{P}_1=M(T_\infty)$, which is given by (3.11. Thus, $\mathbb{P}_2\ll\mathbb{P}_0$, and so $\mathbb{P}_2$ is supported by $\mathcal{R}$. Let E_0 be the set of $\gamma\in\mathcal{R}$ such that $\gamma\cap[v_\infty,v_1]=\emptyset$. For $\gamma\in E_0$, let $D_\infty(\gamma)$ be the connected component of $\mathbb{H}\setminus\gamma$, whose boundary contains $[v_\infty,v_1]$. Let $\widetilde{\rho}_j=\rho_j/\kappa$, $R_j(0)=v_j/v_\infty$, $1\le j\le m$, and $\underline{R}(0)=(R_1(0),\dots,R_m(0))$. Combining (2.6) with (3.11), we get

$$\frac{d\mathbb{P}_2}{d\mathbb{P}_0}=\frac{\mathbf{1}_{E_0}F(\underline{1})}{F(\underline{R}(0))}\prod_{j=1}^m R_j(0)^{-\frac{\rho_j}{\kappa}}\Big(\frac{H_{D_\infty}(v_j,v_\infty)}{|v_j-v_\infty|^{-2}}\Big)^{-\frac{\rho_j(\rho_\infty+\kappa-4)}{4\kappa}}\prod_{1\le j<k\le m}\Big(\frac{H_{D_\infty}(v_j,v_k)}{|v_j-v_k|^{-2}}\Big)^{-\frac{\rho_j\rho_k}{4\kappa}}. \quad (5.1)$$

Let $\mathbb{P}_1^r$ denote the law of the chordal SLE$_\kappa(\underline{\rho}^r,\rho_\infty^r)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{v}^r$. Since $\rho_\infty^r=\sum_{j=1}^m\rho_j\ge\frac{\kappa}{2}-2$, and for any $2\le k\le m$, $\rho_\infty^r+\sum_{j=k}^m\rho_j^r=\sum_{j=1}^{k-1}\rho_j\ge\frac{\kappa}{2}-2$, by Proposition 2.11, $\mathbb{P}_1^r\ll\mathbb{P}_0$. Let E_0^r be the set of $\gamma\in\mathcal{R}$, which do not intersect $[v_\infty^r,v_1^r]$. For $\gamma\in E_0^r$, let $D_\infty^r(\gamma)$ be the connected component of $\mathbb{H}\setminus\gamma$, whose boundary contains $[v_\infty^r,v_1^r]$. Let g_* be a conformal map from D_∞^r onto $\mathbb{H}$ such that $\max(\partial D_\infty^r\cap\mathbb{R})$ is mapped to ∞. By (2.9),

$$\frac{d\mathbb{P}_1^r}{d\mathbb{P}_0}=\mathbf{1}_{E_0^r}\prod_{j\in\mathbb{N}_m^\infty}\frac{g_*'(v_j^r)^{\frac{\rho_j^r(\rho_j^r+4-\kappa)}{4\kappa}}}{|v_j^r|^{\rho_j^r/\kappa}}\prod_{j<k\in\mathbb{N}_m^\infty}\Big(\frac{|g_*(v_j^r)-g_*(v_k^r)|}{|v_j^r-v_k^r|}\Big)^{\frac{\rho_j^r\rho_k^r}{2\kappa}}, \quad (5.2)$$

We now express $d\mathbb{P}_1^r/d\mathbb{P}_0$ in terms of boundary Poisson kernel and conformal radius, but in a way different from (2.6). First, we have $H_{D_\infty^r}(v_j^r,v_k^r)=\frac{g_*'(v_j^r)g_*'(v_k^r)}{|g_*(v_j^r)-g_*(v_k^r)|^2}$, $j<k\in\mathbb{N}_m^\infty$. When D_∞^r is defined, let Ω_∞^r denote the union of D_∞^r, its reflection about $\mathbb{R}$, and the interval $(v_\infty^r,\max(\partial D_\infty^r\cap\mathbb{R}))$. Then g_* extends to a conformal map from Ω_∞^r onto $\mathbb{C}\setminus(-\infty,g_*(v_\infty^r)]$. So we have $\mathrm{crad}^{(4)}_{v_j^r}(\Omega_\infty^r)=\frac{|g_*(v_j^r)-g_*(v_\infty^r)|}{g_*'(v_j^r)}$. By (5.2) and that $\sum_{j\in\mathbb{N}_m^\infty}\rho_j^r=0$,

$$\frac{d\mathbb{P}_1^r}{d\mathbb{P}_0^r}=\frac{\mathbf{1}_{E_0^r}}{\mathcal{Z}^r}\prod_{j=1}^m\Big(\mathrm{crad}^{(4)}_{v_j^r}(\Omega_\infty^r)^{-\frac{\rho_j^r(4-\kappa)}{2\kappa}}H_{D_\infty^r}(v_j^r,v_\infty^r)^{-\frac{\rho_j^r(\rho_\infty^r+4-\kappa)}{4\kappa}}\Big)\prod_{1\le j<k\le m}H_{D_\infty^r}(v_j^r,v_k^r)^{-\frac{\rho_j^r\rho_k^r}{4\kappa}},$$

where $\mathcal{Z}^r>0$ is a constant given by

$$\mathcal{Z}^r:=\prod_{j=1}^m(v_\infty^r/v_j^r)^{\frac{\rho_j^r}{\kappa}}\prod_{j=1}^m|v_j^r-v_\infty^r|^{\frac{\rho_j^r\rho_\infty^r}{2\kappa}}\prod_{1\le j<k\le m}|v_j^r-v_k^r|^{\frac{\rho_j^r\rho_k^r}{2\kappa}}.$$

Recall the definition of M^r in (3.19) and the formula for I^r_j in (3.16). On the event E^r_0, for $j\in\mathbb{N}_m$, as $t\uparrow T^r_\infty=T^r_j$, $R^r_j(t)\to 1$ and $\operatorname{crad}^{(4)}_{v^r_j}(\Omega^r_t)\to\operatorname{crad}^{(4)}_{v^r_j}(\Omega^r_\infty)$ because Ω^r_t tends to $\Omega^r_{T^r_j}=\Omega^r_\infty$ in the Carathéodory topology. Since $\rho_j=-\rho^r_j$, we get

$$M^r(\infty)=M^r(T^r_\infty)=\frac{F(\underline{1})}{F(\underline{R}^r(0))}\prod_{j=1}^m\Big(\frac{\operatorname{crad}^{(4)}_{v^r_j}(\Omega^r_\infty)}{|v^r_j-v^r_\infty|}\Big)^{-\frac{\rho^r_j(\kappa-4)}{2\kappa}},\quad\text{on } E^r_0.$$

Since $R^r_j(0)=v^r_\infty/v^r_j$, we get

$$\frac{d\mathbb{P}^r_1}{d\mathbb{P}_0}M^r(\infty)=\frac{\mathbf{1}_{E^r_0}F(\underline{1})}{F(\underline{R}^r(0))}\prod_{j=1}^m R^r_j(0)^{\frac{\rho^r_j}{\kappa}}\Big(\frac{H_{D^r_\infty}(v^r_j,v^r_\infty)}{|v^r_j-v^r_\infty|^{-2}}\Big)^{-\frac{\rho^r_j(\rho^r_\infty+4-\kappa)}{4\kappa}}\prod_{1\le j<k\le m}\Big(\frac{H_{D^r_\infty}(v^r_j,v^r_k)}{|v^r_j-v^r_k|^{-2}}\Big)^{-\frac{\rho^r_j\rho^r_k}{4\kappa}}. \tag{5.3}$$

We compare (5.1) with (5.3). Note that $E^r_0=J_*(E_0)$, $J(D_\infty(\gamma))=D^r_\infty(J_*(\gamma))$ for $\gamma\in E_0$, $\underline{R}^r(0)=\underline{R}(0)$ and $\rho^r_j=-\rho_j$. By conformal covariance of boundary Poisson kernel, we get

$$H_{D^r_\infty(J_*(\gamma))}(v^r_j,v^r_k)/|v^r_j-v^r_k|^{-2}=H_{D_\infty(\gamma)}(v_j,v_k)/|v_j-v_k|^{-2},\quad 1\le j<k\le m+1.$$

So we get $(d\mathbb{P}^r_1/d\mathbb{P}_0)\cdot M^r(\infty)=(d\mathbb{P}_2/d\mathbb{P}_0)\circ J_*^{-1}$. Since $J_*(\mathbb{P}_0)=\mathbb{P}_0$ and $\mathbb{P}_2$ is a probability measure, we get $\mathbb{E}^r_1[M^r(\infty)]=\mathbb{E}_0[(d\mathbb{P}^r_1/d\mathbb{P}_0)\cdot M^r(\infty)]=\mathbb{E}_0[d\mathbb{P}_2/d\mathbb{P}_0]=1$. Since M^r is a positive supermartingale w.r.t. $\mathbb{P}^r_1$, and $M^r(0)=1$, we then conclude that M^r is a uniformly integrable martingale w.r.t. $\mathbb{P}^r_1$. By Definition 3.4 and Lemma A.1, we have $\mathbb{P}^r_2\ll\mathbb{P}^r_1$, and $d\mathbb{P}^r_2/d\mathbb{P}^r_1=M^r(\infty)$. Thus, $\mathbb{P}^r_2\ll\mathbb{P}_0$, and $d\mathbb{P}^r_2/d\mathbb{P}_0=(d\mathbb{P}^r_1/d\mathbb{P}^r_0)\cdot M^r(\infty)=(d\mathbb{P}_2/d\mathbb{P}_0)\circ J_*^{-1}$. Since $J_*(\mathbb{P}_0)=\mathbb{P}_0$, we then conclude that $J_*(\mathbb{P}_2)=\mathbb{P}^r_2$. This finishes the proof of the case that none of the v_j's takes values $0^\pm$ or $\pm\infty$.

Now suppose $v_1=0^-$ and all other v_j's including v_∞ lie in $(-\infty,0)$. Let $0<l<\min\{|v_j|: j\in\mathbb{N}^\infty_m, j>1\}$. Let τ_l be the first time that η reaches $\{|z|=l\}$. By DMP of $\text{iSLE}_\kappa(\underline{\rho})$ curve, conditionally on $\mathcal{F}_{\tau_l}$, $\eta(\tau_l+\cdot)$ is an $\text{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}\setminus K_{\tau_l}$ from $\eta(\tau_l)$ to ∞ with force points $0^-,v_2,\dots,v_m,v_\infty$. Note that η does not visit $(-\infty,0]$ during the time interval $(0,\tau_l]$. So $\eta([0,\tau_l])$ does not separate any of $v_2,\dots,v_m,v_\infty$ from ∞. Thus, none of the force points for the $\text{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}\setminus K_{\tau_l}$ from $\eta(\tau_l)$ to ∞ is degenerate. By the reversibility result we have derived, the conditional law given $\mathcal{F}_{\tau_l}$ of the time-reversal of η up to the time of hitting $\eta(\tau_l)$ is that of an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}\setminus K_{\tau_l}$ from ∞ to $\eta(\tau_l)$ with force points $0^-,v_2,\dots,v_m,v_\infty$. In particular, this implies that a.s. the time-reversal of $J(\eta)$ up to hitting the circle $\{|z|=1/l\}$ can be parametrized to be a chordal Loewner curve. By letting $l\downarrow 0$, we see that the time-reversal of the complete $J(\eta)$ a.s. can be parametrized to be a chordal Loewner curve. So η a.s. belongs to $\mathcal{R}$, and we may define $J_*(\eta)$. Given $\mathcal{F}_{\tau_l}$, the part of $J_*(\eta)$ up to the time that it reaches $J(\eta(\tau_l))$ is then an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $J(\mathbb{H}\setminus K_{\tau_l})$ from 0 to $J(\eta(\tau_l))$ with force points $+\infty,v^r_2,\dots,v^r_m,v^r_\infty$.

Fix $n\in\mathbb{N}$. Let $f_n(z)=1/(1-nz)$ be a Möbius automorphism of $\mathbb{H}$, which maps 0 and ∞ to 1 and 0, respectively. Let $u_j=f_n(v_j)$, $j\in\mathbb{N}^\infty_m$, and $\underline{u}=(u_1,\dots,u_m,u_\infty)$. Then u_j's lie on $(0,1)$ except that $u_1=1^-$. We may reparametrize the part of $f_n(\eta)$ up to the time that it hits

its target 0 or separates 0 from ∞ by half-plane capacity, and get a chordal Loewner curve η_+, which is an $\text{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from 1 to 0 with force points $\underline{u}$. Let $f_n^r(z) = f_n \circ J(z) = z/(z+n)$. We may reparametrize the part of $f_n^r(J_*(\eta))$ up to the time that it hits its target 1 or separates 1 from ∞, and get a chordal Loewner curve η_-.

Let $\xi_+^l = f_n(\{z \in \mathbb{H} : |z| = l\}) \in \Xi_+$. By the relation between η and $J_*(\eta)$ derived earlier, we know that, for any $\xi_- \in \Xi_-$ such that $(\xi_+^l, \xi_-) \in \Xi$, given the part of η_+ up to $\tau^+_{\xi_+^l}$, up to a time-change, the part of η_- up to $\tau^-_{\xi_-}$ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H} \setminus \text{Hull}(\eta_+([0, \tau^+_{\xi_+^l}]))$ from 0 to $\eta_+(\tau^+_{\xi_+^l})$ with force points $1^-, u_2, \dots, u_m, u_\infty$, also up to $\tau^-_{\xi_-}$. By Lemma 4.7, the part of η_- up to $\tau^-_{\xi_-}$ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to 1 with force points $\underline{u}$, up to $\tau^-_{\xi_-}$. Since this holds for any $\xi_- \in \Xi_-$ such that $(\xi_+^l, \xi_-) \in \Xi$, we then conclude that the part of η_- up to hitting $\overline{\xi_+^l}$ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ under chordal coordinate from 0 to 1 with force points $\underline{u}$ up to the same hitting time. By letting $l \downarrow 0$ and using the definition of η_-, we know that the part of $f_n^r(J_*(\eta))$ up to the time that it hits 1 or separates 1 from ∞ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to 1 with force points $\underline{u}$ up to the same time. Since f_n^r maps $-n$, ∞, and v_j^r to ∞, 1, and u_j, respectively, the part of $J_*(\eta)$ up to the first time that it separates $-n$ from ∞ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{v}^r$ up to the same time. Letting $n \to \infty$, we conclude that the whole $J_*(\eta)$ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{v}^r$. So $J_*(\mathbb{P}_2) = \mathbb{P}_2^r$.

Finally, we consider the case $v_\infty = -\infty$. It suffices to show that the law of $J_*(\eta^r)$ is $\mathbb{P}_2$ in the case that v^r_∞ is degenerate, i.e., 0^+. We have proved that this is true if v^r_∞ is not degenerate. Let $l \in (0, v^r_m)$ and let τ^r_l be the first time that η^r hits $\{|z| = l\}$. By Lemma 3.6 (iv), τ^r_l is strictly less than the lifetime of η^r. By DMP of $\text{iSLE}^r_\kappa(\underline{\rho})$, conditionally on $\mathcal{F}^r_{\tau^r_l}$, $\eta^r(\tau^r_l + \cdot)$ is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H} \setminus K^r(\tau^r_l)$ from $\eta^r(\tau^r_l)$ to ∞ with force points $v^r_1, \dots, v^r_m, 0^+$. Since $\rho^r_\infty \ge \frac{\kappa}{2} - 2$, none of the force points for the $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H} \setminus K^r_{\tau^r_l}$ is degenerate.

By the reversibility result we have derived, the conditional law given $\mathcal{F}^r_{\tau^r_l}$ of the time-reversal of η^r up to the time of hitting $\eta^r(\tau^r_l)$ is that of an $\text{iSLE}_\kappa(\underline{\rho})$ curve in $\mathbb{H} \setminus K^r_{\tau^r_l}$ from ∞ to $\eta^r(\tau^r_l)$ with force points $v^r_1, \dots, v^r_m, 0^+$. In particular, this implies that a.s. the time-reversal of $J(\eta^r)$ up to hitting the circle $\{|z| = 1/l\}$ can be parametrized to be a chordal Loewner curve. By letting $l \downarrow 0$, we see that a.s $\eta^r \in \mathcal{R}$, and we may define $J_*(\eta^r)$. Given $\mathcal{F}^r_{\tau^r_l}$, the part of $J_*(\eta^r)$ up to the time that it reaches $J(\eta^r(\tau^r_l))$ is then an $\text{iSLE}_\kappa(\underline{\rho})$ curve in $J(\mathbb{H} \setminus K^r_{\tau^r_l})$ from 0 to $J(\eta^r(\tau^r_l))$ with force points $v_1, \dots, v_m, -\infty$.

Fix $n \in \mathbb{N}$. Let $f^r_n(z) = \frac{nz}{1+nz}$, which maps 0 and ∞ to 0 and 1, respectively. Let $u_j = f^r_n(v^r_j)$, $j \in \mathbb{N}^\infty_m$, and $\underline{u} = (u_1, \dots, u_m, u_\infty)$. Then u_j's lie on $(0,1)$ except that $u_\infty = 0^+$. We may reparametrize the part of $f^r_n(\eta^r)$ up to the time that it hits its target 1 or separates 1 from ∞ by half-plane capacity, and get a chordal Loewner curve η_-, which is an $\text{iSLE}^r_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to 1 with force points $\underline{u}$. Let $f_n(z) = f^r_n \circ J(z) = \frac{n}{n-z}$. We may reparametrize the part of $f_n(J_*(\eta^r))$ up to the time that it hits its target 0 or separates 0 from ∞ by half-plane capacity, and get a chordal Loewner curve η_+.

Let $\xi^l_- = f^r_n(\{z \in \mathbb{H} : |z| = l\}) \in \Xi_-$. By the relation between η^r and $J_*(\eta^r)$ derived earlier, we know that, for any $\xi_+ \in \Xi_+$ such that $(\xi_+, \xi^l_-) \in \Xi$, given the part of η_- up to $\tau^-_{\xi^l_-}$, up to a time-change, the part of η_+ up to $\tau^+_{\xi_+}$ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}\setminus \mathrm{Hull}(\eta_-([0,\tau^-_{\xi^l_-}]))$ from 1 to $\eta_-(\tau^-_{\xi^l_-})$ with force points $\underline{u}$. By Lemma 4.7, the part of η_+ up to $\tau^+_{\xi_+}$ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 1 to 0 with force points $\underline{u}$ up to the same hitting time. Since this holds for any $\xi_+\in\Xi_+$ such that $(\xi_+,\xi^l_-)\in\Xi$, we then conclude that the part of η_+ up to hitting $\overline{\xi^l_-}$ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 1 to 0 with force points $\underline{u}$ up to the same hitting time. By letting $l\downarrow 0$ and using the definition of η_+, we know that the part of $f_n(J_*(\eta^r))$ up to the time that it hits 0 or separates 0 from ∞ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 1 to 0 with force points $\underline{u}$ up to the same hitting time. Since f_n maps n and ∞ to ∞ and 0, respectively, the part of $J_*(\eta^r)$ up to the first time that it separates n from ∞ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{v}$ up to the same time. Letting $n\to\infty$, we conclude that the whole $J_*(\eta^r)$ is an iSLE$_\kappa(\underline{\rho})$ curve in $\mathbb{H}$ from 0 to ∞ with force points $\underline{v}$. So $J_*(\mathbb{P}^r_2)=\mathbb{P}_2$. The proof is now complete. □

Corollary 5.1. *Let κ, ρ_1 and ρ_2 satisfy (i) or (ii) of Theorem 1.1. Let $v\in\mathbb{R}\setminus\{0\}$, and let $\sigma\in\{+,-\}$ be the sign of v. Let η be a chordal SLE$_\kappa(\rho_1,\rho_2)$ curve in $\mathbb{H}$ started from 0 with force points 0^σ and v. Then the time-reversal of $J\circ\eta$, where $J(z)=-1/z$, can be reparametrized to be a chordal Loewner curve η^r, which is an iSLE$^r_\kappa(\rho_1,\rho_2)$ curve in $\mathbb{H}$ from 0 to ∞ with force points $-\sigma\infty, J(v), 0^{-\sigma}$. Let (K^r_t) be the chordal Loewner hulls generated by η^r; let $\widehat{v}^r_\infty(t)=g^0_{K^r_t}(0^{-\sigma})$, $\widehat{v}^r_2(t)=g^0_{K^r_t}(J(v))$; and let $R^r(t)=\frac{\widehat{w}^r(t)-\widehat{v}^r_\infty(t)}{\widehat{w}^r(t)-\widehat{v}^r_2(t)}$ before the first time that $\widehat{v}^r_2(t)=\widehat{v}^r_\infty(t)$, and equals 1 after that time. Then the driving function $\widehat{w}^r$ for η^r satisfies the SDE*

$$d\widehat{w}^r=\sqrt{\kappa}dB^r+\frac{\rho_1}{\widehat{w}^r-v^r_\infty}\,dt-\Big(\frac{1}{\widehat{w}^r-\widehat{v}^r_2}-\frac{1}{\widehat{w}^r-\widehat{v}^r_\infty}\Big)[\rho_2+\kappa G_*(R^r)]dt, \tag{5.4}$$

where B^r is a standard Brownian motion, $G_*(x)=xF_*'(x)/F_*(x)$, and F_* is the (single-variable) hypergeometric function $F(1-\frac{4}{\kappa},\frac{2\rho_2}{\kappa},\frac{2\rho_1+2\rho_2+4}{\kappa};\cdot)$.

Proof. We apply Theorem 1.2 to the case that $m=2$, $v_1=0^\sigma$, $v_2=v$, and $v_\infty=\sigma\infty$, and derive a statement about the law of η^r. Then the driving function $\widehat{w}^r$ of η^r solves the SDE (3.22). Note that $v^r_1=J(v_1)=-\sigma\infty$, and so $R^r_1\equiv 0$. The R^r_2 in Theorem 1.2 agrees with the R^r here. Also note that for the function F in Theorem 1.2, by (2.13) we have $F(0,\cdot)=F_*$. Thus, $G_1(R^r_1,R^r_2)\equiv 0$ and $G_2(R^r_1,R^r_2)\equiv G_*(R^r)$. Then (3.22) reduces to (5.4). □

Remark 5.2. The η^r in the corollary is an SLE$_\kappa$-type process with two force points, and may be defined using a single-variable hypergeometric function. But it is different from the intermediate SLE$_\kappa(\rho)$ process in [21], which is also defined using a single-variable hypergeometric function.

Appendices

A Laws of Stochastic Processes with Random Lifetime

This appendix can be viewed as a supplement of [19, Section 2], and we use the setup there as follows. Let S be a Polish space, and $\Sigma=\bigcup_{T\in(0,\infty]}C([0,T),S)$. For each $f\in\Sigma$, let $T_\Sigma(f)$

be such that $[0, T_\Sigma(f))$ is the domain of f. Let $\Sigma_t = \{f \in \Sigma : T_\Sigma(f) > t\}$, $0 \le t < \infty$, and $\Sigma_\infty = \bigcap_{0\le t<\infty} \Sigma_t = C([0,\infty), S)$. For $0 \le t < \infty$, let $\mathcal{F}_t$ be the σ-algebra generated by the family $\{f \in \Sigma_s : f(s) \in U\}$ over all $s \in [0,t]$ and $U \in \mathcal{B}(S)$, and let $\mathcal{F}_\infty = \vee_{0\le t<\infty}\mathcal{F}_t$. A probability measure on $(\Sigma, \mathcal{F}_\infty)$ is viewed as the law of a continuous S-valued stochastic process with random lifetime. For two probability measures μ and ν on Σ, we say that ν is locally absolutely continuous w.r.t. μ, and write $\nu \lhd \mu$, if for every $t \ge 0$, $\nu|_{\mathcal{F}_t\cap\Sigma_t} \ll \mu|_{\mathcal{F}_t\cap\Sigma_t}$, which means that for any $A \in \mathcal{F}_t$ with $A \subset \Sigma_t$, $\mu(A) = 0$ implies that $\nu(A) = 0$. Let M_t be the Radon-Nikodym derivative of $\nu|_{\mathcal{F}_t\cap\Sigma_t}$ against $\mu|_{\mathcal{F}_t\cap\Sigma_t}$. We call (M_t) the density process. It is clear that $\nu \ll \mu$ implies that $\nu \lhd \mu$.

Now suppose that μ and ν are probability measures on Σ, μ is supported by Σ_∞, and $\nu \lhd \mu$ with (M_t) being the density process. Then each M_t is μ-integrable, and for any $t_2 > t_1 \ge 0$ and $A \in \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$,

$$\int_A M_{t_2} d\mu = \int_{A\cap\Sigma_{t_2}} M_{t_2} d\mu = \nu(A \cap \Sigma_{t_2}) \le \nu(A \cap \Sigma_{t_1}) = \int_{A\cap\Sigma_{t_1}} M_{t_1} d\mu = \int_A M_{t_1} d\mu.$$

So (M_t) is a nonnegative supermartingale w.r.t. μ. The following lemma provides us the existence of ν given the measure μ and the supermartingale M.

Lemma A.1. *Let μ be a probability measure supported by Σ_∞. Let $(M_t)_{0\le t<\infty}$ be a nonnegative right-continuous $(\mathcal{F}_t)$-supermartingale w.r.t. μ such that $\int M_0 d\mu = 1$. Then there exists a unique probability measure ν on Σ such that $\nu \lhd \mu$, and M is the density process. Moreover, we have the following.*

(i) The ν is supported by Σ_∞ if and only if M is a martingale w.r.t. μ.

(ii) For any $(\mathcal{F}_t)$-stopping time τ, $\nu \ll \mu$ on $\mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$, and M_τ is the Radon-Nikodym derivative.

(iii) For any $(\mathcal{F}_t)$-stopping time τ, $\nu \ll \mu$ on $\mathcal{F}_\tau$ if and only if $M(t \wedge \tau)$, $t \ge 0$, is a uniformly integrable martingale (w.r.t. μ); and then $d\nu|_{\mathcal{F}_\tau}/d\mu|_{\mathcal{F}_\tau} = M_\tau$. Here on the event $\tau = \infty$, M_τ is understood as $M_\infty := \lim_{t\to\infty} M_{t\wedge\tau}$, which μ-a.s. converges. In particular, $\nu \ll \mu$ if and only if M is a uniformly integrable martingale; and then $d\nu/d\mu = M_\infty$.

We say that the measure ν is constructed by locally weighting the measure μ by M.

Proof. Add an extra element, denoted by $*$, to S, and write S_* for the union $S \cup \{*\}$. For $\Lambda \subset [0,\infty)$, an element $f \in S_*^\Lambda$ is called ordered if there do not exist $\lambda_1 < \lambda_2 \in \Lambda$ such that $f(\lambda_1) = *$ and $f(\lambda_2) \in S$. For any finite set $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\} \subset [0,\infty)$, we define a measure ν_Λ on S_*^Λ by the following. For $0 \le k \le n$, let $\pi_{\Lambda,k} : \Sigma_\infty \to S_*^\Lambda$ be defined by $\pi_{\Lambda,k}(f) = (f(t_0), f(t_1), \dots, f(t_k), *, \dots, *)$. For any measurable subset A of S_*^Λ, define

$$\nu_\Lambda(A) = \sum_{k=0}^{n-1} \int_{\pi_{\Lambda,k}^{-1}(A)} (M_{t_k} - M_{t_{k+1}}) d\mu + \int_{\pi_{\Lambda,n}^{-1}(A)} M_{t_n} d\mu.$$

Since M is a nonnegative supermartingale and $\int M_0 d\mu = 1$, ν_Λ is a probability measure. Since $\pi_{\Lambda,n}$ is the projection π_Λ from Σ_∞ onto S^Λ, and $\pi_{\Lambda,k}^{-1}(S^\Lambda) = \emptyset$ for $k < n$, we have $\nu_\Lambda|_{S^\Lambda} \ll \pi_\Lambda(\mu)$, and M_{t_n} is the Radon-Nikodym derivative. We also see that ν_Λ is supported by the set of ordered elements of S_*^Λ since every $\pi_{\Lambda,k}$ takes values in ordered elements.

We now check that $\{\nu_\Lambda : 0 \in \Lambda \subset [0,\infty), |\Lambda| < \infty\}$ is a consistent family. Let $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\}$. Suppose $\Lambda' = \Lambda \cup \{s\} \subset [0,\infty)$ and $s \notin \Lambda$. Let $A \subset S_*^\Lambda$ be measurable, and $A' = A \times S_*^{\{s\}} \subset S_*^{\Lambda'}$. We need to show that $\nu_{\Lambda'}(A') = \nu_\Lambda(A)$. First, suppose $s > t_n$. Then for each $0 \le k \le n$, $\pi_{\Lambda',k}^{-1}(A') = \pi_{\Lambda,k}^{-1}(A)$, and $\pi_{\Lambda',n+1}^{-1}(A') = \pi_{\Lambda,n}^{-1}(A)$. So

$$\nu_{\Lambda'}(A') = \sum_{k=0}^{n} \int_{\pi_{\Lambda',k}^{-1}(A')} (M_{t_k} - M_{t_{k+1}}) d\mu + \int_{\pi_{\Lambda',n+1}^{-1}(A)} M_s d\mu$$

$$= \sum_{k=0}^{n-1} \int_{\pi_{\Lambda,k}^{-1}(A)} (M_{t_k} - M_{t_{k+1}}) d\mu + \int_{\pi_{\Lambda,n}^{-1}(A)} (M_{t_n} - M_s) d\mu + \int_{\pi_{\Lambda,n}^{-1}(A)} M_s d\mu = \nu_\Lambda(A).$$

Next, suppose $t_{k_0-1} < s < t_{k_0}$ for some $1 \le k_0 \le n$. Then for any $k < k_0$, $\pi_{\Lambda',k}^{-1}(A') = \pi_{\Lambda,k}^{-1}(A)$, and for any $k \ge k_0$, $\pi_{\Lambda',k}^{-1}(A') = \pi_{\Lambda,k-1}^{-1}(A)$. Let $t'_k = t_k$ for $0 \le k < k_0$, $t'_{k_0} = s$, and $t'_k = t_{k-1}$ for $k_0 < k \le n+1$. Then $\Lambda' = \{0 = t'_0 < t'_1 < \cdots < t'_{n+1}\}$. So

$$\nu_{\Lambda'}(A') = \sum_{k=0}^{n} \int_{\pi_{\Lambda',k}^{-1}(A')} (M_{t'_k} - M_{t'_{k+1}}) d\mu + \int_{\pi_{\Lambda',n+1}^{-1}(A')} M_{t'_{n+1}} d\mu$$

$$= \sum_{k=0}^{k_0-2} \int_{\pi_{\Lambda,k}^{-1}(A)} (M_{t_k} - M_{t_{k+1}}) d\mu + \int_{\pi_{\Lambda,k_0-1}^{-1}(A)} (M_s - M_{t_{k_0-1}}) d\mu + \int_{\pi_{\Lambda,k_0-1}^{-1}(A)} (M_{t_{k_0}} - M_s) d\mu$$

$$+ \sum_{k=k_0}^{n} \int_{\pi_{\Lambda,k-1}^{-1}(A)} (M_{t_{k-1}} - M_{t_k}) d\mu + \int_{\pi_{\Lambda,n}^{-1}(A)} M_{t_n} d\mu = \nu_\Lambda(A).$$

By Kolmogorov extension theorem, there is an S_*-valued process $(Z_t)_{0 \le t < \infty}$ (defined on some probability space) such that for any finite set $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\} \subset [0,\infty)$, the joint distribution of $(Z_{t_0}, Z_{t_1}, \ldots, Z_{t_n})$ is ν_Λ. We now restrict our attention to $(Z_p)_{p \in \mathbb{Q}_+}$. By the properties of ν_Λ we know that for any $p_1 < p_2 \in \mathbb{Q}$, if $Z_{p_1} = *$ then a.s. $Z_{p_2} = *$. Thus, by excluding an event with probability zero, we may assume that $(Z_p)_{p \in \mathbb{Q}_+}$ takes values in ordered elements. Let $T_\Sigma = \inf\{p \in \mathbb{Q}_+ : Z_p = *\}$. Then T_Σ is a random number such that $Z_t \in S$ for $t \in [0, T_\Sigma) \cap \mathbb{Q}_+$ and $Z_t = *$ for $t \in (T_\Sigma, \infty) \cap \mathbb{Q}_+$.

Suppose $(Y_t)_{t \ge 0}$ is a continuous process with law μ. Let $t_0 \in \mathbb{Q}_+$. By the property of ν_Λ, the law of $(Z_p)_{p \in [0,t_0] \cap \mathbb{Q}_+}$ restricted to the event that $Z_{t_0} \in S$, is absolutely continuous w.r.t. that of $(Y_p)_{p \in [0,t_0] \cap \mathbb{Q}_+}$, and the Radon-Nikodym derivative is M_{t_0}. Since Y is continuous on $[0,\infty)$, this implies that on the event that $Z_{t_0} \in S$, a.s. $(Z_p)_{p \in [0,t_0] \cap \mathbb{Q}_+}$ extends to a (random) continuous function $Z^{(t_0)}$ on $[0, t_0]$. By excluding an event with probability zero, we may assume that this is always true for every $t_0 \in \mathbb{Q}_+$. We may define a continuous function Z' on $[0, T_\Sigma)$ such that for any $p \in \mathbb{Q}_+$, on the event $\{T_\Sigma > p\}$, which is contained in $\{Z_p \in S\}$, we define $Z' = Z^{(p)}$ on $[0,p]$. There is no contradiction in the definition because whenever $p_1 < p_2 \in \mathbb{Q}_+$, on the event $\{T_\Sigma > p_2\}$, which is contained in $\{T_\Sigma > p_1\}$, we have $Z^{(p_1)} = Z^{(p_2)}|_{[0,p_1]}$. Then Z' is a continuous stochastic process with a random lifetime T_Σ.

Let ν be the law of Z'. We claim that ν is the measure that we need. Fix $t_* \geq 0$. We need to show that $\nu(A) = \int_A M_{t_*} d\mu$ for any $A \in \mathcal{F}_{t_*} \cap \Sigma_{t_*}$. For every finite set $\Lambda \subset [0,\infty)$, let π_Λ denote the natural projection from $S_*^{[0,\infty)}$ onto S_*^Λ. We naturally embed Σ into $S_*^{[0,\infty)}$ by understanding the value of $f(t)$ for $t \geq T_\Sigma(f)$ as $*$. So π_Λ is also a mapping from Σ into S_*^Λ. First, assume that there is $\Lambda \subset \mathbb{Q} \cap [0,t_*]$ with $0 \in \Lambda$ and $|\Lambda| < \infty$ such that $A = \pi_\Lambda^{-1}(A_\Lambda) \cap \Sigma_{t_*}$ for some $A_\Lambda \in \mathcal{B}(S^\Lambda)$. Let (p_m) be a sequence in $\mathbb{Q} \cap (t_*,\infty)$ such that $p_m \downarrow t_*$. By the definition of T_Σ, we see that $Z' \in \Sigma_{t_*}$, i.e., $T_\Sigma > t_*$, if and only if there is some m such that $Z_{p_m} \in S$. Also note that $Z'(t) = Z(t)$ for $t \in \Lambda$ because $\Lambda \subset \mathbb{Q}$. Thus,

$$\nu(A) = \mathbb{P}[Z' \in A] = \mathbb{P}[\bigcup_{m=1}^\infty \{\pi_\Lambda(Z) \in A_\Lambda, Z(p_m) \in S\}] = \lim_{m\to\infty} \mathbb{P}[\pi_{\Lambda\cup\{p_m\}}(Z) \in A_\Lambda \times S]$$

For each $m \in \mathbb{N}$, since the law of $\pi_{\Lambda\cup\{p_m\}}(Z)$ is $\nu_{\Lambda\cup\{p_m\}}$, whose restriction to $S^{\Lambda\cup\{p_m\}}$ is absolutely continuous w.r.t. $\pi_{\Lambda\cup\{p_m\}}(\mu)$ with Radon-Nikodym derivative M_{p_m}, we get

$$\mathbb{P}[\pi_{\Lambda\cup\{p_m\}}(Z) \in A_\Lambda \times S] = \nu_{\Lambda\cup\{p_m\}}(A_\Lambda \times S) = \int_{\pi^{-1}_{\Lambda\cup\{p_m\}}(A_\Lambda\times S)} M_{p_m} d\mu = \int_{\pi_\Lambda^{-1}(A_\Lambda)} M_{p_m} d\mu.$$

Here the last equality follows from that μ is supported by Σ_∞. Since $p_m \downarrow t_*$, by right-continuity of M and Fatou's lemma,

$$\int_{\pi_\Lambda^{-1}(A_\Lambda)} M_{t_*} d\mu \leq \liminf_{m\to\infty} \int_{\pi_\Lambda^{-1}(A_\Lambda)} M_{p_m} d\mu.$$

On the other hand, since M is an $(\mathcal{F}_t)$-supermartingale w.r.t. μ, for all $m \in \mathbb{N}$,

$$\int_{\pi_\Lambda^{-1}(A_\Lambda)} M_{p_m} d\mu \leq \int_{\pi_\Lambda^{-1}(A_\Lambda)} M_{t_*} d\mu.$$

Combining the last four displayed formulas and the fact that μ is supported by $\Sigma_\infty \subset \Sigma_{t_*}$, we get $\nu(A) = \int_A M_{t_*} d\mu$. This holds for any A in the π-family

$$\{\pi_\Lambda^{-1}(A_\Lambda) \cap \Sigma_{t_*} : A_\Lambda \in \mathcal{B}(S^\Lambda), \Lambda \subset [0,t_*] \cap \mathbb{Q}, |\Lambda| < \infty\},$$

which generates the σ-algebra $\mathcal{F}_{t_*} \cap \Sigma_{t_*}$ in Σ_{t_*} thanks to the continuity of $f \in \Sigma_{t_*}$ on $[0,t_*]$. By Dynkin's $\pi - \lambda$ theorem, we get $\nu(A) = \int_A M_{t_*} d\mu$ for any $A \in \mathcal{F}_{t_*} \cap \Sigma_{t_*}$. The uniqueness of such ν also follows from Dynkin's $\pi - \lambda$ theorem.

(i) If ν is supported by Σ_∞, then for any $t_2 \geq t_1 \geq 0$ and $A \in \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$, we get $\nu(A) = \nu(A \cap \Sigma_{t_j}) = \int_{A\cap\Sigma_{t_j}} M_{t_j} d\mu = \int_A M_{t_j} d\mu$, $j = 1,2$. So $\mathbb{E}_\mu[M_{t_2}|\mathcal{F}_{t_1}] = M_{t_1}$, i.e., M is a martingale.

On the other hand, if M is a martingale, then for any $t \ge 0$, $\nu(\Sigma_t) = \int M_t d\mu = \int M_0 d\mu = 1$. So $\nu(\Sigma_\infty) = \lim_{t\to\infty} \nu(\Sigma_t) = 1$, i.e., ν is supported by Σ_∞.

(ii) Let τ be an $(\mathcal{F}_t)$-stopping time. Since M is right-continuous and adapted, it is progressive. So M_τ on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable. First assume that τ takes values in $\mathbb{Q}_+$. Let $A \in \mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$. Then for any $t \in \mathbb{Q}_+$, $A \cap \{\tau = t\} \in \mathcal{F}_t \cap \{T_\Sigma > t\}$. So we have

$$\nu(A) = \sum_{t\in\mathbb{Q}_+} \nu(A \cap \{\tau = t\}) = \sum_{t\in\mathbb{Q}_+} \int_{A\cap\{\tau=t\}} M_t d\mu = \sum_{t\in\mathbb{Q}_+} \int_{A\cap\{\tau=t\}} M_\tau d\mu = \int_A M_\tau d\mu.$$

Next, we do not assume that τ takes values in $\mathbb{Q}_+$, but assume that there is a deterministic number $N \in \mathbb{N}$ such that $\tau < N$. For each $n \in \mathbb{N}$, define τ_n such that if $\tau \in [\frac{k-1}{2^n}, \frac{k}{2^n})$ for some $k \in \mathbb{N}$, then $\tau_n = \frac{k}{2^n}$. Then each τ_n is a bounded stopping time taking values in $\mathbb{Q}_+$, and $\tau_n \downarrow \tau$. Let $A \in \mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$, and $A_n = A \cap \{T_\Sigma > \tau_n\}$. Then $A = \bigcup_n A_n$, and $A_n \in \mathcal{F}_{\tau_n} \cap \{T_\Sigma > \tau_n\}$ for each n. So we have $\nu(A) = \lim_{n\to\infty} \nu(A_n) = \lim_{n\to\infty} \int_{A_n} M_{\tau_n} d\mu = \lim_{n\to\infty} \int_A M_{\tau_n} d\mu$, where the last "=' follows from that μ is supported by Σ_∞. By right-continuity of M and Fatou's lemma, $\lim_{n\to\infty} \int_A M_{\tau_n} d\mu \ge \int_A M_\tau d\mu$. Applying Optional Stopping Theorem to the right-continuous supermartingale M and the bounded stopping times $\tau \le \tau_n$, we get $\int_A M_{\tau_n} d\mu \le \int_A M_\tau d\mu$ for each n. So $\lim_{n\to\infty} \int_A M_{\tau_n} d\mu = \int_A M_\tau d\mu$, and we then get $\nu(A) = \int_A M_\tau d\mu$. Finally, we do not assume that τ is uniformly bounded. Let $A \in \mathcal{F}_\tau \cap \{T_\Sigma > \tau\}$. Then for any $N \in \mathbb{N}$, $\tau \wedge N$ is a uniformly bounded stopping time, and $A \cap \{\tau \le N\} \in \mathcal{F}_{\tau\wedge N} \cap \{T_\Sigma > \tau \wedge N\}$. So $\nu(A \cap \{\tau \le N\}) = \int_{A\cap\{\tau\le N\}} M_{\tau\wedge N} d\mu = \int_{A\cap\{\tau\le N\}} M_\tau d\mu$. By monotone convergence theorem, we get $\nu(A) = \lim_{N\to\infty} \nu(A \cap \{\tau \le N\}) = \int_A M_\tau d\mu$, as desired.

(iii) First, suppose $\nu \ll \mu$ on $\mathcal{F}_\tau$ with $\zeta = d(\nu|\mathcal{F}_\tau)/d(\mu|\mathcal{F}_\tau)$. Then for any $t \ge 0$, $\nu \ll \mu$ on $\mathcal{F}_{\tau\wedge t}$ with $d(\nu|\mathcal{F}_{\tau\wedge t})/d(\mu|\mathcal{F}_{\tau\wedge t}) = \mathbb{E}_\mu[\zeta|\mathcal{F}_{\tau\wedge t}]$. By (ii), $d(\nu|\mathcal{F}_{\tau\wedge t})/d(\mu|\mathcal{F}_{\tau\wedge t}) = M(\tau \wedge t)$ on $\Sigma_{\tau\wedge t}$. Since μ is supported by $\Sigma_\infty \subset \Sigma_{\tau\wedge t}$, we get $M_{\tau\wedge t} = \mathbb{E}_\mu[\zeta|\mathcal{F}_{\tau\wedge t}]$ for all $t \ge 0$. Thus, $M_{\tau\wedge t}$, $t \ge 0$, is a uniformly integrable martingale.

Next, suppose that $M_{\tau\wedge t}$, $t \ge 0$, is a uniformly integrable martingale. Then $\lim_{t\to\infty} M_{\tau\wedge t}$ converges a.s. to M_τ, and for any $t \ge 0$, $\mathbb{E}_\mu[M_\tau|\mathcal{F}_{\tau\wedge t}] = M_{\tau\wedge t}$. Define a measure ν_τ on $(\Sigma, \mathcal{F}_\tau)$ by $d\nu_\tau = M(\tau)d\mu$. Since $\mathbb{E}_\mu[M_\tau] = \mathbb{E}_\mu[M_0] = 1$, ν_τ is a probability measure. For any $t \ge 0$, $\nu_\tau \ll \mu$ on $\mathcal{F}_{\tau\wedge t}$, and $d(\nu_\tau|\mathcal{F}_{\tau\wedge t})/d(\mu|\mathcal{F}_{\tau\wedge t}) = \mathbb{E}_\mu[M_\tau|\mathcal{F}_{\tau\wedge t}] = M_{\tau\wedge t}$. On the other hand, by (ii) $d(\nu|\mathcal{F}_{\tau\wedge t})/d(\mu|\mathcal{F}_{\tau\wedge t}) = M_{\tau\wedge t}$ on $\Sigma_{\tau\wedge t}$. Since ν_τ and ν are both probability measures, they must agree on $\mathcal{F}_{\tau\wedge t}$. Since $\bigcup_{t\ge0} \mathcal{F}_{\tau\wedge t}$ is a π-family, by Dynkin's $\pi - \lambda$ theorem, ν_τ and ν agree on $\vee_{t\ge0}\mathcal{F}_{\tau\wedge t} = \mathcal{F}_\tau$. By the definition of ν_τ, we get that $d(\nu|\mathcal{F}_\tau)d(\mu|\mathcal{F}_\tau) = M_\tau$.

The last statement of (iii) follows from the above equivalence by choosing $\tau = \infty$. □

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